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Integral Sliding Mode in Systems Operating under Uncertainty Conditions

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Abstract

The paper generalized a new Sliding Mode design concept, namely *Integral Sliding Mode*. Different from the conventional Sliding Mode design approaches, the order of the motion equation in *Integral Sliding Mode* is equal to the order of the original system, rather than reduced by the number of dimension of the control input. As the result, robustness of the system can be guaranteed throughout an entire response of the system starting from the initial time instance. Uniform formulations of this new Sliding Mode design principle are developed in the paper. It is shown through examples that our generalized *Integral Sliding Mode* scheme enables a wide scope of application areas including control in robotics, electric drives, etc.. In the case that the given control matrix can not be used directly for generating the Sliding Mode, the associated *Decoupling Problem* was also discussed based on the concept of *Sliding Mode Transformation*. To alleviate *Chattering*, often encountered in the conventional Sliding Mode control system, we remove the discontinuous control action from the real control path and insert it to an internal dynamic process for generating the sliding mode.

Introduction

For a general non-linear dynamic system at the presence of modeling uncertainties and external disturbances, Sliding Mode Control approaches originated from the theory of Variable Structure System (VSS) may be found [1]. The term Variable Structure System first made its appearance in the late 1950's. Since that time, the first expectations of such systems have naturally been reevaluated. Their real potential has been revealed. Research directions have been originated to new control methodologies due to the appearance of new control problems, new mathematical methods and recent advances in switching circuits. It has been shown that dominant role in VSS theory is played by Sliding Modes, and the core ideal of designing VSS control algorithms consists of enforcing this type of motion in some manifolds in system spaces. Traditionally, these manifolds are constructed by the intersection of some hyper surfaces in the state space and this intersection domain is normally called switching plane. Once the system state reaches the switching plane, the structure of the feedback loop is adaptively altered to *slide* the system state along the switching plane; the system response depends thereafter on the gradient of the switching plane and remains insensitive to variations of system parameters and external disturbances. This motion is called Sliding Mode. The order of the motion equation in the sliding motion is equal to $n-m$ with n being the dimension of the state space and m the dimension of the control input. However, during the reaching phase (before Sliding Mode occurs), the system possesses no such insensitivity property; therefore, insensitivity can not be ensured throughout an entire response. The robustness during the reaching phase is normally improved by high-gain feedback control. Stability problems that arise inevitably limit the application of such high-gain feedback control schemes [2].

On the other hand, the concept of *Integral Sliding Mode* concentrates on the robustness of the motion in the whole state space. The order of the motion equation in this new type of Sliding Mode is equal to the dimension of the state space. Therefore, the robustness of the system can be guaranteed throughout an entire response of the system starting from the initial time instance. This paper focuses on the generalization of the *Integral Sliding Mode* concept and emphasizes the background philosophy used for developing such a new Variable Structure System. We assume that there exists already an *ideal system* that consists of a known non-linear plant and a proper designed feedback control. A discontinues part, which is designed based on the *Integral Sliding Mode* for rejecting the additional unknown part and/or external disturbances, is then added to this existing control. Design examples in some application areas are given as the natural extension of our design philosophy, although they may be already more or less mentioned in the literature in a particular research area and in a non-general form [3], [5].

As a very important part of this paper, we deal with *Chattering Problem* based on our *Integral Sliding Mode* concept. This leads to a novel approach for constructing a so-called Perturbation Estimator by removing the discontinue control action from the real control path and insert it into an internal dynamic process. As the result, *Chattering* will no more be exited in the controlled plant due to the continuity of the control action while high degree of robustness and high accuracy of the control are preserved. This is the major advantage over the conventional approaches [4] used to overcome *Chattering Problem*.

1. Problem statement

For a given dynamic system represented by the following state space equation

$$\dot{x} = f(x) + B(x)u \quad (1)$$

where $x \in R^n$, $u \in R^m$, suppose there exists a feedback control law $u = u_0(x)$, which may be continuous or discontinuous, such that system (1) can be stabilized in a desired way (e.g. its state trajectory follows a reference trajectory with a given accuracy). We denote this ideal closed loop system as

$$\dot{x}_* = f(x) + B(x)u_0 \quad (2)$$

where x_* denotes the state trajectory of the ideal system under control u_0. However, systems like (1) are normally operating under some uncertainty conditions that may be generated by parameter variations, unmodeled dynamics and external disturbances etc. Under this consideration a real control system may be summarized with

$$\dot{x} = f(x) + B(x)u + h(x,t) \quad (3)$$

in which function $h(x,t)$ represents the whole perturbation described above and it fulfills the *matching condition* i.e.

$$h(x,t) \in span\{B(x)\} \quad (4)$$

or equivalently

$$h(x,t) = B(x)u_h \text{ with } u_h \in R^m. \quad (5)$$

It is assumed here that function h is bounded and with known upper bound

$$|h_i(x,t)| \le k_i(x,t), \quad i = 1 \sim n, \quad (6)$$

with $k_i(x,t)$ being a known positive scalar function.

Now, following question arises: what is u such that the solutions of the system (3) satisfies $x(t) \equiv x_*(t)$, starting from the initial time instance i.e. $x(0) = x_*(0)$?

2. Integral Sliding Mode

For system (3), firstly, we design a control like

$$u = u_0 + u_1, \quad (7)$$

where u_0 is the *ideal control* defined in (2) and u_1 is designed to be discontinuous for rejecting the perturbation term $h(x,t)$. Secondly, we design our *switching function* s as

$$s = s_0(x) + z, \text{ with } s, s_0(x), z \in R^m. \quad (8)$$

This *switching function* consists of two parts; the first part $s_0(x)$ may be designed as the linear combination of the system states (similar to the conventional Sliding Mode design); and, the second part z induces the integral term and will be determined below.

To derive the Sliding Mode equation, the time derivative of s on the system trajectories should be made equal zero; the algebraic equation $\dot{s} = 0$ should be solved with respect to control input and the solution u_{eq} referred to as the Equivalent Control should be substituted into the motion equation for u [1].

Our design philosophy: upon the required performances we propose to design an integral feedback such that the Equivalent Control

$$u_{1eq} = -u_h, \quad (9)$$

and starting from this Equivalent Control we determine the formulation of variable z *and hence the formulation of* s.

Condition (9) holds if

$$\dot{z} = -\frac{\partial s_0}{\partial x}\{f(x) + B(x)u_0(x)\}, \quad z(0) = -s_0(x(0)), \quad (10)$$

where $z(0)$ is determined based on the requirement $s(0) = 0$ (Sliding Mode occurs starting from the initial time instance) . The motion equation of the system in Sliding Mode will be

$$\dot{x} = f(x) + B(x)u_0(x). \quad (11)$$

As one can see, the motion equation of the system in Sliding Mode is just our ideal system (2).

Definition: a Sliding Mode is said to be an ***Integral Sliding Mode*** *if the motion equation of the system in this Sliding Mode is of the same order as the original system.*

Now let us discuss the problem of *Enforcing the Sliding Mode*. For the derivation of (10), we have made the assumption that matrix $\frac{\partial s_0}{\partial x}B(x)$ is *non-singular*. If we design $u_1 = -M(x)sign(s)$ ($M(x)$ is positive definite diagonal matrix and $sign(s) = [sign(s_1)\, sign(s_2) \cdots sign(s_m)]^T$), then the Sliding Mode can be enforced under the condition that the matrix $\frac{\partial s_0}{\partial x}B(x)$ is positive definite and the elements of matrix $M(x)$ are large enough. The proof is straight forward using Lyapunov Second Method by considering $V = 0.5s^Ts$ as *Lyapunov candidate*. For certain class of systems e.g. mechanical systems, the above conditions may be fulfilled, since in these systems consisting of second order equations the control matrix $B(x)$ is positive definite. However, there exist some systems (e.g. electric drive systems) which belong to the category described by equation (3) whose control matrix is non-singular only. In this case the selection of s_0 and the design of u_1 is no more straight forward. This problem can be solved generally by the following *decoupling control* procedure.

3. Decoupling control

Let us define a new vector s^*, a transformation of s through a *non-singular* projector Ω being determined later,

$$s^* = \Omega s \tag{12}$$

with $\Omega \in R^{m\times m}$; it follows

$$s = \Omega^{-1} s^* . \tag{13}$$

Design a *Lyapunov candidate*

$$V = 0.5 s^T s \tag{14}$$

and take the time derivative of V with the substitution of equations (8), (10), (3) and (13), yields

$$\dot{V} = (s^*)^T f^* + (s^*)^T \Psi u_1 , \tag{15}$$

where $f^* = (\Omega^{-1})^T \frac{\partial s_0}{\partial x} h(x,t)$ and $\Psi = (\Omega^{-1})^T \frac{\partial s_0}{\partial x} B(x)$.

Let the discontinuous control input u_1 be of the following form

$$u_1 = -M(x) sign(s^*) , \tag{16}$$

in which $M(x) \in R^{m\times m}$ is a positive definite diagonal matrix. Substitute (16) into (15), gives

$$\dot{V} = (s^*)^T f^* - (s^*)^T \Psi M(x) sign(s^*) . \tag{17}$$

Now we select matrix Ω such that matrix Ψ is positive definite and then select matrix $M(x)$ such that $\dot{V}$ is negative. A reasonable candidate of matrix Ω is

$$\Omega = \left\{ \frac{\partial s_0}{\partial x} B(x) \right\}^T \tag{18}$$

which enables a complete decoupling. In this case matrix Ψ becomes an unit matrix and equation (17) can be simplified as following

$$\dot{V} = (s^*)^T u_h - (s^*)^T M(x) sign(s^*) . \tag{19}$$

Obviously the gain matrix $M(x)$ enforcing the Sliding Mode is of form

$$m_i(x) > UH_i , \quad i = 1 \sim m , \tag{20}$$

where $m_i(x)$ is the ith diagonal element of matrix $M(x)$ and UH_i is the upper bound of ith element of vector u_h defined as

$$UH_i \geq |u_{hi}| , \quad i = 1 \sim m . \tag{21}$$

For practical systems, control inputs to the actuators are always bounded; consequently the maximum allowable perturbation is also bounded, otherwise Sliding Motion can not be enforced. This limitation in terms of u_h is given below

$$|u_{hi}^{max}| < ULIMIT_i - |u_{0i}(t)| , \quad i = 1 \sim m , \tag{22}$$

with $ULIMIT_i$ is the control limitation of the ith control input u_i.

4. Examples

4.1. A linear time invariant system [5]

Consider a controllable linear time invariant system with scalar control

$$\dot{x} = Ax + B[u + f(x,t)] , \tag{23}$$

where $x \in R^n$, $u \in R^1$, A and B are known matrix and vector respectively, and $f(x,t)$ is a non-linear perturbation with known upper bound $f_0(x,t)$ i.e.

$$|f(x,t)| < f_0(x,t) . \tag{24}$$

Design control u as stated in equation (7): $u = u_0 + u_1$, where u_0 is predetermined such that system $\dot{x} = Ax + Bu_0$ follows a given trajectory with satisfactory accuracy. For example, u_0 may be obtained through linear static feedback control, like $u_0 = -k^T x$, $k \in R^{n\times 1}$, in which gain vector k can be determined by *Pole Placement* or *LQG* methods.

According to (8) and (10), design the switching function

$$s = C^T x + z , \qquad C \in R^{n\times 1} , \tag{25}$$

$$\dot{z} = -C^T \{Ax + Bu_0\}, \; z(0) = -C^T x(0) ; \tag{26}$$

then the motion equation of the Sliding Mode coincides with that of the ideal system $\dot{x} = Ax + Bu_0$ without perturbation. Further more, since $s(0) = C^T x(0) + z(0) = 0$, Sliding Mode will occur from the initial time instance $t = 0$. For a linear system like (23) with full rank matrix B the switching function (25) may be designed such that $C^T B > 0$; so that our second part of the control i.e. u_1 can be designed as following

$$u_1 = -m_0(x) sign(s) \tag{27}$$

where $m_0(x) > |f_0(x,t)|$, and associated $\dot{V}$ will be negative.

4.2. Control of robot manipulators

Dynamic equation of a rigid body robot manipulator with n degree of freedom is given as

$$\hat{M}(q)\ddot{q} + \hat{N}(q,\dot{q}) + h(q,\dot{q},\ddot{q}) = \tau \tag{28}$$

where $\hat{M} \in R^{n\times n}$ is the known mass matrix; $\hat{N} \in R^{n\times 1}$ is the known vector including centrifugal, coriolis and gravity forces ; $h \in R^{n \times 1} = \Delta M\ddot{q} + \Delta N + d(t)$ is the whole perturbation with ΔM and ΔN denoting the error between known and real quantities and $d(t)$ denoting the external disturbance vector. Variable q represents the joint angle vector and τ the joint torque vector.

Using the so called *Computed Torque Method* based on the model without perturbation, the required joint torque for the tracking control of the joint position will be

$$\tau_0 = \hat{M}(q)(\ddot{q}_d + K_v\dot{e} + K_p e) + \hat{N}(q,\dot{q}) \qquad (29)$$

where $K_p, K_v \in R^{n \times n}$ are positive definite diagonal matrices determining the closed loop performance, and $e = q_d(t) - q(t)$ with $q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ being the desired trajectories. Substituting (29) into (28) the resulted error dynamics is

$$\ddot{e} + K_v\dot{e} + K_p e = \hat{M}^{-1}(q)h(q,\dot{q},\ddot{q}). \qquad (30)$$

As one can see from (30), no matter how the constant matrices K_p, K_v are chosen, the tracking error e will not tend to zero or even not be stable. Now according to the proposed *Integral Sliding Mode* design method, i.e. equations (8) and (10), let the switching function be $s = s_0 + z$ in which

$$s_0 = Ce + \dot{e}, \qquad (31)$$
$$\dot{z} = -C\dot{e} + K_v\dot{e} + K_p e,\ z(0) = -Ce(0) - \dot{e}(0), \qquad (32)$$

where C is a positive definite matrix (as can be seen below, matrix C here is not necessary in this manipulator control problem). After integration of (32) we get our switching function

$$s = \dot{e} + K_v e + K_p\int_0^t e(\zeta)d\zeta - K_v e(0) - \dot{e}(0) \qquad (33)$$

If the Sliding Mode could be enforced by proper designed joint torque, then $\dot{s} = 0$ as well. From (33) this leads to

$$\ddot{e} + K_v\dot{e} + K_p e = 0. \qquad (34)$$

It is just the ideal error dynamics, as if perturbations did not exist.

Now let us design the joint torque needed for generating the Sliding Mode: $\tau = \tau_0 + \tau_1$ with τ_0 already defined in (29) and τ_1 the discontinuous part to reject the disturbances. Design

$$\tau_1 = -\Gamma sign(s) \qquad (35)$$

where $\Gamma \in R^{n \times n}$ is again a positive definite diagonal matrix with its *ith* diagonal element satisfying $\Gamma_i > |h_i(q,\dot{q},\ddot{q})|$, $i = 1 \sim n$ (It should be noted that this condition may not always be fulfilled since the right-hand side depends on $\ddot{q}$ and as a result on Γ_i). Because matrix $\frac{\partial s_0}{\partial x}B$ ($x = [e\ \dot{e}]^T$ here) is equal to $\hat{M}^{-1}(q)$ which is known to be positive definite, so that the decoupling procedure discussed in section 3 is not necessary.

4.3. Control of electric drive systems - implementation of Pulse-Width Modulation [6]

Different from the examples in section 4.1 and 4.2 we use here directly our design philosophy in stead of applying equation (8), (10) to implement the so called *Pulse Width Modulation* (PWM) in an electric drive system. Without loss of generality, an electric machine can be described by a dynamic system like

$$\dot{x} = f(x) + B(x)u,\ x \in R^n,\ u \in R^m \qquad (36)$$

where x may represents current and/or flux components of the stator or rotor and u is the DC-link voltage which takes only two values, namely, $-u_0$ and $+u_0$. For field oriented control design, equation (36) is often transformed into a rotating coordinate aligned with one of the flux vector (rotor flux, stator flux or airgap flux). Denoting the transformation matrix as T, a non-linear projector with sinusoidal elements, the system equation (36) may be transformed into a new coordinate system y:

$$\dot{y} = f_y(y) + B_y(y)u_y \qquad (37)$$

where u_y is the new control input in coordinate y. Suppose that the control u_y has been determined satisfying the given specifications. Now the question arises: how to obtain the control u taking only discontinuous values $-u_0$ and $+u_0$ and being equivalent to u_y? Let us now transform the control u_y back to the original coordinate system using the inverse transformation T^{-1}, and denote this transformed control as u^*

$$u^* = T^{-1}u_y. \qquad (38)$$

Now we would like to make the equivalent value of the control u to be equal to the equivalent value of u^* i.e. $u_{eq} = u^*_{eq}$. Under this consideration we may design a Sliding Mode

$$s = \int_0^t [u(\zeta) - u^*(\zeta)]d\zeta = 0 \qquad (39)$$

and associated control u should be

$$u = -u_0 sign(s). \qquad (40)$$

It can be easily proven that the Sliding Mode can be enforced if $u_0 > \|u^*\|$, or in other words, the DC-link voltage is high enough to enforce the desired motion. Fig. 1 shows the results of a PWM implementation in the control of an induction motor using this Sliding Mode control principle.

5. Alleviation of Chattering - Integral Sliding Mode in Perturbation Estimation

Due to the nature of Sliding Mode control schemes, switched control action inevitably results in high frequency chattering in practical implementation. To alleviate this phenomenon, the continuos approximation of the discontinuous sign function is the most direct approach [4], in which the sign function is simply replaced by a saturation function. However, chattering is diminished by confining the gain inside the layer that is inversely proportional to the layer thickness. Neither robustness nor tracking error meet the specifications. Hence, chattering is alleviated at the expense of robustness and accuracy. To overcome this major drawback of traditional Sliding Mode schemes, we are going to reformulate our *Integral Sliding Mode* principle in terms of *Perturbation Estimator*.

In stead of equation (7) we change our control input to

$$u = u_0 + u_{1eq}. \tag{41}$$

However, the equivalent value of a discontinuous control can not be obtained directly, because it depends on unknown disturbances. It is shown in [1] that the equivalent value is equal to the average value measured by a first order liner filter, with the real discontinuous control as its input, if the time constant of the filter is coordinated with the width of the area where motion in real Sliding Mode takes place. Therefore we can write $u_{1eq} = u_{1av}$ with u_{1av} defined in

$$\mu \dot{u}_{1av} + u_{1av} = u_1 \tag{42}$$

in which the time constant μ should be made so small that the linear filter should not distort the slow component of the switched action which is equal to u_{1eq}. Normally the spectrum of the perturbation does not overlap with the high frequency components of the switching unit.

To this moment one may ask the questions: if the discontinuity in the real control path is smoothed, how could the Sliding Mode be generated? and, does u_{1av} ($= u_{1eq}$) still cancel the perturbation u_h? As it is developed following, these questions are answered positively.

Similarly to equations (8) and (10) we redesign the switching function

$$s = s_0(x) + z, \tag{43}$$

with z defined in

$$\dot{z} = -\frac{\partial s_0}{\partial x}\{f(x) + B(x)u - B(x)u_1\}, \quad z(0) = -s_0(x(0)). \tag{44}$$

Following the Lyapunov proof in section 3, it can be easily shown that the Sliding Mode defined in (43) will be enforced if the same condition as stated in (20) holds. Further more, solving u_1 from $\dot{s} = 0$ shows that $u_{1eq} = -u_h$ holds as well.

In this case equation (44) can be interpreted as an internal process for generating the Sliding Mode given in (43); discontinuity appears only in the internal process, thus no chatters are excited in the real control path. Moreover, since u_{1av} cancels the perturbation u_{1h} without knowing the precise knowledge of the system model and associated parameters, the high degree of robustness is maintained. Another advantage of this perturbation estimation scheme over the traditional methods is that the time derivative of the state vector is not necessary; the only information needed here is the upper bound of the perturbation. From the concept point of view *Integral Sliding Mode* is utilized here only for the estimation of the system perturbation rather than for the propose of control. The control action to the real controlled system will be continuous enhanced by the perturbation compensator.
Fig. 2 shows the effectiveness of this control scheme in the torque control of a flexible robot joint [8]. In this application the mass of inertia of the motor-rotor and of the arm, the joint stiffness and the frictions at both motor and joint sides are assumed to be not known precisely.

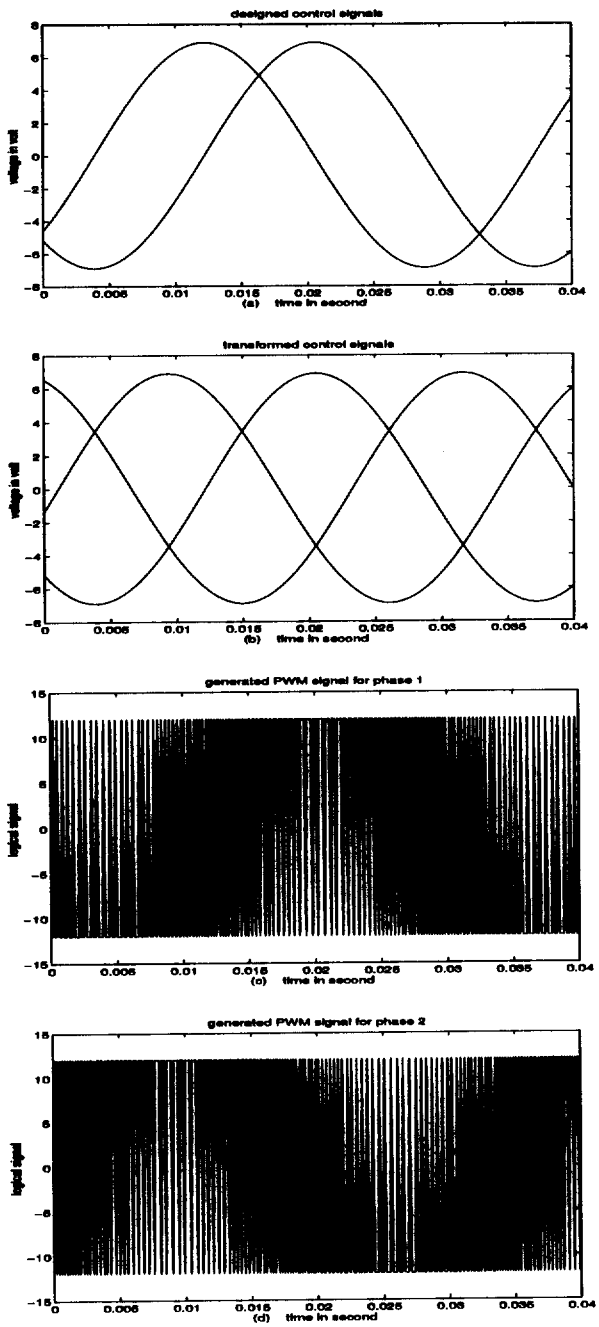
Conclusion

The new Sliding Mode design concept - *Integral Sliding Mode* is generalized in this paper. The proposed uniform formulation of the *Integral Sliding Mode* enables a wide scope of application areas. The most advantage of this new design principle is that the robustness provided by Sliding Mode can be guaranteed throughout an entire response of the system starting from the initial time instance. Here, we emphasis the basic ideal and the background philosophy used to develop such a new Sliding Mode design approach. And, we also pay attentions to its application problems in practical systems. The examples presented in the paper are the natural extension of our Sliding Mode design principle, although some of them were more or less mentioned in the literature. The *Chattering Problem* which was the major drawback in the context of the Sliding Mode control is solved using proposed algorithms, while the robustness provided by Sliding Mode design and the accuracy of the control system are preserved.

References

[1] V. I. Utkin, "Sliding Modes in Control and Optimization," Springer-Verlag, 1992.

[2] Harashima, F. H. Hashimoto, and S.Kondo, "MOSFET converter-fed position servo system with sliding mode control," IEEE Trans. Ind. Electron., 32, 238-244, 1985.
[3] J. -J. E. Slotine, "The robust control of robot manipulator". Int. J. Robotics Res., 4, 49-64, 1985.
[4] J. -J. E. Slotine, "Sliding controller design for non-linear system," Int. J. Control, 40, 421-434, 1984.
[5] J. Ackermann, V.I. Utkin, "Sliding Mode Control Design Based On Ackermann Formula," CDC'94, Florida, December 14-16, 1994, pp. 3622-3627.
[6] J.Shi, V.I. Utkin, "Sliding Mode Control in Electric Drives," Lecture Notes in DLR seminar, 1993.
[7] V. I. Utkin, "Sliding mode control design principles and applications to electric drives," IEEE Trans. Indust. Electron., pp. 2-22, Feb. 1993.
[8] J. Shi, Y. S. Lu, "Chatter Free Variable Structure Perturbation Estimator on the Torque Control of Flexible Robot Joints with Disturbance and Parametric Uncertainties," IEEE IECON'96, 1996.



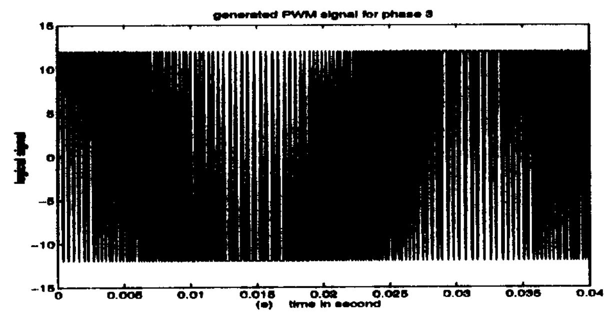


Fig.1 PWM implementation based on the Integral Sliding Mode for the control of a real induction motor; (a) designed the control signals in the two phases stator coordinate; (b) transformed control signals in the three phases stator coordinate; (c), (d) and (e) PWM signals generated by the Integral Sliding Mode, feed to the gates of inverters.

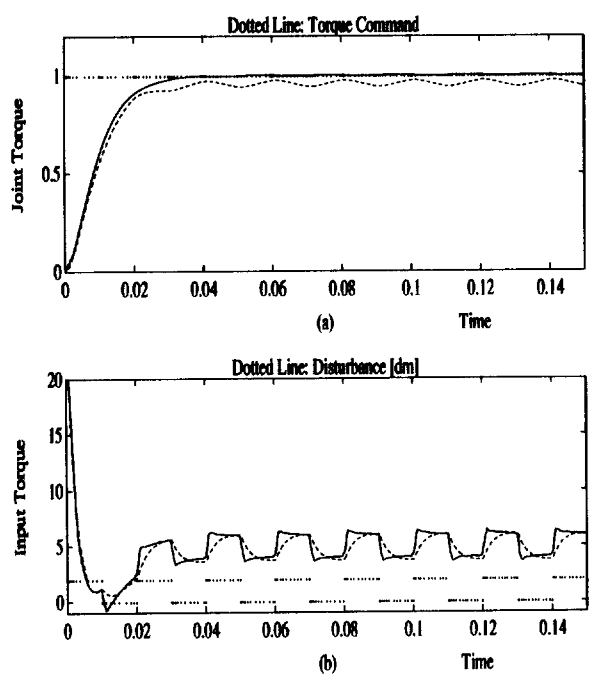


Fig. 2. Dynamical responses: (a) Output torque. Solid line: with perturbation compensation. Dashed line: without perturbation compensation. Dotted line: the torque command. (b) Input torque. Solid line: with perturbation compensation. Dashed line: without perturbation compensation. Dotted line: the disturbance torque of the motor side.