Feedback Limitations in Nonlinear Systems: From Bode Integrals to Cheap Control*

M.M. Seron
seron@seidel.ece.ucsb.edu

J.H. Braslavsky
julio@seidel.ece.ucsb.edu

P.V. Kokotović
petar@ece.ucsb.edu

D.Q. Mayne
mayne@seidel.ece.ucsb.edu

Center for Control Engineering and Computation
University of California, Santa Barbara, CA 93106-9560

Abstract

This paper links the Bode integral characterization of performance limitations in linear control systems with the limiting cost of a particular linear cheap control. This link allows us to establish analogous feedback limitations in nonlinear systems. We show how unstable zero dynamics impose unavoidable obstructions to the closed-loop performance.

Keywords: Feedback limitations, Performance limitations, Bode integrals, Nonlinear cheap control, Zero dynamics, Nonminimum phase systems.

1 Introduction

The frequency-domain analysis of limitations to the performance of linear feedback systems was initiated by Bode [2]. Since then, many authors have further clarified these limitations (e.g., [6, 7, 15]). In particular, Francis & Zames [6] and Freudenberg & Looze [7] made explicit the role of nonminimum phase (NMP) zeros and open-loop unstable poles in limiting the closed-loop performance. All these works rely on the analyticity of closed-loop transfer functions, typically the sensitivity function S and the complementary sensitivity function T, to obtain integral formulas that express feedback limitations. Sensitivity functions have also been used in input-output operator frameworks to study limitations in nonlinear systems [25, 23]. These and other recent contributions to the frequency-domain / input-output operator analysis of feedback limitations are reviewed in [21].

With the development of the linear quadratic optimal control design in the 1960's, a different approach to performance limitations was pursued by Chang [3], Kalman [11], Kwakernaak & Sivan [13, 14], Shaked [24], and, most recently, by Qiu & Davison [16] and Chen, Qiu & Toker [4]. Letting the cost of control tend to zero ("cheap control"), these studies evaluated the best achievable performance.

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It is surprising that the relationship between the limiting cost of optimal linear control and the Bode integral characterization of performance limitations has not been investigated. Such an investigation is undertaken in this paper. For linear systems, we show in Section 2 that the "missing link" is a relation between the limiting optimal cost for cheap control, obtained by Qiu & Davison [16], and the Bode T-integral formula due to Middleton [15].

The linear analysis in Section 2 prepares the path for an investigation of feedback limitations in nonlinear systems, developed in Section 3. We present a novel singular perturbation analysis of a nonlinear cheap control problem and explain how the unstable nonlinear zero dynamics determine the limiting cost. Our analysis reveals that *the cheap control problem reduces to the minimum energy problem in which the system output is used to stabilize the nonlinear zero dynamics.* We also discuss a nonlinear analog of the fact that for linear cheap control some closed-loop poles converge to the mirror image of NMP zeros [3, 11, 13, 14].

The results of a related investigation of the lowest achievable level of L_2 disturbance attenuation were reported by Schwartz, Isidori & Tarn [19] after the submission of the original version of this paper. They show that the lowest level of attenuation is determined by the unstable zero dynamics of the system. The crucial role of the unstable zero dynamics is thus revealed in both disturbance attenuation and disturbance-free problems.

2 Feedback Limitations in Linear Systems

2.1 Cheap Control

Consider the linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + \tilde{B}u(t), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y(t) &= Cx(t), \quad y \in \mathbb{R}^m, x(0) = x_0 \end{aligned} \tag{1}$$

and the cost functional

$$J_\varepsilon = \frac{1}{2}\int_0^\infty \left(y^T(t)y(t) + \varepsilon^2 u^T(t)u(t)\right)dt \tag{2}$$

We are interested in the *cheap control problem*, that is, in finding a state feedback control u that minimizes J_ε when $\varepsilon > 0$ is small. It was shown by Chang [3] and Kalman [11] that, as $\varepsilon \to 0$, some closed-loop poles converge to the *mirror image* of the NMP zeros, while the remaining poles tend to infinity in Butterworth patterns. This fact reveals a fundamental limitation: the closed-loop poles that tend to the mirror image of the NMP zeros are not canceled. Instead, they remain in the observable part of the closed-loop dynamics.

Kwakernaak & Sivan [13, 14] (see also [5, 18]) proved that the lowest achievable cost is zero *if and only if the system is minimum phase and right invertible*. Shaked [24] (see also [8]) obtained an explicit expression for the lowest achievable cost as a function of the NMP zeros. Qiu & Davison [16] provided a simpler formula for particular initial conditions. A complete treatment of linear cheap control can be found in [17].

Assumption 1.

(i) (1) is stabilizable and detectable.

(ii) (1) has relative degree one, that is, rank $C\tilde{B} = m$.

(iii) All zeros of $C(sI - \tilde{A})^{-1}\tilde{B}$ are NMP. ◦

Assumption (i) is necessary and sufficient for a meaningful linear optimal control problem. Assumptions (ii) and (iii), which represent the case with the maximum number of NMP zeros, are made only for simplicity of exposition and can be removed.

By a change of coordinates $\tilde{x} = [M^T \; M_0^T]^T x$, the full rank input matrix $\tilde{B}$ can be brought to the form $[B_1^T \; 0]^T$, where B_1 is nonsingular. Then, with the same M_0 and with the full rank matrix C, the change of coordinates $[y^T \, z^T]^T = [C^T \, M_0^T]^T x$ transforms (1) to the form

$$\begin{aligned} \dot{y} &= A_1 y + A_2 z + B u, \qquad B = C\tilde{B} , \\ \dot{z} &= B_0 y + A_0 z . \end{aligned} \tag{3}$$

This form displays the zeros of $C(sI - \tilde{A})^{-1}\tilde{B}$ as the eigenvalues of A_0. We observe that (A_0, B_0) is controllable, because (3) is stabilizable and A_0 is *antistable* (that is $-A_0$ is Hurwitz) by (iii), Assumption 1.

The associated algebraic Riccati equation (ARE) for the optimal control of (3) with respect to the cost (2) is

$$\begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix}^T P + P \begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix} + \begin{bmatrix} 0 & \\ 0 & \end{bmatrix} = \frac{1}{\varepsilon^2} P \begin{bmatrix} BB^T & 0 \\ 0 & \end{bmatrix} P . \tag{4}$$

For this singularly perturbed equation Jameson & O'-Malley [10] (see also [27]) provided a solution in the form of a power series in ε. Its leading term is

$$P = \begin{bmatrix} \varepsilon\bar{P}_1 & \varepsilon\bar{P}_2 \\ \varepsilon\bar{P}_2^T & \bar{P}_0 + \varepsilon\bar{P}_3 \end{bmatrix} + O(\varepsilon^2) . \tag{5}$$

$\bar{P}_0$, $\bar{P}_1$ and $\bar{P}_2$, which are independent of ε, are obtained explicitly from the substitution of (5) into (4):

$$I - \bar{P}_1 BB^T \bar{P}_1 + O(\varepsilon) = 0 , \tag{6}$$

$$B_0^T \bar{P}_0 - \bar{P}_1 BB^T \bar{P}_2 + O(\varepsilon) = 0 , \tag{7}$$

$$A_0^T \bar{P}_0 + \bar{P}_0 A_0 - \bar{P}_2^T BB^T \bar{P}_2 + O(\varepsilon) = 0 . \tag{8}$$

Setting $\varepsilon = 0$, (6) gives $\bar{P}_1 = (BB^T)^{-1/2}$, and using this in (7) gives $\bar{P}_2 = (BB^T)^{-1/2} B_0^T \bar{P}_0$. Substituting in (8) we find that $\bar{P}_0 = \bar{P}_0^T$ is the positive definite solution of

$$A_0^T \bar{P}_0 + \bar{P}_0 A_0 = \bar{P}_0 B_0 B_0^T \bar{P}_0 . \tag{9}$$

The value function for the cheap control problem is

$$V(y, z, \varepsilon) = \frac{1}{2} \begin{bmatrix} y \\ z \end{bmatrix}^T \begin{bmatrix} \varepsilon\bar{P}_1 & \varepsilon\bar{P}_2 \\ \varepsilon\bar{P}_2^T & \bar{P}_0 + \varepsilon\bar{P}_3 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + O(\varepsilon^2) , \tag{10}$$

which tends in the limit, as $\varepsilon \to 0$, to

$$V(y, z, 0) = \frac{1}{2} z^T \bar{P}_0 z . \tag{11}$$

This value is the best achievable performance for system (3) with respect to cost (2).

To interpret (11) we consider the *minimum energy problem* for the zero-dynamics subsystem: find y to stabilize $\dot{z} = A_0 z + B_0 y$ and to minimize

$$J = \frac{1}{2} \int_0^\infty y^T(t) y(t) \, dt. \tag{12}$$

Here the system output y acts as the control variable for the zero-dynamics subsystem. The value function for this problem, denoted by $\bar{V}_s(z)$, is $\bar{V}_s(z) = \frac{1}{2} z^T \bar{P}_0 z$, where $\bar{P}_0 = \bar{P}_0^T$ is the positive definite solution of (9). Thus we have established

$$V(y, z, 0) = \bar{V}_s(z) = \frac{1}{2} z^T \bar{P}_0 z , \tag{13}$$

that is, *the limiting cost for the cheap control problem is equal to the optimal cost for the minimum energy problem for the zero-dynamics subsystem*.

The optimal closed-loop zero-dynamics subsystem is

$$\dot{z} = -\bar{P}_0^{-1} A_0^T \bar{P}_0 z$$

and its eigenvalues are the mirror image of the NMP zeros of (3). Hence the cheap control places $n - m$ eigenvalues of the closed-loop system at the mirror image of the NMP zeros of (3), and the remaining m eigenvalues at infinity.

2.2 Singular Perturbation Analysis

A further insight into the properties of the cheap control is provided by a singular perturbation analysis of the closed-loop system

$$\begin{aligned} \varepsilon\dot{y} &= -BB^T (BB^T)^{-1/2} (y + B_0^T \bar{P}_0 z) + O(\varepsilon) , \\ \dot{z} &= B_0 y + A_0 z , \end{aligned} \tag{14}$$

obtained when the optimal feedback control

$$
\begin{aligned}
u^* &= -\frac{1}{\varepsilon}\, B^T(\bar{P}_1 y + \bar{P}_2 z + O(\varepsilon)) \\
&= -\frac{1}{\varepsilon}\, B^T(BB^T)^{-1/2}(y + B_0^T \bar{P}_0 z + O(\varepsilon))
\end{aligned}
\tag{15}
$$

is applied to (3). It is well known (see [12]) that the slow subsystem of (14) is approximated by $\dot{z} = -\bar{P}_0^{-1} A_0^T \bar{P}_0 z$, resulting from the substitution of $y = -B_0^T \bar{P}_0 z$ into the z-equation. The slow dynamics evolve in the *slow invariant subspace* $y = \phi(\varepsilon) z$ of (14), where $\phi(\varepsilon) = -B_0^T \bar{P}_0 + O(\varepsilon)$. This means that $y = -B_0^T \bar{P}_0 z$ is an $O(\varepsilon)$ approximation of the slow invariant subspace, and $y - \phi(\varepsilon) z = y + B_0^T \bar{P}_0 z + O(\varepsilon)$ is the fast variable which rapidly converges to zero.

Due to the two-time-scale property of (14), the value function (10) can be expressed as

$$
V(y, z, \varepsilon) = \varepsilon \bar{V}_f(y, z) + \bar{V}_s(z) + \frac{\varepsilon}{2} z^T \bar{P}_3 z + O(\varepsilon^2) \,, \tag{16}
$$

where $\bar{V}_f(y, z) = \frac{1}{2} y^T \bar{P}_1 y + y^T \bar{P}_2 z = \frac{1}{2} y^T (BB^T)^{-1/2} y + y^T (BB^T)^{-1/2} B_0^T \bar{P}_0 z$, and $\bar{V}_s(z)$ defined in (13), correspond to the *fast* and *slow* parts of the system, respectively. We observe that (14) can be expressed as

$$
\begin{aligned}
\varepsilon \dot{y} &= -BB^T \frac{\partial^T \bar{V}_f}{\partial y} + O(\varepsilon) \,, \\
\dot{z} &= B_0 y + A_0 z
\end{aligned}
\tag{17}
$$

because $y = \phi(0) z = -B_0^T \bar{P}_0 z$ is the subspace where

$$
\frac{\partial \bar{V}_f(y, z)}{\partial y} = 0 \,. \tag{18}
$$

In our nonlinear analysis, the slow dynamics will evolve in the *slow invariant manifold* $y = \phi(z, \varepsilon)$, the nonlinear analog of $y = \phi(\varepsilon) z$. To define this slow manifold we will use the fact that its $O(\varepsilon)$ approximation $y = \phi(z, 0)$ is given by $\partial \bar{V}_f(y, z)/\partial y = 0$.

2.3 Cheap Control and Bode T-Integral

Let us now consider the problem of regulating the output y of (3) to a constant setpoint r. We employ a feedforward term $\bar{u}$ to guarantee that the equilibrium of (3) is at $y = r$. Solving for this equilibrium gives $\bar{u} = -B^{-1}(A_1 - A_2 A_0^{-1} B_0) r$. Introducing the error variables $e = y - r$, $\tilde{z} = z + A_0^{-1} B_0 r$, and $\tilde{u} = u - \bar{u}$, we rewrite (3) as

$$
\begin{aligned}
\dot{e} &= A_1 e + A_2 \tilde{z} + B\tilde{u} \,, \\
\dot{\tilde{z}} &= A_0 \tilde{z} + B_0 e \,.
\end{aligned}
$$

The optimal control $\tilde{u}$ that minimizes

$$
\tilde{J}_\varepsilon = \frac{1}{2} \int_0^\infty \left(e^T(t) e(t) + \varepsilon^2 \tilde{u}^T(t) \tilde{u}(t) \right) dt \,,
$$

is of the same form as (15), and the value function is of the same form as (10), but with $(e, \tilde{z})$ replacing (y, z). From (13) we know that the limit of the value function $V(e(0), \tilde{z}(0), \varepsilon)$ as $\varepsilon \to 0$ is $V(e(0), \tilde{z}(0), 0) = \bar{V}_s(\tilde{z}(0))$. What is of interest here is the value of this limit for the initial conditions $e(0) = r$ and $\tilde{z}(0) = A_0^{-1} B_0 r$, which correspond to y being transferred from 0 to r. The answer is given by a formula due to Qiu & Davison [16].

Lemma 2.1 (Qiu-Davison Formula). For the initial condition $e(0) = -r$ and $\tilde{z}(0) = -A_0^{-1} B_0 r$, the limit of the value function $V(e(0), \tilde{z}(0), \varepsilon)$ for the cheap control problem as $\varepsilon \to 0$ satisfies

$$
\bar{V}_s(\tilde{z}(0)) = \frac{1}{2} r^T H r \,, \quad \text{where} \;\; \operatorname{trace} H = 2 \sum_{i=1}^{k} \frac{1}{\alpha_i} \,, \tag{19}
$$

and $\alpha_1, \ldots, \alpha_k$, are the NMP zeros of system (3). ◦

Thus, the lowest achievable cost to steer the system from rest to the setpoint $y = r$ is an explicit function of the NMP zeros, independent of the system realization. This limiting cost will be larger when the NMP zeros are closer to the imaginary axis.

How is this result related to performance limitations defined by well-known Bode integrals [2, 7, 15]? These hold for the sensitivity and complementary sensitivity functions of the feedback loop in Fig. 1, where L is the open-loop transfer function, formed by the series connection of plant and controller. We first concentrate on the complementary sensitivity function $T = L\,(I + L)^{-1}$ which maps reference r to output y.

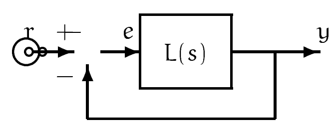

Figure 1: Unitary feedback loop.

A Bode integral for T was derived by Middleton [15]. We review this formula next. Consider single-input single-output (SISO) systems described by rational proper transfer functions. Let $\{\alpha_1, \ldots, \alpha_k\}$ be the set of open right half plane (ORHP) zeros of the open-loop system L, and assume that $L(0) \neq 0$. Then, if the closed-loop system in Fig. 1 is stable and $T(0) = 1$,

$$
\frac{1}{\pi} \int_0^\infty \log |T(j\omega)| \, \frac{d\omega}{\omega^2} + \frac{1}{2K_v} = \sum_{i=1}^{k} \frac{1}{\alpha_i} \,, \tag{20}
$$

where $K_v = \lim_{s \to \infty} sL(s)$ is the *velocity constant* [26, p. 286]. When the controller is minimum phase, the set $\{\alpha_1, \ldots, \alpha_k\}$ consists of the plant's ORHP zeros. In this case, we refer to the left hand side of (20) as the *complementary sensitivity invariant*, briefly, the T-*invariant*.

The connection between the Middleton formula (20) and the Qiu-Davison formula (19) has, apparently, remained unnoticed. We make this connection explicit.

Lemma 2.2 (Cheap Control and T-Invariant). Let L, with $L(0) \neq 0$, be a SISO open-loop system formed by the series connection of a plant and a linear, minimum phase controller. Suppose that $T = L(1+L)^{-1}$ is stable and such that $T(0) = 1$. Then

$$\frac{1}{\pi}\int_0^\infty \log|T(j\omega)| \frac{d\omega}{\omega^2} + \frac{1}{2K_v} = \lim_{\varepsilon\to 0}\frac{1}{2}\int_0^\infty e^2(t)\,dt,$$

where $e(t) = y(t) - r$ is the error in the cheap control problem transferring the system from rest to the setpoint $r = 1$.

Proof. Immediate from (20), the scalar version of Lemma 2.1, and $\lim_{\varepsilon\to 0}\frac{1}{2}\int_0^\infty e^2(t)\,dt = \bar{V}_s(\tilde{z}(0))$. □

Thus the T-invariant is the lowest achievable cost in the cheap control problem transferring the system from rest to a unit setpoint.

In the problem of setpoint regulation described before, let T be the transfer function from r to y. The analysis in Section 2.1 shows that T has poles at the mirror image of its NMP zeros, and the feedforward part of the controller guarantees that $T(0) = 1$. Hence in the limit, as $\varepsilon \to 0$, T satisfies $|T(j\omega)| = 1$ for all ω (see also [13, Theorem 3.13]), which in turn implies that $\int_0^\infty \log|T(j\omega)|\,d\omega/\omega^2 = 0$. Therefore, Lemma 2.2 implies that the cheap control yields the velocity constant

$$\lim_{\varepsilon\to 0}\frac{1}{K_v} = 2\sum_{i=1}^{n-1}\frac{1}{\alpha_i}. \tag{21}$$

On the other hand, for any relative-degree-one T,

$$\frac{1}{K_v} = \sum_{i=1}^{n-1}\frac{1}{\alpha_i} - \sum_{i=1}^{n}\frac{1}{\beta_i},$$

where $\alpha_1,\dots,\alpha_{n-1}$, and $\beta_1,\dots,\beta_n$, are respectively the zeros and poles of T [26, p. 282]. When T is the result of the above cheap control problem, then, as $\varepsilon \to 0$, one of the poles goes to $-\infty$, and the remaining $n-1$ poles converge to the mirror image of the NMP zeros, which again gives (21).

A dual S-invariant can be defined from the Bode integral formula for the sensitivity function. This S-invariant is the minimum energy required to transfer the system to rest from the initial condition originating from a unit impulse at the input [22].

3 Feedback Limitations in Nonlinear Systems

3.1 Cheap Control

We now analyze the cheap control problem to determine feedback limitations for systems of the form

$$\begin{aligned}\dot{y} &= f(y,z) + g(y,z)u, \quad y \in \mathbb{R}^m,\\ \dot{z} &= f_0(z) + g_0(z)y, \quad z \in \mathbb{R}^{n-m},\end{aligned} \tag{22}$$

where f, g, f_0, g_0 are C^1 vector fields, $f(0,0) = 0$, $f_0(0) = 0$ and $g(y,z)$ is nonsingular everywhere. The system (22) is the nonlinear analog of (3). It is in the *Isidori normal form* [9] for systems with relative degree one, in which the zero-dynamics subsystem $\dot{z} = f_0(z)$ is obtained by setting $y(t) \equiv 0$ in the z-equation.

We are interested in finding a feedback control u that, for any initial state $(y(0), z(0))$, asymptotically stabilizes (22) at $(0,0)$ and minimizes the cost functional

$$J_\varepsilon = \frac{1}{2}\int_0^\infty \left(y^T(t)y(t) + \varepsilon^2 u^T(t)u(t)\right)dt \tag{23}$$

in the limiting case when $\varepsilon \to 0$.

It is well-known (e.g., [20, Theorem 3.19]) that for $\varepsilon > 0$ this problem has a solution if there exists a C^1 positive semidefinite function $V(y,z,\varepsilon)$ which satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\partial V}{\partial y}f(y,z) + \frac{\partial V}{\partial z}(f_0(z) + g_0(z)y) + \frac{1}{2}y^T y - \frac{1}{2\varepsilon^2}\frac{\partial V}{\partial y}g(y,z)g^T(y,z)\frac{\partial^T V}{\partial y} = 0,\; V(0,0) = 0, \tag{24}$$

and such that the feedback control

$$u(y,z,\varepsilon) = -\frac{1}{\varepsilon^2}g^T(y,z)\frac{\partial^T V}{\partial y} \tag{25}$$

asymptotically stabilizes (22) at the origin. Then (25) is the optimal stabilizing control that minimizes J_ε and $V(y,z,\varepsilon)$ is the optimal value function.

Assumption 2.

(i) $\dot{z} = f_0(z)$ is *antistable*, that is, $\dot{z} = -f_0(z)$ is asymptotically stable at the origin.

(ii) There exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$, the above problem of stabilizing (22) and minimizing (23) has the solution $V(y,z,\varepsilon) \geq 0$, with $u(y,z,\varepsilon)$ given by (25). ∘

3.2 Singular Perturbation Analysis

From the linear analysis in Section 2.2 we expect that the cheap control problem of Section 3.1 exhibits the two-time-scale behavior. Therefore, we seek a solution to (24) in the power series form

$$V(y,z,\varepsilon) = \varepsilon\bar{V}_f(y,z) + \bar{V}_s(z) + O(\varepsilon^2), \tag{26}$$

where $\bar{V}_f$ and $\bar{V}_s$ are the fast and slow parts of V, respectively.

After substituting (26) in (25), we obtain the closed-loop system (22), (25) in the standard singular perturbation form

$$\begin{aligned} \varepsilon\dot{y} &= -g(y,z)g^T(y,z)\frac{\partial^T\bar{V}_f}{\partial y} + O(\varepsilon) \\ \dot{z} &= f_0(z) + g_0(z)y \ , \end{aligned} \tag{27}$$

which is the nonlinear analog of (17). The fast subsystem is the $\varepsilon\dot{y}$-equation with z treated as a constant parameter. To derive the slow subsystem we approximate the *slow invariant manifold* $y = \phi(z,\varepsilon)$ of (27) by the manifold

$$\frac{\partial\bar{V}_f(y,z)}{\partial y} = 0 \ , \tag{28}$$

which itself is an $O(\varepsilon)$ approximation of the equilibrium manifold $\varepsilon\dot{y} = 0$. We denote the graph of the manifold (28) by $y = \phi(z,0)$. Then, the slow subsystem is $\dot{z} = f_0(z) + g_0(z)\phi(z,0) + O(\varepsilon)$.

We need to verify that for every initial condition away from the slow manifold, the solution rapidly converges to the manifold. A Lyapunov function for this fast transient is $\varepsilon\bar{V}_f(y,z)$, treated as a function of y for each fixed z. The negativity of its derivative,

$$\varepsilon\dot{\bar{V}}_f = \varepsilon\frac{\partial\bar{V}_f}{\partial y}\dot{y} = -\frac{\partial\bar{V}_f}{\partial y}g(y,z)g^T(y,z)\frac{\partial^T\bar{V}_f}{\partial y} + O(\varepsilon) < 0 \ ,$$

shows that the slow manifold is attractive provided $\bar{V}_f(y,z)$ is positive definite in $y - \phi(z,\varepsilon)$ for each z. This is the case because, by (ii), Assumption 2, the stabilizing solution for the cheap control problem exists for each $\varepsilon \in (0,\varepsilon^*]$.

3.3 The Nonlinear Analog of the T-Invariant

We are ready to study the limiting behavior in the slow manifold $y = \phi(z,\varepsilon)$. Using (26) in (24) yields

$$\frac{\partial\bar{V}_s}{\partial z}\left(f_0(z) + g_0(z)y\right) + \frac{1}{2}y^Ty - \frac{1}{2}\frac{\partial\bar{V}_f}{\partial y}g(y,z)g^T(y,z)\frac{\partial^T\bar{V}_f}{\partial y} + O(\varepsilon) = 0 \ . \tag{29}$$

As $\varepsilon \to 0$ the slow manifold $y = \phi(z,\varepsilon)$ becomes $\partial\bar{V}_f/\partial y = 0$ and the HJB equation (29) reduces to

$$\frac{\partial\bar{V}_s}{\partial z}\left(f_0(z) + g_0(z)y\right) + \frac{1}{2}y^Ty = 0 \ . \tag{30}$$

It is readily observed that (30) is the HJB optimality condition for the following minimum energy problem: find y to stabilize the zero dynamics

$$\dot{z} = f_0(z) + g_0(z)y \tag{31}$$

and to minimize the cost (12). In this problem the system output y is treated as the control variable, and the optimal control is $y = -g_0^T(z)\frac{\partial^T\bar{V}_s}{\partial z}$, which substituted in (30) yields the HJB equation

$$\frac{\partial\bar{V}_s}{\partial z}f_0(z) = \frac{1}{2}\frac{\partial\bar{V}_s}{\partial z}g_0(z)g_0^T(z)\frac{\partial^T\bar{V}_s}{\partial z} \ . \tag{32}$$

If $\dot{z} = f_0(z)$ were asymptotically stable, then the optimal solution of (32) would be $\bar{V}_s(z) \equiv 0$. However, in this case $\dot{z} = f_0(z)$ is antistable and, in order to asymptotically stabilize the zero-dynamics subsystem, $\bar{V}_s(z)$ must be positive definite.

We summarize our findings as follows.

Theorem 3.1 (Limiting Optimal Cost).

(i) Under Assumption 2, for every initial state (y,z) the limiting optimal cost of the nonlinear cheap control problem (22), (23) is

$$\lim_{\varepsilon\to 0} V(y,z,\varepsilon) = \bar{V}_s(z) \ ,$$

where $\bar{V}_s(z)$ is the optimal value function of the minimum energy problem for the zero dynamics (31) controlled by the output y.

(ii) The optimal closed-loop z-subsystem is

$$\dot{z} = f_0^{CL}(z) \triangleq f_0(z) - g_0(z)g_0^T(z)\frac{\partial^T\bar{V}_s}{\partial z} \ , \tag{33}$$

and along the trajectories of the closed- and open-loop systems the value function $\bar{V}_s(z)$ satisfies

$$\dot{\bar{V}}_s^{OL}(z) = -\dot{\bar{V}}_s^{CL}(z) \ , \tag{34}$$

where

$$\begin{aligned} \dot{\bar{V}}_s^{CL}(z) &\triangleq \frac{\partial\bar{V}_s}{\partial z}(z)f_0^{CL}(z) \ , \\ \dot{\bar{V}}_s^{OL}(z) &\triangleq \frac{\partial\bar{V}_s}{\partial z}(z)f_0(z) \ . \end{aligned}$$ ◦

The "mirroring" property (34) follows from (32) and (33), and is a consequence of the Hamiltonian decomposition into a stable and an unstable manifold [1]. For antistable systems, the unstable manifold corresponds to the solution $\bar{V}_s(z) \equiv 0$ of (32).

Thus, in the cheap control problem for NMP nonlinear systems, the output y cannot be steered arbitrarily rapidly to zero, because it must first attend to the task of stabilizing the zero dynamics. The fast part $\varepsilon\bar{V}_f$ of the value function $V(y,z,\varepsilon)$ is negligible, so that the lowest achievable cost is given by $\bar{V}_s(z)$, where z is the initial state of the zero-dynamics subsystem. The linear property that the optimal poles are placed as the mirror image of the NMP zeros has its nonlinear analog in the mirroring property (34).

As in the linear case, we can formulate the cheap control problem to steer the output $y(t)$ to the setpoint $r = 1$. Including a feedforward control term $\bar{u} = g^{-1}(r, \bar{z}) f(r, \bar{z})$, where $\bar{z}$ is determined from $f_0(\bar{z}) + g_0(\bar{z}) r = 0$, we can introduce the new variables $e = y - r$, $\tilde{z} = z - \bar{z}$ and $\tilde{u} = u - \bar{u}$. The limiting cost $\bar{V}_s(\tilde{z})$ of this setpoint problem would then yield the nonlinear analog of the T-invariant.

4 Conclusions

We have linked the two sets of results that quantify feedback limitations in linear control systems. The T-invariant is the cost achieved by the cheap control transferring the system output to a unit setpoint.

After establishing this link for linear systems, we have shown that the cheap control problem for nonlinear systems becomes, in the limiting case, the minimum energy problem for the zero dynamics driven by the output of the system. Thus the lowest achievable cost is the amount of control energy needed to stabilize the unstable zero dynamics. For nonlinear systems this a fundamental limitation of feedback analogous to that given by the T-invariant.

Our analysis has also provided a definite answer to the heretofore unanswered question whether the feedback limitation for linear systems, expressed by the T-invariant, can be reduced (or even eliminated) with an unconstrained nonlinear controller. The answer to this question is negative, because the analysis of the cheap control problem proves that the lowest cost achieved with a linear controller cannot be further reduced by any nonlinear controller. This limitation of feedback control is caused by the zero dynamics in both linear and nonlinear systems.

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