COMPUTATIONALLY EFFICIENT IMPLEMENTATION OF HYPERCOMPLEX DIGITAL FILTERS

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ABSTRACT

Hypercomplex digital filters have an attractive advantage of the order reduction, however, also have a drawback that multiplication requires a large amount of computations. This paper proposes a novel implementation of hypercomplex digital filters. By decomposing hypercomplex number multiplication, we show that it can be realized as two parallel complex multiplications. Using this technique, any types of hypercomplex digital filters can be implemented with less than half computations of the direct approach.

1. INTRODUCTION

Hypercomplex numbers are generally defined as an expansion of complex numbers[1]. Among hypercomplex numbers, Hamilton's quaternion is well-known, however, Schütte et al. suggest in [2] that quaternions are not suited for digital signal processing and they have proposed the modified version of quaternions as "reduced biquaternions" (RBs).

Recently several researches concerning hypercomplex coefficient digital filters have been reported [2, 3, 4, 5]. One of the most significant advantages of using hypercomplex numbers is the order reduction in digital filters. For example, the fourth order IIR digital filter is reduced to the first order one [2]. Another example is that the power complementary complex coefficient filter set of the second order is composed of the first order hypercomplex all-pass filter [3]. Furthermore, in [4, 5], hypercomplex coefficient digital filters have been applied to various types of transfer functions.

Despite of such interesting features, this number system have a serious drawback that multiplication is cost ineffective, i.e., one hypercomplex multiplication requires 16 real multiplications and 12 real additions. Due to this drawback, hypercomplex digital filters have not often been employed in digital signal processing.

To overcome this problem, several approaches have been studied. On the multiplication of quaternions, it is known that ten real multiplications is needed in [6]. Similarly, Dimitrov et al. have shown that in an RB multiplication, the number of real multiplications can be reduced to ten or nine [7]. As another approach, Mizukami et al. have employed a residue number system (RNS) and have realized quarter size reduction [8]. However, in RNS multiple multiplication modules and auxiliary hardware modules such as residue to binary and binary to residue converters are needed, which lead to complex implementation.

In this paper, we basically consider the above mentioned hypercomplex number for digital signal processing, however, we simply call it a "bicomplex number" in connection with its derivation. In the following, bicomplex multiplication is reviewed and by decomposing the multiplication algorithm, we show that it can be realized as two parallel complex multiplications. Furthermore, we apply to hypercomplex coefficient digital filters and evaluate the efficiency of our algorithm.

2. BICOMPLEX NUMBERS

Hypercomplex numbers are generally defined as the numbers whose components are combined with more than two different imaginary units. Among hypercomplex numbers, Hamilton's quaternion is well known as

$$q = q_1 + iq_2 + jq_3 + kq_4, \tag{1}$$

where q_1, q_2, q_3, and q_4 are real numbers and i, j, and k are the imaginary units satisfying that

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

However, it has been pointed out in [2] that quaternions are not suited for digital signal processing mainly because they are not commutative in multiplication.

In this paper, we consider another number system as follows. Let a_1 and a_2 be complex numbers as

$$a_1 = a_{11} + ja_{12},$$
$$a_2 = a_{21} + ja_{22}, \quad (2)$$

where j is the imaginary unit such that $j^2 = -1$. Then we consider

$$A = a_1 + ia_2 = (a_{11} + ja_{12}) + i(a_{21} + ja_{22}), \quad (3)$$

where i is a different imaginary unit called a vector unit such that $i^2 = -1$. The number A is consequently composed of four real numbers. We will call it a "bicomplex number" hereafter. To avoid confusion, it should be noted that bicomplex numbers treated in this paper are substantially the same as RBs [2], which are derived from Hamilton's biquaternions. As is reported in [2], bicomplex numbers or RBs are commutative in addition and multiplication, and so they can be applied to most of digital signal processing algorithms directly.

3. MULTIPLICATION OF BICOMPLEX NUMBERS

Given two bicomplex numbers A and B,

$$A = a_1 + ia_2 = (a_{11} + ja_{12}) + i(a_{21} + ja_{22}), \quad (4)$$

$$B = b_1 + ib_2 = (b_{11} + jb_{12}) + i(b_{21} + jb_{22}). \quad (5)$$

Multiplication of A and B is defined as follows;

$$\begin{aligned} C &= A \cdot B = B \cdot A \\ &= c_1 + ic_2 = (c_{11} + jc_{12}) + i(c_{21} + jc_{22}), \quad (6) \end{aligned}$$

where

$$c_{11} = a_{11}b_{11} - a_{12}b_{12} - a_{21}b_{21} + a_{22}b_{22},$$
$$c_{12} = a_{11}b_{12} + a_{12}b_{11} - a_{21}b_{22} - a_{22}b_{21},$$
$$c_{21} = a_{11}b_{21} - a_{12}b_{22} + a_{21}b_{11} - a_{22}b_{12},$$
$$c_{22} = a_{11}b_{22} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{11}.$$

In the direct computation, the number of arithmetic units amounts to 16 real multiplications and 12 real additions.

From the result of the direct bicomplex multiplication, let $\hat{c}_{1r}$, $\hat{c}_{2r}$, $\hat{c}_{1i}$, and $\hat{c}_{2i}$ be

$$\hat{c}_{1r} = c_{11} + c_{22},$$
$$\hat{c}_{2r} = c_{11} - c_{22},$$
$$\hat{c}_{1i} = c_{12} - c_{21},$$
$$\hat{c}_{2i} = c_{12} + c_{21}.$$

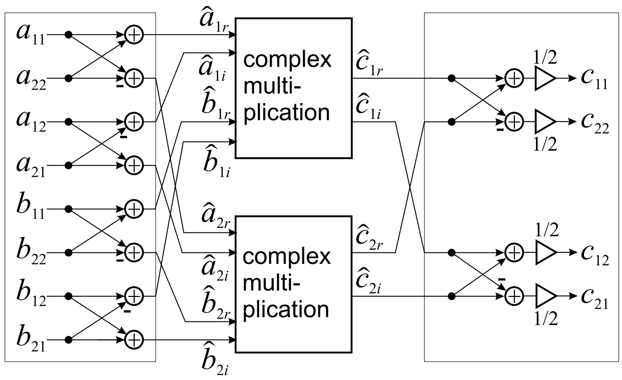

Figure 1: Algorithm 1.

By easy inspection, each variable is factorized as

$$\hat{c}_{1r} = (a_{11} + a_{22})(b_{11} + b_{22}) - (a_{12} - a_{21})(b_{12} - b_{21}),$$
$$\hat{c}_{2r} = (a_{11} - a_{22})(b_{11} - b_{22}) - (a_{12} + a_{21})(b_{12} + b_{21}),$$
$$\hat{c}_{1i} = (a_{11} + a_{22})(b_{12} - b_{21}) + (a_{12} - a_{21})(b_{11} + b_{22}),$$
$$\hat{c}_{2i} = (a_{11} - a_{22})(b_{12} + b_{21}) + (a_{12} + a_{21})(b_{11} - b_{22}).$$

Similarly defining $\hat{A}$ and $\hat{B}$ as

$$\hat{a}_{1r} = a_{11} + a_{22},$$
$$\hat{a}_{2r} = a_{11} - a_{22},$$
$$\hat{a}_{1i} = a_{12} - a_{21},$$
$$\hat{a}_{2i} = a_{12} + a_{21},$$
$$\hat{b}_{1r} = b_{11} + b_{22},$$
$$\hat{b}_{2r} = b_{11} - b_{22},$$
$$\hat{b}_{1i} = b_{12} - b_{21},$$
$$\hat{b}_{2i} = b_{12} + b_{21},$$

$\hat{C}$ can be simplified as

$$\hat{c}_{1r} = \hat{a}_{1r}\hat{b}_{1r} - \hat{a}_{1i}\hat{b}_{1i},$$
$$\hat{c}_{2r} = \hat{a}_{2r}\hat{b}_{2r} - \hat{a}_{2i}\hat{b}_{2i},$$
$$\hat{c}_{1i} = \hat{a}_{1r}\hat{b}_{1i} + \hat{a}_{1i}\hat{b}_{1r},$$
$$\hat{c}_{2i} = \hat{a}_{2r}\hat{b}_{2i} + \hat{a}_{2i}\hat{b}_{2r}.$$

These equations are the same as complex number multiplications such as

$$\hat{c}_{kr} + j\hat{c}_{ki} = (\hat{a}_{kr} + j\hat{a}_{ki})(\hat{b}_{kr} + j\hat{b}_{ki}) \text{ for } k = 1, 2.$$

And finally each element of C is computed by

$$c_{11} = (\hat{c}_{1r} + \hat{c}_{2r})/2,$$

Table 1: Comparison of computations per bicomplex multiplication

	Real mults.	Real adds.
Direct method	16	12
Method 1	8	16
Method 2	6	22

$$c_{12} = (\hat{c}_{2i} + \hat{c}_{1i})/2,$$
$$c_{21} = (\hat{c}_{2i} - \hat{c}_{1i})/2,$$
$$c_{22} = (\hat{c}_{1r} - \hat{c}_{2r})/2.$$

This procedure is shown in Figure 1. Traditionally complex multiplication requires four real multiplications and two real additions. Or it is also known that complex multiplication is realized with three real multiplications and five real additions as

$$\hat{c}_{kr} = \hat{b}_{kr}(\hat{a}_{kr} - \hat{a}_{ki}) + \hat{a}_{ki}(\hat{b}_{kr} - \hat{b}_{ki}),$$
$$\hat{c}_{ki} = \hat{b}_{ki}(\hat{a}_{kr} + \hat{a}_{ki}) + \hat{a}_{ki}(\hat{b}_{kr} - \hat{b}_{ki}).$$

Here we call the former one the method 1 and the latter one the method 2. In the method 1, 8 real multiplications and 16 real additions are required in total. And in the method 2, 6 real multiplications and 22 real additions are required. Table 1 shows this comparison. It is noted that a multiplication by 1/2 can be treated as one bit shift under the assumption of binary operation, and so it is need not to be counted as an arithmetic unit.

Concerning bicomplex addition, $C = A + B$ is defined as

$$\begin{aligned} C &= A + B = B + A \\ &= \{(a_{11} + b_{11}) + j(a_{12} + b_{12})\} \\ &+ i\{(a_{21} + b_{21}) + j(a_{22} + b_{22})\}. \end{aligned} \quad (7)$$

In this case, it can be easily shown that

$$\hat{c}_{kr} + j\hat{c}_{ki} = (\hat{a}_{kr} + j\hat{a}_{ki}) + (\hat{b}_{kr} + j\hat{b}_{ki}) \text{ for } k = 1, 2.$$

This equation suggests that a bicomplex addition is also decomposed as two complex additions.

Through the above observation, we have shown the efficient implementation of multiplication itself, however, we can show more interesting property as follows. First we define the vector notations as

$$\boldsymbol{A} = [\ a_{11} \quad a_{12} \quad a_{21} \quad a_{22}\ ]^T,$$
$$\hat{\boldsymbol{A}} = [\ \hat{a}_{1r} \quad \hat{a}_{1i} \quad \hat{a}_{2r} \quad \hat{a}_{2i}\ ]^T.$$

Using vector-matrix representation, $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$ can be expressed by

$$\hat{\boldsymbol{A}} = \boldsymbol{R}\boldsymbol{A}, \quad (8)$$
$$\hat{\boldsymbol{B}} = \boldsymbol{R}\boldsymbol{B}, \quad (9)$$

where

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Furthermore, $\boldsymbol{C}$ is restored by

$$\boldsymbol{C} = \boldsymbol{R}^{-1}\hat{\boldsymbol{C}}. \quad (10)$$

Therefore,

$$\boldsymbol{C} = \boldsymbol{A} \odot \boldsymbol{B} = \boldsymbol{R}^{-1}(\hat{\boldsymbol{A}} \odot \hat{\boldsymbol{B}}), \quad (11)$$

where $\boldsymbol{A} \odot \boldsymbol{B}$ denotes bicomplex addition, subtraction, or multiplication, and $\hat{\boldsymbol{A}} \odot \hat{\boldsymbol{B}}$ denotes two parallel complex additions, subtractions, or multiplications. This means that if the input and the output of a hypercomplex number system are transformed by $\boldsymbol{R}$ and $\boldsymbol{R}^{-1}$, complex arithmetic can be used in the internal system. This property can reduce the complexity of hypercomplex number system.

4. IMPLEMENTATION OF HYPERCOMPLEX DIGITAL FILTERS

As a general model of hypercomplex digital filters, we assume that the input and the output of the hypercomplex number system are bicomplex numbers as

$$X(n) = \{x_{11}(n) + jx_{12}(n)\} + i\{x_{21}(n) + jx_{22}(n)\},$$
$$Y(n) = \{y_{11}(n) + jy_{12}(n)\} + i\{y_{21}(n) + jy_{22}(n)\}.$$

As described in the previous section, bicomplex arithmetic can be decomposed of two complex arithmetic with termination of the transform matrix. Therefore, if constant hypercomplex coefficients are modified by $\boldsymbol{R}$ in advance, bicomplex signal filtering can be performed by complex signal arithmetic. Figure 2 shows the proposed structure of hypercomplex digital filters. Using this technique, bicomplex adders and multipliers can be replaced by two parallel complex adders and multipliers, respectively. Table 2 shows the comparison of components per one bicomplex multiplier.

As an example, we give the first order hypercomplex digital filter[2], which is described as

$$Y(n) = AY(n-1) + C\{X(n) - BX(n-1)\}, \quad (12)$$

where A, B, and C are also bicomplex numbers. Since this filter is composed of three bicomplex multipliers,

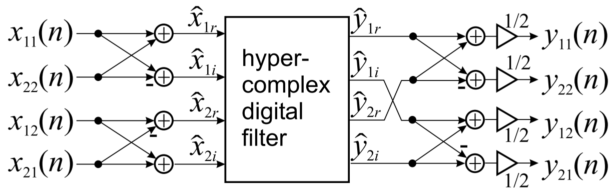

Figure 2: Proposed implementation of hypercomplex digital filter.

Table 2: Comparison of constant coefficient bicomplex multipliers

	Real mults.	Real adds.
Direct method	16	12
Method 1	8	4
Method 2	6	10

two bicomplex adders, and one bicomplex delay unit, in the direct implementation, total number of computations per sample is 48 real multiplications and 44 real additions. In the proposed implementation, as a bicomplex multiplier can be reduced to two parallel complex multipliers, in the case of the method 1, 24 real multiplications and 28 real additions are required. If the method 2 is taken, only 18 real multiplications with 46 additions are required. Table 3 summarizes this comparison. Traditionally hardware complexity of multipliers is considerably larger than that of adders, so the proposed approach could reduce the total complexity of the system to less than half of the direct method.

5. CONCLUSIONS

We have proposed the computationally efficient implementation of hypercomplex digital filters. By decomposing hypercomplex number multiplication, we have shown that it is realized as two parallel complex multiplications. Using this result, hypercomplex number systems can be reduced to simple complex number systems with termination of the transformer. As a result, hypercomplex digital filters can be implemented with less than half computations of the direct approach. This technique could be applied to any kinds of hypercomplex digital filters.

Table 3: Comparison of components of first order hypercomplex digital filters

	Real mults.	Real adds.
Direct method	48	44
Method 1	24	28
Method 2	18	46

6. REFERENCES

[1] I.L. Kantor and A.S. Solodovnikov, "Hypercomplex Numbers: an Elementary Introduction to Algebras," Springer-Verag, 1989.

[2] H.D. Schütte and J. Wenzel, "Hypercomplex numbers in digital signal processing," Proc. IEEE Int. Symp. Circuits and Syst., pp.1557–1560, 1990.

[3] H. Toyoshima, "Complex IIR digital filters composed of hypercomplex all-pass filters," Proc. 1995 IEEE Singapore Int. Conf. Signal Processing, Circuits & Systems, pp.178–183, 1995.

[4] K. Ueda and S. Takahashi, "Digital filters with hypercomplex coefficients," Scripta Technica Journals, Electronics and Communications in Japan, part 3, vol.76, no.9, pp.85–98, 1994.

[5] K. Ueda and S. Takahashi, "Realization of two transfer functions with complex coefficients by a transfer function with hypercomplex coefficients," Scripta Technica Journals, Electronics and Communications in Japan, part 3, vol.79, no.7, pp.65–80, 1996.

[6] H.F. de Groote, "On the complexity of quaternion multiplication," Infor. Process. Lett., vol.3, no.6, pp.177–179, 1975.

[7] V.S. Dimitrov, T.V. Cooklev, and B.D. Donevsky, "On the multiplication of reduced biquaternions and applications," Infor. Process. Lett. vol.43, no.3, pp.161–164, 1992.

[8] K. Mizukami, H. Toyoshima, and S. Takahashi, "A residue number system for the hypercomplex arithmetic," Proc. Joint Technical Conference on Circuits/Systems, Computers and Communications, pp.163–167, 1992.

[9] H. Toyoshima, "Computationally Efficient Bicomplex Multipliers for Digital Signal Processing," IEICE Trans. Information and Systems, to be appeared.