PERFORMANCE ANALYSIS OF CYCLIC ESTIMATORS FOR HARMONICS IN MULTIPLICATIVE AND ADDITIVE NOISE

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ABSTRACT

The problem of interest is the estimation of the parameters of harmonics in the presence of additive and multiplicative noise. Expressions for the asymptotic performance of the cyclic-variance (CV) based method are derived when the multiplicative noise has non-zero mean. We show that the CV-based method may yield more accurate results than methods based on the cyclic mean (CM), depending upon the color of the noise and the intrinsic and local SNRs. Performance is analyzed in detail for several special cases of the multiplicative noise, such as white Gaussian, AR and generalized-Gaussian noise.

1. INTRODUCTION

Consider the discrete-time harmonic signal in multiplicative and additive noise

$$x(t) = s(t)\cos(\omega_0 t + \phi_0) + \nu(t), \qquad t = 0, 1, ..., T-1 \quad (1)$$

We make the following assumptions: (AS1) ω_0 and ϕ_0 are deterministic constants in $(0, \pi/2)$ and $(-\pi, \pi]$ respectively; (AS2) $s(t)$ is a real stationary mixing process with mean μ_s and variance σ_s^2; (AS3) $\nu(t)$ is a real stationary mixing process with zero-mean and variance σ_ν^2; (AS4) $s(t)$ and $\nu(t)$ are mutually independent. The mixing property implies that samples of the process that are well separated in time are approximately independent [1, pp. 25-27]. The mixing condition we assume is that the cumulants are absolutely summable $\sum_\tau |c_{ks}(\tau)| < \infty, \ \sum_\tau |c_{k\nu}(\tau)| < \infty, \ \forall k$.

2. CYCLIC ESTIMATORS

The observed signal $x(t)$ is cyclostationary since its moments are periodically time-varying. Cyclic moments and their consistent estimates have been proposed in [2]. Since $s(t)$ and $\nu(t)$ are assumed mixing, these estimates converge in mean square and are asymptotically normal [3]. Using the two first-order cyclic statistics, consistent estimators of the harmonic parameters have been proposed in [4]. This cyclic approach is based on the Cycle-Mean (CM) when $\mu_s \neq 0$, and on the Cyclic Variance (CV) when $\mu_s = 0$.

The cyclic mean is estimated via the normalized DFT,

$$\hat{M}_{1x}(\alpha) = \frac{1}{T}\sum_{t=0}^{T-1} x(t) e^{-j\alpha t} \ . \qquad (2)$$

The unknown parameters are estimated in [4] via

$$\hat{\omega}_0^{(1)} = \arg\max_{\alpha>0} \left|\hat{M}_{1x}(\alpha)\right| \qquad (3)$$

$$\hat{\phi}_0^{(1)} = \arg\left[\hat{M}_{1x}(\hat{\omega}_0^{(1)})\right] \ , \ \hat{\mu}_s^{(1)} = 2\left|\hat{M}_{1x}(\hat{\omega}_0^{(1)})\right| \ .$$

If $\mu_s = 0$, the CM contains no information about the harmonic parameters. Spectral analysis of the squared data enables us to recover the parameters of the harmonic signal. The CV-based estimator was studied in [4] for this case.

The parameter vector to estimate using the CV is $\theta_0 = [a_0, \phi_0, T\omega_0]$ where

$$a_0 = \mu_s^2 + \sigma_s^2 \ . \qquad (4)$$

The cyclic-variance is estimated via the normalized DFT of the squared,

$$\hat{M}_{2x}(\alpha; 0) = \frac{1}{T}\sum_{t=0}^{T-1} x^2(t) e^{-j\alpha t} \qquad (5)$$

We will call this 'CV' even though M_{2x} is actually the second-order cyclic moment, since the mean value has not been removed. It is worth noting that the CV method requires ω to be in $(0, \pi/2)$ in order to avoid ambiguity. The CV-based estimator is

$$\hat{\omega}_0^{(2)} = \frac{1}{2}\arg\max_{\alpha>0} \left|\hat{M}_{2x}(\alpha; 0)\right| \qquad (6)$$

$$\hat{\phi}_0^{(2)} = \frac{1}{2}\arg\left[\hat{M}_{2x}(2\hat{\omega}_0^{(2)}; 0)\right] \ , \ \hat{a}_0^{(2)} = 4\left|\hat{M}_{2x}(\hat{\omega}_0^{(2)}; 0)\right| \ .$$

Note that the CV can be used even when $\mu_s \neq 0$.

The analysis of the CV method has been largely restricted to the $\mu_s = 0$ case. In [5], the noises were assumed to be iid, and it was shown that when $\mu_s \neq 0$, the CV method may yield better detection performance than the CM method. In this paper, we consider the asymptotic performance analysis, i.e., we evalute the variance of the CV-based estimators for $\mu_s \neq 0$, and we compare them with those for the CM-based estimators. We show that the CV-based estimator may be more accurate that the CM-based estimator. This is true not only when $\mu_s = 0$ but also for values of the mean ranging from 0 to a threshold which depends upon the noise pdfs, and the noise color. We quantify this for the case where $s(t)$ is a low-pass Gaussian AR process; such a model has been used in [6] to model returns from an on-board radar on a train.

3. ASYMPTOTIC PERFORMANCE ANALYSIS

3.1. Cyclic Mean

The asymptotic performance of the CM-based estimators in (3) has been studied in [4]. The large sample variance of the frequency estimate is given by

$$\mathrm{var}(\hat{\omega}_0^{(1)}) \approx \frac{1}{T^3}\left[\frac{24S_{2\nu}(\omega_0)}{\mu_s^2}+\frac{6S_{2s}(2\omega_0)}{\mu_s^2}\right] \quad (7)$$

where S_{2s} and S_{2v} are the power spectra of the two noise processes.

3.2. Cyclic Variance

The large sample performance of the CV-based estimators has been studied in [4] for the zero mean multiplicative noise case, i.e. $\mu_s = 0$, $\sigma_s^2 \neq 0$. Here, we address the more general case where $\mu_s \neq 0$ and $\sigma_s^2 \neq 0$. Some of the development in [4] is applicable, by substituting a_0 of (4) for σ_s^2. We will omit those details.

It is straightforward to show that the CV-based estimates in (6) are equivalent to the NLLS estimates:

$$(\hat{a}_0^{(2)}, \hat{\phi}_0^{(2)}, \hat{\omega}_0^{(2)}) = \arg\min_{a,\phi,\omega} Q_T(a,\phi,\omega)\ , \quad (8)$$

$$Q_T(a,\phi,\omega) = \frac{1}{T}\sum_{t=0}^{T-1}\left[x^2(t)-\frac{a}{2}\cos(2\omega t+2\phi)\right]^2 . \quad (9)$$

To carry out the small error analysis of the CV-based estimator, we use the first-order Taylor expansion of $a\cos(2\omega t+2\phi)$, substitute it into (9), to obtain

$$\hat{\theta}-\theta_0 = \mathbf{a}+\mathbf{H}\mathbf{b} \quad (10)$$

$$\mathbf{a} = [-a_0, 0, 0]' \quad (11)$$

$$\mathbf{H} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -8a_0^{-1} & 12a_0^{-1} \\ 0 & 12a_0^{-1} & -24a_0^{-1} \end{bmatrix} \quad (12)$$

$$\mathbf{b} = \begin{bmatrix} \frac{1}{T}\sum_{t=0}^{T-1}\cos(2\omega_0 t+2\phi_0)x^2(t) \\ \frac{1}{T}\sum_{t=0}^{T-1}\sin(2\omega_0 t+2\phi_0)x^2(t) \\ \frac{1}{T}\sum_{t=0}^{T-1}\left(\frac{t}{T}\right)\sin(2\omega_0 t+2\phi_0)x^2(t) \end{bmatrix} \quad (13)$$

The estimate $\hat{\theta}$ is asymptotically unbiased:

$$\lim_{T\to\infty} E[\hat{\theta}-\theta_0] = \mathbf{0}\ . \quad (14)$$

The asymptotic covariance of the CV-based estimator is

$$\Sigma_{\hat{\theta}} = \lim_{T\to\infty} T\mathrm{cov}(\hat{\theta}) = \mathbf{H}\mathbf{B}\mathbf{H}' \quad (15)$$

$$\mathbf{B} = \lim_{T\to\infty} T\mathrm{cov}(\mathbf{b})\ . \quad (16)$$

A closed-form expression for $\mathbf{B}$ is derived in the Appendix. Let us define

$$h_1(\tau) = c_{4s}(0,\tau,\tau)+2c_{2s}^2(\tau)$$
$$h_2(\tau) = c_{4\nu}(0,\tau,\tau)+2c_{2\nu}^2(\tau)$$
$$h_3(\tau) = 4c_{2s}(\tau)c_{2\nu}(\tau)$$
$$h_4(\tau) = c_{3s}(\tau,\tau)+c_{3s}(0,\tau)$$

$$H_k(\lambda) = \sum_{\tau=-\infty}^{\infty} h_k(\tau)e^{-j\lambda\tau}, \quad k=1,...,4 \quad (17)$$

$$D_1(\lambda) = 4\mu_s^2 S_{2s}(\lambda)+2\mu_s H_4(\lambda)+H_1(\lambda)$$
$$D_2(\lambda) = 4\mu_s^2(S_{2\nu}(\lambda)+S_{2\nu}(3\lambda))+H_3(\lambda)+H_3(3\lambda)$$
$$D(\lambda) = \frac{1}{8}D_1(2\lambda)+\frac{1}{32}D_1(4\lambda) +\frac{1}{8}D_2(\lambda)+\frac{1}{2}H_2(2\lambda) \quad (18)$$

Functions $H_k(.)$, $k = 1,...,4$, are real valued since $h_k(.)$, $k = 1,...,4$, are even functions.

The large sample covariance matrix of the CV-based estimate is given by

$$\mathbf{\Sigma}_{\hat{\theta}} = \begin{bmatrix} D_1(0)+16D(\omega_0) & 0 & 0 \\ 0 & \frac{16}{a_0^2}D(\omega_0) & -\frac{24}{a_0^2}D(\omega_0) \\ 0 & -\frac{24}{a_0^2}D(\omega_0) & \frac{48}{a_0^2}D(\omega_0) \end{bmatrix} \quad (19)$$

Eq (19) follows from (12), (15) and (32).

The asymptotic variance of the frequency estimate is

$$\mathrm{var}(\hat{\omega}_0^{(2)}) = \frac{1}{T^3}\frac{48}{a_0^2}D(\omega_0)\ . \quad (20)$$

When $\mu_s = 0$, our formulas reduce to those established in [4]. If the multiplicative noise is symmetrically distributed, its third-order cumulants vanish, as do $h_4(.)$ and $H_4(.)$.

In the special case where both $s(t)$ and $\nu(t)$ are white Gaussian, we obtain

$$\mathrm{var}(\hat{\omega}_0^{(2)}) = \frac{1}{T^3}\frac{15(2R_1R_2+R_2^2)+48(R_1+R_2)+48}{(R_1+R_2)^2}$$

where

$$R_1 = \mu_s^2/\sigma_\nu^2; \qquad R_2 = \sigma_s^2/\sigma_\nu^2\ .$$

4. CM VERSUS CV

In [5], the accuracy of the peak estimate was used to compare CV with CM; this is important for the detection problem, particularly for small-to-moderate sample size and/or SNR. But if both CM and CV yield clean peaks, then one should focus on the estimation problem and compare the variances of the two frequency estimates.

Equations (7) and (20) give the expressions for the asymptotic variances of the estimates of ω_o based on the cyclic mean and the cyclic variance. The CM is inapplicable when $\mu_s = 0$ (note that the variance blows up); on the other hand, the CV is not useful when $\sigma_s^2 = 0$ (purely additive noise) case, becuse the CM-based estimator has a smaller variance regardless of the color/pdf of $\nu(t)$. In order to get insight into the CV versus CM tradeoff, we consider some important cases.

We will focus on frequency estimation. The asymptotic relative efficiency (ARE) of the CV-based estimator wrt the CM-based estimator can be defined as

$$ARE = \mathrm{var}(\hat{\omega}_0^{(1)})/\mathrm{var}(\hat{\omega}_0^{(2)}) \quad (21)$$

4.1. White noise case

Let $c_{4s} := c_{4s}(0,0,0)$ and let c_{3s} and $c_{4\nu}$ be similarly defined. When both the multipicative and additive noise are white, we obtain

$$h_1(\tau) = (c_{4s} + 2\sigma_s^4)\delta(\tau); \quad h_2(\tau) = (c_{4s} + 2\sigma_s^4)\delta(\tau);$$
$$h_3(\tau) = 4\sigma_s^2\sigma_\nu^2\delta(\tau) \quad h_4(\tau) = 2c_{3s}\delta(\tau); \qquad (22)$$

Below, we limit our study to multiplicative noise sources such that $c_{3s} = 0$. Let

$$\eta := \mu_s^2/\sigma_s^2; \quad \gamma := \sigma_\nu^2/\sigma_s^2 \ .$$

The ARE is then found to be

$$ARE = \frac{6(\eta+1)^2(4\gamma+1)/\eta}{(48\gamma+30)\eta + \frac{15}{2}\kappa_s + 15 + 24\gamma^2(\kappa_\nu+2) + 48\gamma} \qquad (23)$$

where κ_s and κ_ν are the kurtoses of the multiplicative and additive noise respectivelly:

$$\kappa_s = c_{4s}/\sigma_s^4; \quad \kappa_\nu = c_{4\nu}/\sigma_\nu^4 \qquad (24)$$

Before addressing the multiplicative noise case, it is worth noting that the ARE in the pure additive noise case is at most equal to 0.5:

$$ARE|_{\sigma_s^2=0} = \frac{\mu_s^2/\sigma_\nu^2}{2\mu_s^2/\sigma_\nu^2 + \kappa_\nu + 2} < 0.5 \qquad (25)$$

since $\kappa_s \geq -2$ for continuous valued random processes. Thus, in the additive noise case the CM-based estimator is by far the best estimator.

4.1.1. Pure multiplicative noise case

Let the additive noise power be zero, so that the harmonic signal is contaminated by a pure multiplicative noise. In this case, the CV-based estimator will be more efficient than the CM-based estimator if η (the coherent-to-non coherent harmonic power ratio) satifies

$$\eta < \tilde{\eta} = \frac{\sqrt{(\frac{5}{2}\kappa_s+1)^2 + 64} - (\frac{5}{2}\kappa_s+1)}{16} \qquad (26)$$

For Gaussian noise, $\kappa_s = 0$, so that the threshold of efficiency $\tilde{\eta} \approx 0.44$. The ARE versus η is depicted in figure 1. It is seen that even if $\mu_s \neq 0$, we should use the CV-based estimator provided $\mu_s < 0.66\sigma_s$.

To study the influence of the kurtosis on the ARE, we model the multiplicative noise pdf by the generalized Gaussian density

$$p(s) = (\alpha/2\beta\Gamma(1/\alpha))\exp(-|(s-\mu_s)/\beta|^\alpha) \qquad (27)$$

where $\Gamma(.)$ is the gamma function. The Gaussian case is obtained for $a = 2$, whereas for $a = 1$, we obtain the Laplace distribution. The kurtosis for this distribution is given by

$$\kappa_s = \Gamma(1/a)\Gamma(5/a)\Gamma^{-2}(3/a) - 3 \qquad (28)$$

Figure 2 displays $\tilde{\eta}$ versus the shape parameter α. It is seen that $\tilde{\eta}$ is an increasing function of α. We then conclude that as α increases (lighter-tailed pdf), so does the performance gain using the CV-based estimator (for fixed η).

It is worth noting that since $\kappa_s \geq -2$, we have that

$$\max(\tilde{\eta}) = \sqrt{5} + 1/4 \ \approx 0.81 \qquad (29)$$

This implies that the CM-based estimator outperforms the CV-based estimator if $\eta > 0.81$, regardless of the pdf of the multiplicative noise (provided $c_{3s} = 0$).

4.1.2. Multiplicative and additive noise case

Here, we study the influence of the additive noise on the threshold $\tilde{\eta}$, thus generalizing the results in section 4-1-1. From eq. (23), we can derive a condition on η similar to (26). Figure 3 displays the behavior of the threshold $\tilde{\eta}$ as a function of γ when the multiplicative noise is generalized non-Gaussian (α of 1, 2 and 10), and the additive noise is Gaussian. As expected, the performance gain using the CV-based estimator decreases as γ increases.

4.2. Colored noise case

Let $s(t)$ be a linear non-Gaussian process, $s(t) = H(z)u(t)$, where $u(t)$ is iid non-Gaussian, whose k-th order cumulants at zero lag are denoted by γ_{ku}; we will continue to use μ and σ^2 to denote mean and variance. Consider the purely multiplicative noise so that $\sigma_v^2 = 0$. We can express the cumulants of $s(t)$ in terms of γ_{ku} and the impulse response $h(t)$. In typical applications, the multiplicative noise is well modeled as a low-pass AR process [6]. Consider the AR(1) model, $H(z) = (1 - \rho z^{-1})$, $-1 < \rho < 1$, $h(t) = \rho^t$, $t \geq 0$. We evaluated $D(\lambda)$ for this case; due to lack of space, we omit the explicit values of (17)-(18).

As we increase μ_s^2/σ_s^2, the intrinsic SNR becomes larger (the harmonic eventually becomes constant amplitude), and the CM based method will be preferred. Because of the color of $s(t)$, there is a frequency dependent behavior as well: for a given μ_s^2/σ_s^2, and ρ, the CM method will be preferred for frequencies greater than a threshold λ_o which depends upon ρ. In figure 4, we plot $\lambda_o/2\pi$ as a function of μ^2/σ^2, for various values of ρ. For a given ρ, the threshold decreases from 0.25 to 0, as μ_s^2/σ_s^2 increases; a sharp transition point or knee is visible in the figure; this transition point moves to the right as ρ increases. For white noise, the transition point is at $\mu_s^2/\sigma_s^2 = 0.4414$ (the root of $8x^2 + x - 1 = 0$). The effect of increasing ρ is to make the curve flatter, and the knee sharper.

5. CONCLUSIONS

We evaluated the performance of the CV-based method when the multiplicative noise has non-zero mean; we compared its performance with that of the CV method; for special cases such as white noise, AR noise, and generalized-Gaussian noise, we showed that the CV-based method yields more accurate estimates than the CM-based method provided that the intrinsic SNR is below a threshold (frequency-dependent in the case of colored noise).

APPENDIX

In this appendix, we derive an expression for the asymptotic covariance matrix $\mathbf{B}$ of (16), which is required to obtain the covariance matrix (19).

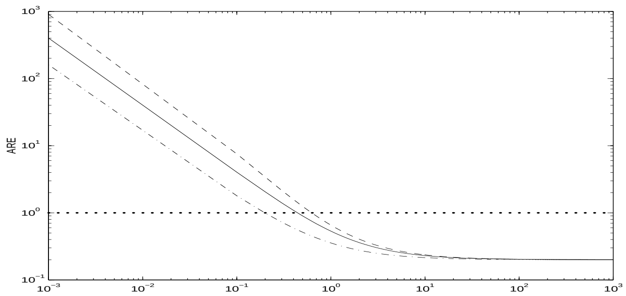

Figure 1. ARE vs η for different shape parameters of the multiplicative noise pdf; $\alpha = 1$ (dashed dotted line), $\alpha = 2$ (solid line) and $\alpha = 10$ (dashed line).

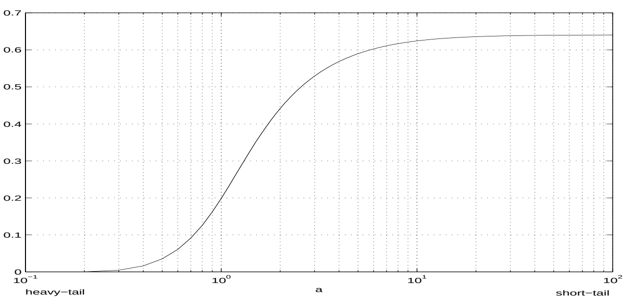

Figure 2. Threshold $\tilde{\eta}$ versus the shape parameter of the the multiplicative noise pdf.

Let $y(t) = s(t) - \mu_s$, which has zero mean; also, let

$$\begin{aligned} z_1(t) &= y^2(t) - \sigma_s^2 + 2\mu_s y(t)\ ,\ z_2(t) = 2\nu(t)(y(t) + \mu_s) \\ z_3(t) &= \nu^2(t) - \sigma_\nu^2\ ,\ \mathrm{DC} = 0.5(\mu_s^2 + \sigma_s^2) + \sigma_\nu^2 \\ \xi(t) &= z_1(t)\cos^2(\omega t + \phi) + z_2(t)\cos(\omega t + \phi) + z_3(t) \end{aligned}$$

Processes $z_1(t)$, $z_2(t)$, $z_3(t)$ and $\xi(t)$ are zero-mean, and

$$x^2(t) = 0.5 a_0 \cos(2\omega t + 2\phi) + \xi(t) + \mathrm{DC}\ . \qquad (30)$$

Using standard large sample trigonometric formula, it turns out that the computation of matrix $\mathbf{B}$ only involves term $\xi(t)$ in $x^2(t)$, so that $\mathbf{B} = \lim_{T\to\infty} T\mathrm{cov}(\bar{\mathbf{b}})$ where

$$\bar{\mathbf{b}} = \begin{bmatrix} \frac{1}{T}\sum_{t=0}^{T-1}\cos(2\omega_0 t + 2\phi_0)\xi(t) \\ \\ \frac{1}{T}\sum_{t=0}^{T-1}\sin(2\omega_0 t + 2\phi_0)\xi(t) \\ \\ \frac{1}{T}\sum_{t=0}^{T-1}\left(\frac{t}{T}\right)\sin(2\omega_0 t + 2\phi_0)\xi(t) \end{bmatrix} \qquad (31)$$

$z_1(t)$ is decorrelated from $z_2(t)$ and $z_3(t)$, while $z_2(t)$ and $z_3(t)$ are, in general, correlated. After some calculations, the first element of the matrix $\mathbf{B}$ is found to be

$$B(1,1) = \frac{1}{32}\sum_{\tau=-\infty}^{\infty} c_{2z_1}(\tau)\left[2 + 4\cos(2\omega\tau) + \cos(4\omega\tau)\right]$$
$$4c_{2z_2}(\tau)\left[\cos(\omega\tau) + \cos(3\omega\tau)\right] 16 c_{2z_3}(\tau)\cos(2\omega\tau)$$

where $c_{2z_i}(\tau)$ is the covariance function of $z_i(t)$,

$$\begin{aligned} c_{2z_1} &= 4\mu_s^2 c_{2s}(\tau) + 2\mu_s[c_{3s}(0,\tau) + c_{3s}(\tau,\tau)] \\ &\quad + c_{4s}(0,\tau,\tau) + 2c_{2s}^2(\tau) \\ c_{2z_2} &= 4c_{2\nu}(\tau)[\mu_s^2 + c_{2s}(\tau)] \\ c_{2z_3} &= c_{4\nu}(0,\tau,\tau) + 2c_{2\nu}^2(\tau) \end{aligned}$$

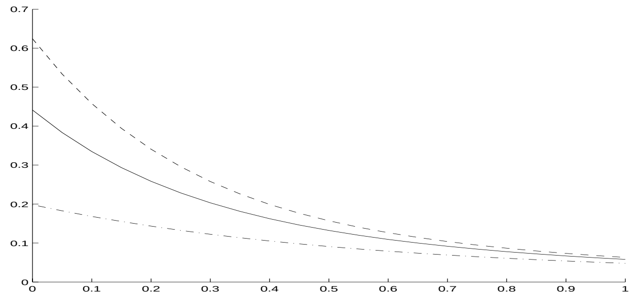

Figure 3. Threshold $\tilde{\eta}$ vs γ for different shape parameters of the multiplicative noise pdf; $\alpha = 1$ (dashed dotted line), $\alpha = 2$ (solid line) and $\alpha = 10$ (dashed line); $\nu(t)$ was AWGN.

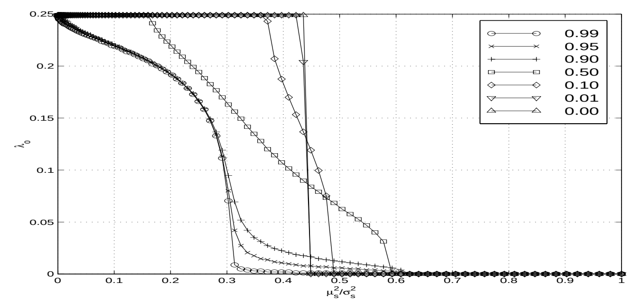

Figure 4. Threshold $\lambda_o/2\pi$ vs μ_s^2/σ_s^2 for various ρ's.

The other elements of the $\mathbf{B}$ matrix are derived similarly; using the notations in (17) and (18), we obtain

$$\mathbf{B} = \begin{bmatrix} D(\omega_0) + \frac{1}{16}D_1(\omega_0) & 0 & 0 \\ 0 & D(\omega_0) & \frac{1}{2}D(\omega_0) \\ 0 & \frac{1}{2}D(\omega_0) & \frac{1}{3}D(\omega_0) \end{bmatrix} \qquad (32)$$

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