Stability of Fuzzy Systems based on an Approximation Argument

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Abstract

The paper is concerned with the stability in the sense of Lyapunov of nonlinear control systems, which are of Sugeno-Takagi type. We briefly recall a construction method for piecewise affine-linear control functions and then point out how the controllers can be viewed as Sugeno-Takagi controllers. The main result is a stability theorem that is of the following type. We compare two control systems. We assume that the stability behavior of the first one is known and the difference between the two control functions is bounded. Moreover we assume that one control function (the fuzzy controller) is Lipschitz continuous. Then we give conditions for the stability of the system with the fuzzy controller.

1 INTRODUCTION

The problem of the stability of fuzzy systems is the problem of stability of nonlinear differential equations, that might be arbitrary difficult to solve. Here stability is understood in the sense of Lyapunov, relations to other notions of stability like L_p-stability are well known [11]. Even if in general, stability of nonlinear differential equations is hard to prove, there are many criteria of stability for special situations. And indeed fuzzy systems as well as systems with fuzzy controllers are such special situations. Several work has been done on this topic, e.g. [2, 6, 7] and the references therein. In [3] the author tackles the stability problem by using the approximating properties of the fuzzy controller. This is our main idea too. We show the stability of fuzzy systems by comparing them to some other systems with known behaviour. Now let us come to an overview of our method.

In recent papers [9, 10] we have developed a method to design controllers for nonlinear control systems. Conditions for the stability of these control systems were shown and they were tested in a number of experiments, one of which is described in [8].

The first step of our method is to generate (with methods from linear control theory) stable affine-linear controllers and to glue them together in order to get a first, switching controller. More information on the method will be given in section 2. Since, in general, these controllers are not continuous, this might lead to some trouble in applications, although we have given conditions for the stability of these systems [9]. The present paper shows how to get continuous or even Lipschitz continuous controllers from our original switching controllers. Indeed, these continuous controllers are nothing but Sugeno-Takagi fuzzy controllers. It must be mentioned here that the stability of switched systems is itself a challenging problem [1].

The second step is to blend the piecewise continuous controllers and as well the corresponding systems by means of some appropriate blending functions. The idea behind this is well known in the context of computer aided geometric design (CAGD). It was described already in 1964, according to [5]. In context of fuzzy mathematics, the controller or system corresponding with this principle is commonly referred to as Sugeno-Takagi controller or Sugeno-Takagi system [4], respectively. We will discuss the principle briefly in section 3.

Applying this blending technique to our discontinuous, piecewise affine-linear control functions will lead to the desired continuous control function as shown in Lemma 3.1. Finally, in section 4, we will show that under additional conditions, the resulting Sugeno-Takagi system is asymptotically stable once the switching system is stable.

2 CONSTRUCTION OF A PIECEWISE LINEAR CONTROLLER

For a detailed description of the contruction method for piecewise linear controllers see [10]. The main idea of the method is as follows.

- Split the relevant subset of the state space $U \subset \mathbb{R}^n$ in disjunct areas U_ν, $\nu = 0, \ldots, N$, with $U = \bigcup_{\nu=0}^{N} U_\nu$.
- Find an affine-linear approximation of $\mathbf{f}$ on each U_ν

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \approx \mathbf{A}_\nu(\mathbf{x} - \mathbf{x}_\nu) + \mathbf{B}_\nu(\mathbf{u} - \mathbf{u}_\nu) + \mathbf{f}(\mathbf{x}_\nu, \mathbf{u}_\nu) \qquad \text{for } \mathbf{x} \in U_\nu, \tag{1}$$

- Compute matrices $\mathbf{K}_\nu$ for a linear state feedback controller on each U_ν, such that the closed loop is

$$\dot{\mathbf{x}} \;\; = \;\; (\mathbf{A}_\nu + \mathbf{B}_\nu \mathbf{K}_\nu)\, \mathbf{x} \;, \quad \mathbf{x} \in U_\nu \;, \quad \nu = 0, \ldots, N, \tag{2}$$

  i.e. linear. Constructive conditions for the existence of such $\mathbf{K}_\nu$ are given in [10, Lemma 1].

- Combine the local control functions defined on each U_ν to a global one on U, by

$$\mathbf{u}(\mathbf{x}) = \mathbf{K}_\nu \; (\mathbf{x} - \mathbf{x}_\nu) + \mathbf{u}_\nu \qquad \text{if } \mathbf{x} \in U_\nu.$$

  This procedure results in a switching controller.

In order to make this controller continuous we apply the Sugeno-Takagi method as shown in the next Section.

3 CONTINUOUS SUGENO-TAKAGI FUZZY CONTROLLER

We briefly recall the well known formula of the Sugeno-Takagi fuzzy controller [4]. Let $N+1$ be the number of rules of the fuzzy controller. The mapping of the controller $\mathbf{u} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is defined componentwise by

$$u_i(\mathbf{x}) = \sum_{\nu=0}^{N} u_\nu^{(i)}(\mathbf{x}) \; a_\nu(\mathbf{x}) \tag{3}$$

where $a_\nu : \mathbb{R}^n \longrightarrow \mathbb{R}$ are nonnegative continuous functions with $\sum_{\nu=0}^{N} a_\nu(\mathbf{x}) \equiv 1$, and $u_\nu^{(i)}$ are the conclusion functions for the rule ν and the output component i of the controller.

Now let us consider an affine-linear state space controller associated with each rule, i.e. for some area U_ν we have a matrix

$$\mathbf{K}_\nu \in \mathbb{R}^{m \times n} \qquad \text{for } \nu = 0, \ldots, N.$$

Denote the i-th row of this matrix by $\mathbf{k}_\nu^{(i)\,T}$, i.e.

$$\mathbf{K}_\nu = \begin{pmatrix} \mathbf{k}_\nu^{(1)\,T} \\ \vdots \\ \mathbf{k}_\nu^{(m)\,T} \end{pmatrix}.$$

Now define the above conclusion functions as

$$u_\nu^{(i)}(\mathbf{x}) = \mathbf{k}_\nu^{(i)\,T} \, (\mathbf{x} - \mathbf{x}_\nu) + (u_\nu)_i.$$

Obviously

$$\begin{pmatrix} u_\nu^{(1)}(\mathbf{x}) \\ \vdots \\ u_\nu^{(m)}(\mathbf{x}) \end{pmatrix} = \mathbf{K}_\nu \, (\mathbf{x} - \mathbf{x}_\nu) + \mathbf{u}_\nu, \tag{4}$$

that means each rule $u_\nu^{(i)}$ of the Sugeno-Takagi fuzzy controller realizes one of the affine-linear controllers.

Inserting (4) into (3) gives

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u_1(\mathbf{x}) \\ \vdots \\ u_m(\mathbf{x}) \end{pmatrix} = \sum_{\nu=0}^{N} \left[\mathbf{K}_\nu \left(\mathbf{x} - \mathbf{x}_\nu\right) + \mathbf{u}_\nu\right] \cdot a_\nu(\mathbf{x}). \tag{5}$$

This shows how the affine-linear control functions $\mathbf{x} \mapsto \mathbf{K}_\nu (\mathbf{x} - \mathbf{x}_\nu) + \mathbf{u}_\nu$ for the domain U_ν can be blended together to obtain a Sugeno-Takagi controller. For this, we give the subsequent result, which will be needed in the next section.

Lemma 3.1 *If the functions $a_\nu : \mathbb{R}^n \longrightarrow \mathbb{R}$ are Lipschitz continuous, then the Sugeno-Takagi controller (5) is Lipschitz continuous.*

PROOF: Obviously, the functions $\mathbf{x} \mapsto \mathbf{K}_\nu (\mathbf{x} - \mathbf{x}_\nu) + \mathbf{u}_\nu$, $\nu = 0, \ldots, N$, are Lipschitz continuous. Since the product and the sum of Lipschitz continuous functions is Lipschitz continuous the Sugeno-Takagi controller is Lipschitz continuous.

$\square$

4 PROOF OF STABILITY

As announced above, in this section we state and prove a stability result for continuous Sugeno-Takagi controllers. All norms $\|\cdot\|$ that occur in this section are of some arbitiary, but fixed l_p type, $1 \le p \le \infty$. The space for the norm can be read off from its argument in each case.

Theorem 4.1 *Let the system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{6}$$

be given where $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and let $\mathbf{f}$ satisfy the Lipschitz condition

$$\|\mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{y}, \mathbf{v})\| \le L \cdot \|(\mathbf{x}, \mathbf{u}) - (\mathbf{y}, \mathbf{v})\|$$

for all $(\mathbf{x}, \mathbf{u}), (\mathbf{y}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^m$. Let $\hat{\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^m$ and let $\tilde{\mathbf{u}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a Lipschitz continuous mapping with a Lipschitz constant $\tilde{L}$ satisfying

$$\|\hat{\mathbf{u}} - \tilde{\mathbf{u}}\|_\infty < \varepsilon$$

for some $\varepsilon > 0$.

Define the closed loop systems

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \hat{\mathbf{f}}(\hat{\mathbf{x}}) := \mathbf{f}(\hat{\mathbf{x}}, \hat{\mathbf{u}}(\hat{\mathbf{x}}))\ , \qquad (7) \\ \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}}) := \mathbf{f}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}(\tilde{\mathbf{x}}))\ , \qquad (8) \end{aligned}$$

and let $\hat{\mathbf{u}}$ be such that the system (7) is exponentially stable, i.e. the solution $\hat{\mathbf{x}}$ of (7) satisfies

$$\|\hat{\mathbf{x}}(t, \mathbf{x}_0)\| \le e^{Mt} \|\mathbf{x}_0\|$$

with some $M < 0$.

Then the following holds:

i) The function $\tilde{\mathbf{f}}$ satisfies a Lipschitz condition, i.e. there exists some constant $\bar{L} > 0$ such that

$$\|\tilde{\mathbf{f}}(\mathbf{x}) - \tilde{\mathbf{f}}(\mathbf{y})\| \le \bar{L} \|\mathbf{x} - \mathbf{y}\|$$

holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

ii) There is a constant $\bar{\varepsilon} > 0$ such that for each initial value $\mathbf{x}_0$, the solutions $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ of (7) and (8) respectively, satisfy

$$\|\hat{\mathbf{x}}(t, \mathbf{x}_0) - \tilde{\mathbf{x}}(t, \mathbf{x}_0)\| \le \bar{\varepsilon} \frac{e^{\bar{L}t} - 1}{\bar{L}}$$

for any $t \ge 0$.

iii) Let $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ be the linearization of (7) in zero, and assume that some ball $B_d(\mathbf{0})$ is in the domain of attraction for this linearized system. Assume further that $\tilde{\mathbf{u}}|_{B_d(\mathbf{0})} = \hat{\mathbf{u}}|_{B_d(\mathbf{0})}$ and define

$$\delta(t) = \bar{\varepsilon}\frac{e^{\bar{L}t} - 1}{\bar{L}} + e^{Mt}\|\mathbf{x}_0\| \; .$$

(a) If δ is increasing for $t > 0$, then $\|\mathbf{x}_0\| = \delta(0) \leq d$ is sufficient for the asymptotic stability of (8).

(b) Otherwise

$$\delta\left(\frac{1}{M - \bar{L}} \ln\left(\frac{-\bar{\varepsilon}}{M\|\mathbf{x}_0\|}\right)\right) \leq d$$

is sufficient for the asymptotic stability of (8).

For the proof of this theorem we need an auxiliary result from [9, Lemma 2.3].

Lemma 4.2 *Let*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}), && (9) \\ \dot{\tilde{\mathbf{x}}} &= \mathbf{g}(\tilde{\mathbf{x}}) && (10) \end{aligned}$$

be two closed loop systems with $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let $\mathbf{f}$ fulfill the Lipschitz condition $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in U \subseteq \mathbb{R}^n$, where $L > 0$. Let further be

$$\|\mathbf{f} - \mathbf{g}\|_\infty < \varepsilon \, .$$

Then for the solutions $\mathbf{x}$ and $\tilde{\mathbf{x}}$ of (9) and (10) the following statement holds

$$\|\mathbf{x}(t, \mathbf{x}_0) - \tilde{\mathbf{x}}(t, \mathbf{x}_0)\| \leq \varepsilon \, \frac{e^{Lt} - 1}{L} \, , \qquad (11)$$

where $\mathbf{x}_0$ denotes the initial value of (9) and (10).

□

Proof of theorem 4.1: Since $\mathbf{f}$ satisfies a Lipschitz condition with Lipschitz constant L, we have

$$\begin{aligned} \|\tilde{\mathbf{f}}(\mathbf{x}) - \tilde{\mathbf{f}}(\mathbf{y})\| &= \|\mathbf{f}(\mathbf{x}, \tilde{\mathbf{u}}(\mathbf{x})) - \mathbf{f}(\mathbf{y}, \tilde{\mathbf{u}}(\mathbf{y}))\| \\ &\leq L \cdot \|(\mathbf{x}, \tilde{\mathbf{u}}(\mathbf{x})) - (\mathbf{y}, \tilde{\mathbf{u}}(\mathbf{y}))\| \\ &= L \cdot \|(\mathbf{x}, \mathbf{0}) + (\mathbf{0}, \tilde{\mathbf{u}}(\mathbf{x})) - (\mathbf{y}, \mathbf{0}) - (\mathbf{0}, \tilde{\mathbf{u}}(\mathbf{y}))\| \\ &\leq L \cdot (\|(\mathbf{x}, \mathbf{0}) - (\mathbf{y}, \mathbf{0})\| + \|(\mathbf{0}, \tilde{\mathbf{u}}(\mathbf{x})) - (\mathbf{0}, \tilde{\mathbf{u}}(\mathbf{y}))\|) \\ &\leq L \cdot (\|\mathbf{x} - \mathbf{y}\| + \|\tilde{\mathbf{u}}(\mathbf{x}) - \tilde{\mathbf{u}}(\mathbf{y})\|) \\ &\leq L \cdot (\|\mathbf{x} - \mathbf{y}\| + \tilde{L} \cdot \|\mathbf{x} - \mathbf{y}\|) \\ &= \bar{L} \cdot \|\mathbf{x} - \mathbf{y}\| \end{aligned}$$

with $\bar{L} := L(1 + \tilde{L})$. This shows our first assertion.

As we have shown in *i)* above, $\tilde{\mathbf{f}}$ satisfies a Lipschitz condition. Therefore, from Lemma 4.2, all that is left to prove is

$$\|\tilde{\mathbf{f}} - \hat{\mathbf{f}}\|_\infty < \varepsilon_1 \text{ with some } \varepsilon_1 > 0 \; . \qquad (12)$$

But this is seen from the following:

$$\begin{aligned} \|\tilde{\mathbf{f}}(\mathbf{x}) - \hat{\mathbf{f}}(\mathbf{x})\| &= \|\mathbf{f}(\mathbf{x}, \tilde{\mathbf{u}}(\mathbf{x})) - \mathbf{f}(\mathbf{x}, \hat{\mathbf{u}}(\mathbf{x}))\| \\ &\leq L \cdot \|(\mathbf{x}, \tilde{\mathbf{u}}(\mathbf{x})) - (\mathbf{x}, \hat{\mathbf{u}}(\mathbf{x}))\| \\ &= L \cdot \|\tilde{\mathbf{u}}(\mathbf{x}) - \hat{\mathbf{u}}(\mathbf{x})\| \\ &\leq L \cdot \|\tilde{\mathbf{u}} - \hat{\mathbf{u}}\|_\infty \\ &\leq L \cdot \varepsilon \\ &=: \varepsilon_1 \; . \end{aligned}$$

This proves (12), thus Lemma 4.2 concludes the proof for *ii)*.

In order to prove *iii)* we observe that *ii)* above gives

$$\left| \|\hat{\mathbf{x}}(t,\mathbf{x}_0)\| - \|\tilde{\mathbf{x}}(t,\mathbf{x}_0)\| \right| \le \|\hat{\mathbf{x}}(t,\mathbf{x}_0) - \tilde{\mathbf{x}}(t,\mathbf{x}_0)\| \le \bar{\varepsilon}\frac{e^{\bar{L}t}-1}{\bar{L}}$$

for the solutions $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ of (7) and (8), respectively.

Now for t with

$$\|\tilde{\mathbf{x}}(t,\mathbf{x}_0)\| < \|\hat{\mathbf{x}}(t,\mathbf{x}_0)\| \ , \tag{13}$$

the estimate for the asymptotic stability of (8) follows from that of (7). If (13) is not satisfied, we observe

$$\begin{aligned} \|\tilde{\mathbf{x}}(t,\mathbf{x}_0)\| &\le \|\tilde{\mathbf{x}}(t,\mathbf{x}_0) - \hat{\mathbf{x}}(t,\mathbf{x}_0)\| + \|\hat{\mathbf{x}}(t,\mathbf{x}_0)\| \\ &\le \bar{\varepsilon}\frac{e^{\bar{L}t}-1}{\bar{L}} + e^{Mt}\|\mathbf{x}_0\| \\ &= \delta(t) \end{aligned} \tag{14}$$

for some $M < 0$, since $\hat{\mathbf{x}}$ is stable.

Now if δ is an increasing function for $t > 0$, then $\delta(0) = \|\mathbf{x}_0\| \le d$, together with (14), implies asymptotic stability of $\tilde{\mathbf{x}}$. If δ is not increasing, then its unique minimum for $t > 0$ is attained at

$$t_0 = \frac{1}{M-\bar{L}} \ln\left(\frac{-\bar{\varepsilon}}{M\|\mathbf{x}_0\|}\right) \ .$$

Thus, if $\delta(t_0) \le d$, this implies that for $t = t_0$, we have

$$\|\tilde{\mathbf{x}}(t,\mathbf{x}_0)\| \le \delta(t_0) \le d \ ,$$

i.e. $\tilde{\mathbf{x}}$ has reached the domain of attraction for $\mathbf{0}$. Again, with the usual arguments, this implies the asymptotic stability of $\tilde{\mathbf{x}}$.

□

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