The Method of Convex Decomposition
for Stability Analysis with Stepwise Mapping

Rainer Knicker, Peter Krause
University of Dortmund
Faculty of Electrical Engineering
Otto-Hahn-Straße 4, D-44221 Dortmund Germany
Phone: +49-231-755-3998, Fax: +49-231-755-2752
email:{knicker, krause}@esr.e-technik.uni-dortmund.de

ABSTRACT: The computer aided method of convex decomposition allows to strictly establish stability for control systems consisting of a piecewise linear plant model and a piecewise estimable nonlinear discrete-time controller. For complex control laws and small sampling times the computational expenditure of the method can become very high. In this paper, general improvements of the method are presented, which overcome these limitations. The efficiency of the new strategy elements is demonstrated on a fuzzy control system.

KEYWORDS: stability analysis, method of convex decomposition, fuzzy control, multilinear control laws, linear plants, $G_{H,N}$-stability, discrete-time systems

INTRODUCTION

The basic demand of a control system is stability. Advanced nonlinear controllers, like fuzzy controllers and neural networks, have complex control laws to supply an excellent performance of the control system. With increasing complexity of nonlinear control systems proving stability becomes more difficult. In general, a strictly analytical stability proof is very complicated or even impossible, so that computer aided methods are applied. When applying computer aided methods, we have to restrict the class of control systems to which these methods can be applied and formulate stability definitions, that are equivalent to the analytical stability definitions and can be established computer aided.

In this paper, the considered class of control systems is given first, followed by the stability definition that can be established by the method of convex decomposition introduced in [Kiendl 1987] and further developed in [Karweina 1989, Rumpf 1997, Knicker 1999, Kiendl 1999]. Then a short description of the procedure of the method for stability analysis is given before new strategy elements and realisations of concepts introduced in [Kiendl 1997] are presented to establish stability for complex control systems.

CONSIDERED CONTROL SYSTEMS

The method can be applied to control systems consisting of a piecewise linear plant model with an input vector $\mathbf{u}_k$ and an estimable nonlinear discrete-time controller (Figure 1).

The controller has to supply a control law that is piecewise linear or at least piecewise estimable by a linear function with a known estimation error [Knicker 1999]. Due to simplicity, without affecting the generality, in this paper only control systems consisting of one linear plant model

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k + \mathbf{h}u_l(\mathbf{x}_k) = \mathbf{F}_a(\mathbf{x}_k) \tag{1}$$

with one input variable and a piecewise linear discrete-time control law

$$u_l(\mathbf{x}) = \begin{cases} u_{l,1}(\mathbf{x}) = \mathbf{a}_1^T \mathbf{x} + a_0, & \mathbf{x} \in Z_1 \\ \vdots & \vdots \\ u_{l,r}(\mathbf{x}) = \mathbf{a}_r^T \mathbf{x} + a_r, & \mathbf{x} \in Z_r \end{cases}, \quad \mathbf{x} \in D \tag{2}$$

are considered, where

$$D = \bigcup_{i=1}^{r} Z_i \tag{3}$$

with

$$Z_i \cap Z_j = \{\}, \ \forall i \neq j \tag{4}$$

is the zone on which $u_l(\mathbf{x})$ is defined and $\mathbf{x}$ is the n-dimensional state vector (Figure 2).

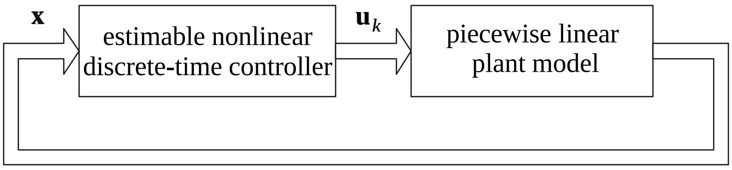

Figure 1: Control system consisting of an estimable nonlinear discrete-time controller and a piecewise linear plant model to which the method can be applied.

This results in a piecewise defined description

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_i \mathbf{x}_k + \mathbf{d}_i = \mathbf{F}_{a,i}(\mathbf{x}_k), \ \mathbf{x}_k \in Z_i \tag{5}$$

of the control system, where the zones Z_i are convex polytopes bounded by hyperplanes.

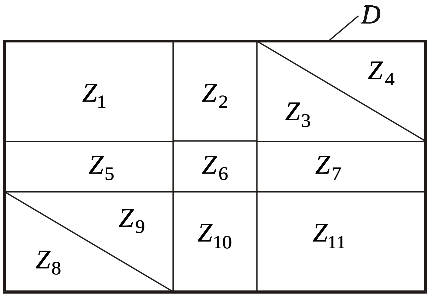

Figure 2: Illustration of the definition domain D for a 2-dimensional nonlinear control system consisting of $r = 11$ convex zones $Z_1, \ldots, Z_{11}$, where a uniform affine mapping induced by a linear plant model and a piecewise linear control law is valid.

All zones and domains in this paper are convex polytopes

$$P = \left\{ \mathbf{x} \middle| \mathbf{C}\mathbf{x} \leq \mathbf{d} \right\} \tag{6}$$

bounded by hyperplanes, described by the intersection of m half-spaces where $\mathbf{C}$ is an $n \times m$ matrix. Each row vector $\mathbf{c}_i^T$ of $\mathbf{C}$ together with the corresponding scalar d_i defines a half-space. By means of linear programming [Danzig 1997] the vertices of the polytopes are determined. The mapping of a polytope is defined as

$$P_{k+1} = \mathbf{F}(P_k) = \{\mathbf{F}(\mathbf{x}) \mid \mathbf{x} \in P_k\}\,, \tag{7}$$

which is a natural extension of (1). For an affine mapping applied to a convex polytope bounded by hyperplanes the result is a convex polytope bounded by hyperplanes again. With this definition, we can construct so called domain trajectories

$$P_k = \xrightarrow{\mathbf{F}} P_{k+1} = \mathbf{F}(P_k) \xrightarrow{\mathbf{F}} P_{k+2} = \mathbf{F}^2(P_k) = \mathbf{F}(P_{k+1}) \xrightarrow{\mathbf{F}} \ldots \tag{8}$$

defined by

$$\mathbf{F}^n(P) = \begin{cases} P & if \quad n = 0 \\ \mathbf{F}(\mathbf{F}^{n-1}(P)) & if \quad n > 0 \end{cases} \tag{9}$$

THE METHOD OF CONVEX DECOMPOSITION FOR STABILITY ANALYSIS

In applications, a general question is whether all states of the workspace of the control system reach the equilibrium state or a sufficient small neighbourhood of the equilibrium state in a finite number of steps and stay there. This is formulated as follows:

Definition of $G_{H,N}$-stability:

> A control system is called $G_{H,N}$-stable if all trajectories starting in the domain G reach the domain H in not more than N steps and stay inside of H for all further steps [Kiendl 1987]. If N is not known the control system is called G_H-stable.

$$\mathbf{F}^n(G) \subseteq H, \qquad \forall n \geq N\,. \tag{10}$$

If the target domain H is a Lyapunov domain, condition (10) is hold for $\mathbf{F}^N(G) \subseteq H$, otherwise criterions to hold condition (10) based on the method of convex decomposition are given in [Kiendl 1997, 1999]. In this paper, without affecting the generality, we consider that H is a Lyapunov domain.

Consequently, each polytope P which reaches H in N steps is called $P_{H,N}$-stable and each polytope which reaches H is called P_H-stable.

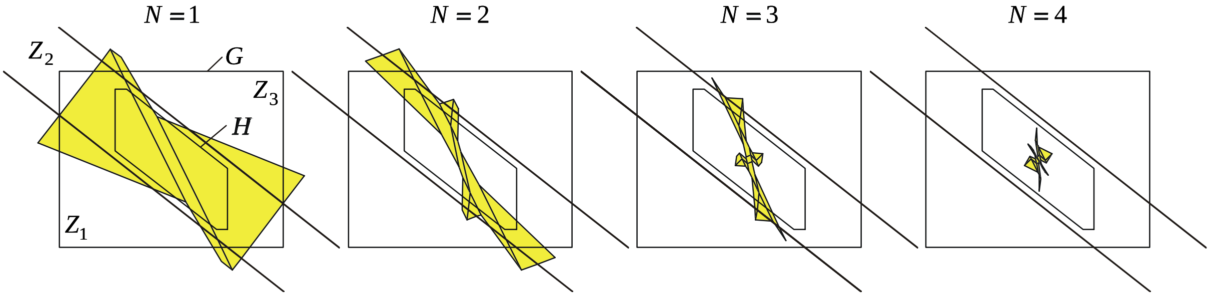

Figure 3: Illustration of the method of convex decomposition for stability analysis proving $G_{H,N}$-stability for a control system consisting of $r = 3$ zones Z_1, Z_2 and Z_3 in $N = 4$ steps. The grey polytopes are $\mathbf{F}^N(G)$.

The basic idea of the method of convex decomposition introduced in [Kiendl 1987] is:

> Let P_k be a convex polytope and $\mathbf{F}(\mathbf{x}) = \mathbf{\Phi}\mathbf{x} + \mathbf{d}$ an affine mapping, then $P_{k+1} = \mathbf{F}(P_k)$ is a convex polytope again and P_{k+1} contains the results of the mappings of the infinite continuum of states situated inside of P_k.

For the considered class of control systems, we can determine the mapping of a polytope and, thus, the mapping of the infinite continuum of states situated inside of the polytope by just considering the vertices of the polytope and the bounding hyperplanes respectively. Additionally, the following is valid:

> We consider a polytope P_k, an affine mapping $\mathbf{F}(\mathbf{x})$ and $P_{k+1} = \mathbf{F}(P_k)$. Each $P \subseteq P_k$ and $\mathbf{x} \in P_k$ respectively is mapped into P_{k+1}.

For control systems consisting of disjunct zones Z_i where an affine mapping (5) or a nonlinear, especially multiaffine mapping, estimable by an affine mapping, is valid, the method can be applied to prove $G_{H,N}$-stability [Scheel; Kiendl 1995, Knicker 1999, Kiendl 1999]. Therefore, the convex polytopes

$$G_i = Z_i \cap G \qquad (11)$$

situated inside of G are mapped and the results are decomposed into polytopes situated in only one of the r disjunct zones Z_i. All decomposed polytopes are mapped and decomposed again until they are situated inside the target domain H or the given number of step N is exceeded. If all polytopes are mapped inside the domain H in not more than N steps, $G_{H,N}$-stability is established (Figure 3). $G_{H,N}$-stability cannot be established if one polytope leaves the definition domain D or N is exceeded.

Thus, the convex decomposition method for stability analysis consists of three basic steps:

S1: Mapping

S2: Decomposition

S3: Checking of termination conditions

C1: polytope inside of H ⇒ polytope is $P_{H,N}$-stable
C2: polytope outside of D ⇒ polytope is not $P_{H,N}$-stable
C3: N exceeded ⇒ polytope is not $P_{H,N}$-stable

THE CONCEPT OF MAPPING OF DOMAINS

In the case of a large number of zones, each zone Z_i in general is mapped into only a very few zones. Therefore, it is useful to determine these zones at the start of the method. In [Rumpf 1997], a method to determine the zones and the half-spaces that decompose $\mathbf{F}(Z_i)$ is given.

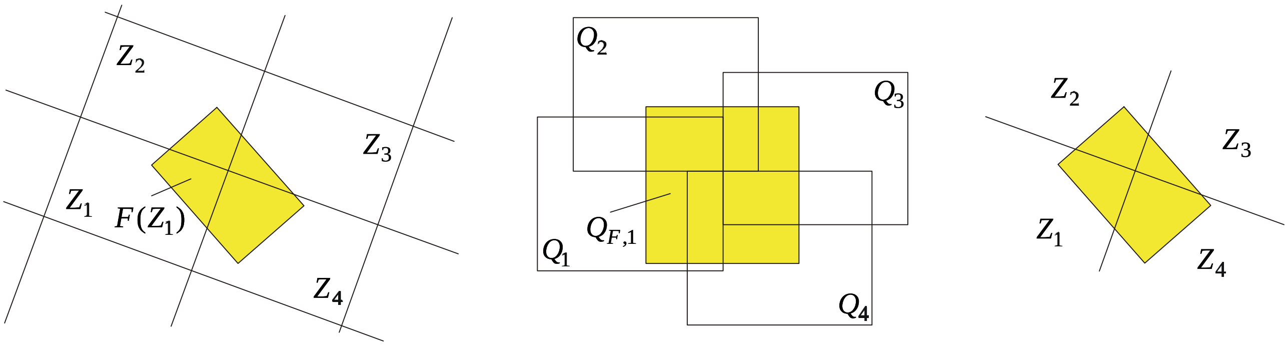

Figure 4: Determination of the half-spaces that decompose the mapping of a zone and, thus, any polytope mapped from this zone.

A more simple method is introduced now:

all zones $Z_i \not\subseteq H$ and all corresponding mapped polytopes $\mathbf{F}(Z_i)$ are enclosed by axis parallel cuboids Q_i and $Q_{F,i}$ respectively. If Q_j and $Q_{F,i}$ have a not empty intersection, which can be determined very easily compared to the calculation of the intersection of polytopes, the intersection of Z_j and $\mathbf{F}(Z_i)$ is determined (Figure 4). If the intersection of the two polytopes again is not empty, the half-spaces of Z_j which decompose $\mathbf{F}(Z_i)$ are determined by means of linear programming [Danzig 1997, Rumpf 1997]. All polytopes $P_F = \mathbf{F}(P \cap Z_i)$ are situated inside of $\mathbf{F}(Z_i)$ and decomposed completely only by those half-spaces that decompose Z_i. For zones Z_j with the shape of a cuboid the decomposing hyperplanes are orthogonal to each other which supplies nice properties for the determination of the vertices of the decomposed polytope. Then all polytopes $G_i \neq Z_i$ not situated completely inside of H are mapped and decomposed. After this first step of the method all neighbourhood zones of each Z_i and each G_i, together with the decomposing half-spaces are determined.

With this information, additional properties of the zones are known: If $\mathbf{F}(Z_i) \cap H$ is empty, all polytopes P mapped from Z_j are outside of H and, consequently, if $\mathbf{F}(Z_i) \subseteq D$ is valid, all polytopes mapped from Z_j are situated inside of D and, therefore, the termination conditions (C1 and C2) need not be checked for polytopes mapped from zones that hold the conditions concerning $\mathbf{F}(Z_i)$.

Another new strategy element that reduces the computational expenditure of the method notably is introduced now:

MAPPING OF DOMAINS:

We consider a domain $G_i = Z_i$ situated completely inside of G and the mapped domain $\mathbf{F}(G_i)$ with

$$\mathbf{F}(G_i) \cap G_i \neq 0 \,. \tag{12}$$

$\mathbf{F}(G_i)$ is decomposed into n polytopes P_j each situated in only one zone Z_j (Figure 5). If P_{j_H}-stability is established for all polytopes $P_j \not\subseteq G_i$, it just has to be shown that an M exists for which

$$\mathbf{F}_{G_i}^{M}(G_i) \cap G_i = 0 \tag{13}$$

with

$$\mathbf{F}_G^m(P) = \begin{cases} \mathbf{F}(P \cap G) & m = 0 \\ \mathbf{F}(\mathbf{F}_G^{m-1}(P) \cap G) & m > 0 \end{cases} \tag{14}$$

is valid to establish P_{j_H}-stability for $P_j \subseteq G_i$ and therewith G_{i_H}-stability. Condition (13) ensures that all trajectories starting from inside of the polytope G_i leave the polytope G_i.

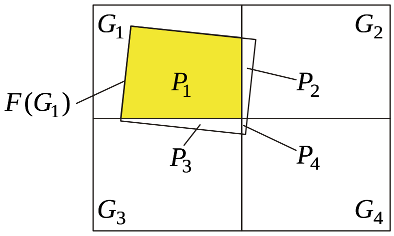

Figure 5: Mapping of a domain $G_i = Z_i$ completely situated inside of G.

For domains G_i with a linear control law condition (13) can be checked very simple. Therefore, the fixed point

$$\mathbf{x}_{f,i} = (\mathbf{E} - \mathbf{\Phi}_i)^{-1}\mathbf{d}_i \tag{15}$$

of equation (5) is determined. We distinguish two cases:

$\mathbf{x}_{f,i} \notin G_i$:

All trajectories starting in G_i will leave G_i and condition (13) is fulfilled.

$\mathbf{x}_{f,i} \in G_i$:

$\mathbf{x}_{f,i}$ is an equilibrium state of the control system.

If all eigenvalues of $\mathbf{\Phi}_i$ are situated inside the unit circle , $\mathbf{x}_{f,i}$ has a certain domain of attraction. G_H -stability can only be established if $\mathbf{x}_{f,i} \in H$ is valid, too.

If at least one eigenvalue of $\mathbf{\Phi}_i$ is situated outside the unit circle, $\mathbf{x}_{f,i}$ is a singular unstable equilibrium state of the control system and G_H -stability cannot be established if $\mathbf{x}_{f,i} \in G$ and $G_i \not\subseteq H$.

If $\mathbf{E}-\mathbf{\Phi}_i$ is singular, a subspace of the state space contains a continuum of fixed points so that we distinguish the same two cases.

For small sampling times the part of $\mathbf{F}(G_i)$ mapped into G_i again is by far the greatest in most cases, so the computational expenditure of the method is decreased notably. This new strategy element can only be applied to zones Z_i that are completely inside of G .

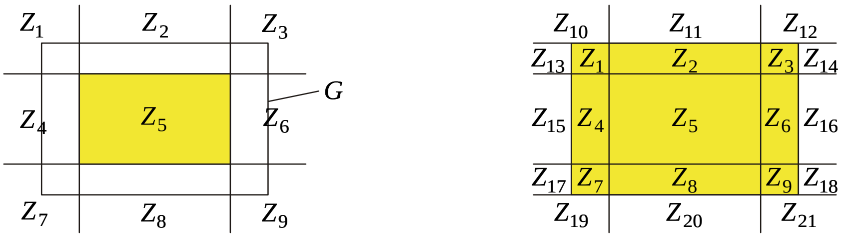

Figure 6: Control law with only one zone completely inside of G (left). Decomposition of all zones Z_i in convex zones completely inside and completely outside, respectively of G (right).

For zones Z_i with parts inside and outside of G, Z_i can be decomposed into convex zones completely inside of G and convex zones completely outside of G, with the same system equation (5) (Figure 6). The control system then is called G -decomposed. Consequently the control system can also be H -decomposed, which has the advantage, that the part of a zone situated inside of H has not to be considered as Z_i -stability is already established.

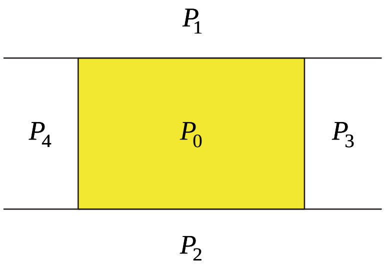

Figure 7: Construction of the convex complement of a polytope P_0 .

To determine a P_0 -decomposed control system we construct the convex complement (Figure 7)

$$P_c = P_1 \cap \ldots \cap P_n \tag{16}$$

with the properties

$$\bigcap_{i=0}^{n} P_i = Z \tag{17}$$

and

$$P_i \cap P_j = \{\}, \ \forall i \neq j, \tag{18}$$

where n is the number of bounding hyperplanes of P_0.

Finally, all zones Z_i which have a not empty intersection with P_0 are decomposed into zones situated inside of only one polytope P_j.

THE CONCEPT OF STEPWISE MAPPING

Advanced controllers, like fuzzy controllers and neural networks, have complex control laws. In special cases the control laws are piecewise multilinear. Otherwise, they can be approximated in a reasonable way by n-dimensional look-up tables. In both cases a large number of domains has to be investigated. For control systems with small sampling times and a large number of domains the number of decompositions for a polytope until it is mapped inside the domain H can become very high.

The basic idea of the stepwise mapping introduced in [Kiendl 1997] can be described as follows:

> If a mapped polytope P is completely situated inside of a union of polytopes for which for each polytope P_{iH}-stability is already established the polytope P also is P_H-stable.

STEPWISE MAPPING:

> We consider a Lyapunov domain P_0 and a family of convex polytopes $P_1, \ldots, P_i$ for which
>
> $$\mathbf{F}^{N_i}(P_i) \subseteq P_0 \cup \ldots \cup P_{i-1}, \ \forall i > 0 \tag{19}$$
>
> is valid. Thus,
>
> $$\mathbf{F}^{N_1+N_2+\ldots+N_i}(P_1 \cup \ldots \cup P_i) \subseteq P_0 \tag{20}$$
>
> is valid, too.

If we choose $P_0 = H$ and $P_i = G_i$ and, thus, $G = G_1 \cup G_2 \cup \ldots G_r$ and $N = N_1 + N_2 + \ldots + N_r$ equation (20) becomes

$$\mathbf{F}^{N}(G) \subseteq H, \tag{21}$$

which is the definition of $G_{H,N}$-stability as H is a Lyapunov domain.

To hold condition (20) it is obvious that the order in which we determine the polytopes G_i is essential for the number of steps N_i and therewith N which we need to establish G_H-stability. The maximum number of steps in which all trajectories starting in G are situated inside of H is not determined directly but can be estimated by the upper boundary N, which is the only drawback of this new strategy element.

All zones Z_j into which G_i is mapped are known after only one mapping of Z_i at the start of the method. If for all these zones Z_{j_H}-stability is established and Z_i holds condition (13) Z_{i_H}-stability is also established. In the case that only G_{j_H}-stability is established for Z_j, which cannot occur for a G-decomposed control system, it has to be checked whether $\mathbf{F}(Z_i) \cap Z_j$ is situated inside of G_j.

After G_{i_H} -stability is established for a domain it is checked whether therewith for another domain G_{j_H} -stability is established, because for all domains into which G_j is mapped G_{i_H} -stability is already established.

For small sampling times it is useful to divide the domains G_i into smaller domains, with the same system description (5), for which G_{j_H} -stability can be established in a few number of steps or in the ideal case with only the first step at the start of the method.

SELCETION OF THE DOMAINS

Using the method with stepwise mapping the order in which the domains are considered is essential for the computational expenditure of the method. Obviously, domains in the neighbourhood of H should be considered first, but depending on the dynamic of the control system and on the domains for which G_{i_H} -stability is established G_{i_H} -stability can be established for domains with a great distance from H within only a few steps.

A first approach to determine an optimal order is introduced here. We consider that the zones into which any Z_j is mapped are known and, thus, trajectories of states can be computed very easily. A state $\mathbf{x}_t \in G_i$ is mapped until it is situated inside of H or the given number of steps N is exceeded (Figure 8).

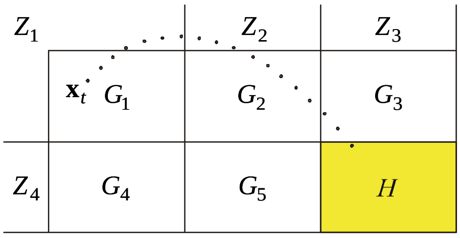

Figure 8: Trajectory starting at the state $\mathbf{x}_t$ mapped into H in 17 steps passing 5 domains and zones, respectively.

The zones into which the state is mapped and the number of steps $\mathbf{x}_t$ stays inside each zone, are noted in a list (Table I). The number of steps until $\mathbf{x}_t$ is mapped into H or a domain for which G_{i_H} -stability is established, is the measure for the domain. This notation has the advantage, that the trajectory has to be determined only once at the start of the method and that the measure can be adapted dynamically, if for a polytope passed by the trajectory G_{i_H} -stability is established. Furthermore, the size of the list for a trajectory is independent of the sampling time and the dimension of the problem. For the trajectory of Figure 8 the measure of domain G_1 decreases from 17 to 10, if G_{2_H} -stability is established. After G_{i_H} -stability is established for a domain the measures for all domains for which G_{i_H} -stability is not established are updated and the domain with the least measure is considered next. If the measure of two or more domains is equal, the domain which is closest to H is considered next.

Zone/Domain	G_1	Z_1	Z_2	G_2	G_3	Σ
Steps	3	4	3	5	2	17

Table I: Resulting list of domains and steps for the trajectory in (Figure 8), with a measure of 17.

To calculate one ore more trajectories which are characteristic for the domain we have to specify starting states $\mathbf{x}_t$ for the trajectories. A characteristic staring point $\mathbf{x}_t$ of the trajectory of a domain is the centre of gravity of the domain, but it has shown out that the mean value of the steps of trajectories starting in the vertices of the domain (mean of vertices), is a much better measure, which results in a slightly increased number of trajectories to be calculated. If one of the trajectories starting from a domain does not reach H or a domain for which G_{i_H} -stability is established the domain gets the measure $N+1$. In general this measure generates a very good order for the domains to be considered.

In the case that for one of the mapped subpolytopes P, starting from the domain G_i currently considered, P_H-stability cannot be established in the given number of steps the domain with the next best measure is considered. If there is no domain left with a measure less or equal to N it is sure that there are no more polytopes G_i left for which $G_{i_{H,N}}$-stability can be established.

This criterion for the order in which the domains are considered is based on trajectories. Other criterions currently under investigation are based on the neighbourhood of the domains, which for example are:

- Into how many zones is G_i mapped?
- How many domains G_j are mapped into Z_i?
- ...

The information is available after the first mapping of each zone, but it has already shown out that none of these criterions alone is better than the criterion based on trajectories. Nevertheless, measures which combine a number of different criterions, for example by means of fuzzy logic or other methods from the field of computational intelligence, can supply a better order. The calculation of the optimal order is not possible due to the curse of dimensionality.

EXAMPLE

The above given strategy elements are illustrated on a second order control system consisting of a cascaded fuzzy controller with a linear plant model as given in [Kiendl 1999]. The sampling time for the controller is 0.1 seconds. The initial domain G (Figure 9) and the target domain H are given as:

$$G = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} -6 \\ -6 \end{pmatrix} \le \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 6 \\ 6 \end{pmatrix} \right\} \tag{22}$$

$$H = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix} \le \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \right\} \tag{23}$$

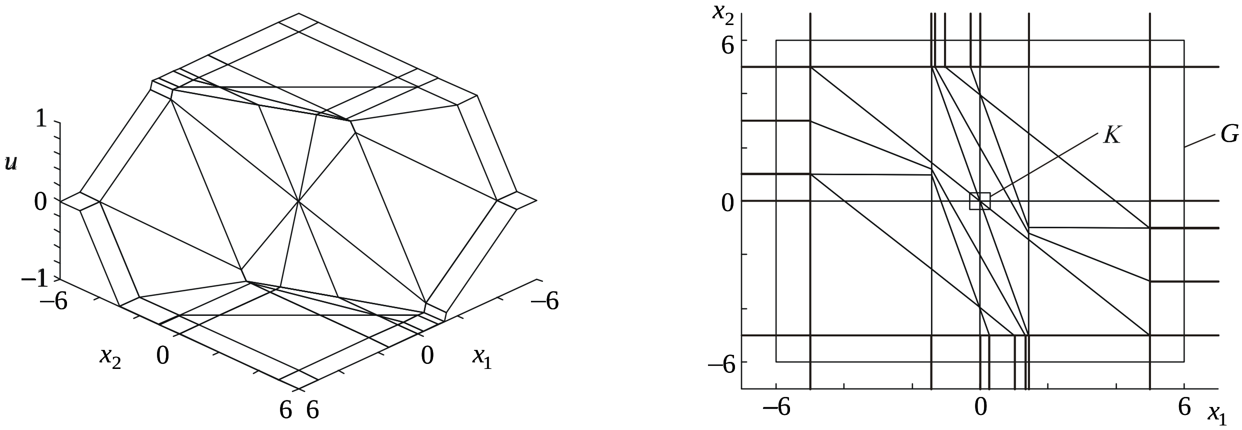

Figure 9: Characteristic surface of the piecewise linear control law of the cascaded fuzzy controller (left) and the corresponding 70 zones with a uniform system description (right).

Condition (10) is hold if all trajectories reach the smaller termination domain (Figure 9)

$$K = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} -0.3 \\ -0.3 \end{pmatrix} \le \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix} \right\}. \tag{24}$$

Figure 9 shows the characteristic surface of the cascaded controller with a saturation of $|u| \le 1$ and the 70 zones where a uniform linear control law and hence a linear system description (5) is valid.

With the method $G_{H,N}$-stability is established for the considered control system. All results presented here were obtained on a Pentium PC with 200 MHz. Table II impressively demonstrates the efficiency of the new strategy elements. For all settings the neighbourhood zones together with the decomposing half-spaces for each zone are determined in the way described above at the start of the method.

settings			results	
stepwise mapping	domain mapping	selection of domains	decompositions	time [min:sec]
off	off		6325672	55:32
off	on		3745568	26:39
on	off	distance	48285	3:02
on	off	mean of vertices	14531	1:20
on	on	distance	18770	0:24
on	on	mean of vertices	8188	0:11

Table II: Computational expenditure for different settings.

With the stepwise mapping and the mapping of domains (Figure 10 left) the number of decompositions and, thus, the computational expenditure of the method is decreased notably. Furthermore these results document that the determination of an optimal order in which the domains have to be considered is a main task.

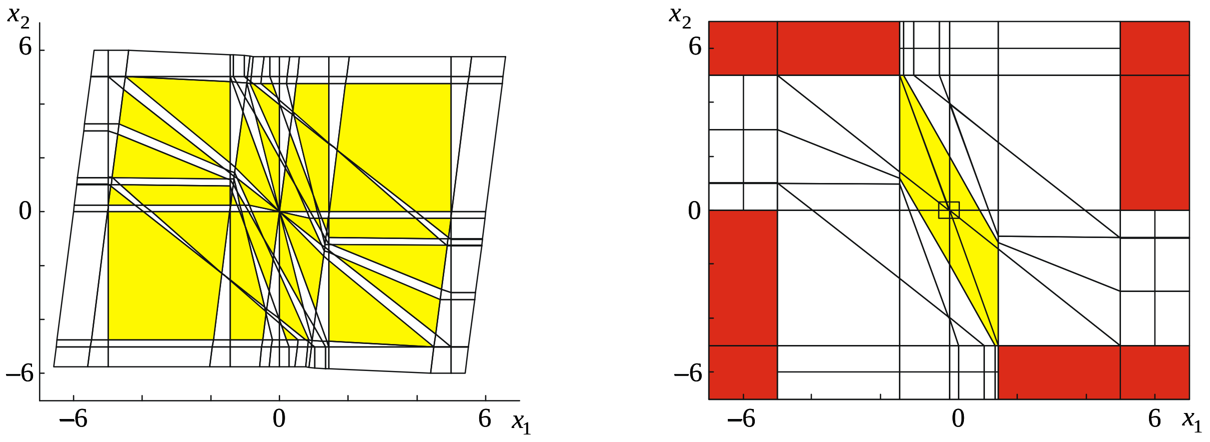

Figure 10: The mapping $\mathbf{F}(G)$ of the initial domain G (left). The shaded polytopes are the parts of the domains $Z_i \subseteq G$ which are mapped into Z_i. Eight zones at the boundary of G (shaded dark) for which condition C1 has to be checked and eight zones in the neighbourhood of H (shaded light) for which condition C2 has to be checked.

Figure 10 (right) illustrates that the termination conditions C1 and C2 are checked only for very few zones of the control system, which again decreases the computational expenditure of the method.

The convex decomposition method for stability analysis has been successfully applied to other higher dimensional control systems [Karweina 1989, Rumpf 1997] as well as to control systems consisting of more complex nonlinear controllers applied to piecewise linear plant models derived from nonlinear plants [Knicker 1999].

CONCLUSIONS

In this paper general improvements of the method of convex decomposition for stability analysis are introduced and the realisations of concepts introduced in [Kiendl 1997] are given. With these new strategy elements, especially the stepwise mapping and the mapping of domains, the method can be applied to very complex control systems with a large number of zones and small sampling times.

The stepwise mapping decreases the computational expenditure of the method essentially. As it has been shown that the order in which G_i –stability is established for each domain G_i is an important factor for the computational expenditure when applying stepwise mapping. A first very satisfying approach to find a good order is a dynamically adapted measure based on state trajectories. Further improvements could be achieved applying advanced strategies based on fuzzy logic or other methods from the field of the computational intelligence.

REFERENCES

Kiendl, Harro, 1987, “Robustheitsanalyse von Regelungssystemen mit der Methode der konvexen Zerlegung”, Automatisierungstechnik 35, pp 192-202.

Kiendl, Harro, 1997, “Fuzzy Control methodenorientiert”, Oldenbourg Verlag, Munich, Germany.

Kiendl, Harro, 1999, “Stability Analysis for Advanced Fuzzy Systems”, 7th EUFIT, Aachen, Germany.

Knicker, Rainer, 1999, “Stability Analysis of Fuzzy and Other Nonlinear Systems based on the Method of Convex Decomposition”, 6th Fuzzy Days, Dortmund, Germany, Springer Verlag.

Karweina, Dieter, 1989, “Rechnergestützte Stabilitätsanalyse für nichtlineare zeitdiskrete Regelungssysteme, basierend auf der Methode der konvexen Zerlegung”, Fortschrittsbericht VDI Reihe 8 Nr. 181, VDI Verlag, Düsseldorf, Germany.

Rumpf, Oliver, 1997, “Stabilitätsanalyse zeitdiskreter nichtlinearer dynamischer Systeme, auf der Basis der konvexen Zerlegung mit paralleler Implementierung”, Fortschrittsbericht VDI Reihe 8 Nr. 651, VDI Verlag, Düsseldorf, Germany.

Scheel, Thomas; Kiendl, Harro, 1995, “Stability Analysis of Fuzzy and other nonlinear Systems using Integral Lyapunov Functions ”, 3rd EUFIT, Aachen, Germany, pp. 765-770.

Dantzig, G.B., 1997, “Linear Programming 1: Introduction”, Springer Verlag.