PIECEWISE AFFINE NEURAL NETWORKS AND NONLINEAR CONTROL : STABILITY RESULTS

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Introduction

Design of a nonlinear control is a difficult task, especially when an exact model of the system to control in not known (Hunt and Su (1983) or Sussmann and Brockett (1982)). The possibilities of tuning artificial neural networks to control have been explored for instance by S. Jagannarthan Jagannarthan (1996) and E.D. Sontag Sontag (1992).

The purpose of this paper is to expose stability properties of artificial neural networks tuned to control. The neural networks used are Piecewise Affine Perceptrons (PAP), a subclass of perceptrons. It has been shown in Lehalle and Azencott (1998) that they have properties that can be used to initialize them to control a given nonlinear system. Besides they have the same useful properties as classical perceptrons (Lehalle and Azencott (1998)) : the universal approximation property and the generalization property.

The stability results given here are obtained by constructing piecewise quadratic Lyapunov functions. The basis of this kind of functions can be found for example in Sontag (1981) and Johansson and Rantzer (1997) and references therein.

This paper will at first establish a result about PAP that will be use to adapt a result about stability of piecewise affine continuous-time systems, then a similar result will be set about discrete-time ones, after that a methodology to tune PAP for control of nonlinear systems will be exposed and finaly this will be illustrated by an example : the control of an engine combustion model by a PAP.

Border matrices of a PAP

Definition 1 (Piecewise Affine Perceptron (PAP)) *a PAP is a function Ψ_W from $\mathbb{R}^d$ in $\mathbb{R}^a$ that can be writen like :*

$$\Psi_W(x) = \Phi\left(W^{(2)}\Phi\left(W^{(1)}\cdot x + b^{(1)}\right) + b^{(2)}\right) \tag{1}$$

*In the framework of a thesis directed by Pr. R. Azencott at the CMLA (DIAM research group) Ecole Normale Supérieure de Cachan, the thesis is carried out at the Research Center of Renault.

where :

- *W is the set of parameters $W^{(1)}$, $b^{(1)}$, $W^{(2)}$, $b^{(2)}$ with $W^{(1)}$ in $\mathcal{M}_{d,c}(\mathbb{R})$, $b^{(1)}$ in $\mathbb{R}^c$, $W^{(2)}$ in $\mathcal{M}_{c,a}(\mathbb{R})$, and $b^{(2)}$ in $\mathbb{R}^a$.*

- *Φ is a function which applies the response function g to each coordinate of a given vector. For PAPs : $g(x) = x$ for $-1 \leq x \leq 1$, $g(x) = 1$ for $x > 1$, and $g(x) = -1$ for $x < -1$.*

c is called the number of hidden units of the PAP.

So the only difference between a one hidden layer perceptron and a PAP is their activation function which is g for the PAP rather than $\tanh(x)$ or $(1 + e^x)^{-1}$ for a classical perceptron.

From Lehalle and Azencott (1998), it is known that a PAP generates a partition of the input space $\mathbb{R}^d$ in polyhedral cells $(C_i)_{i\in I}$. Each of those cells C_i is characterized by six sets of indices $I_+^{(1)}(i)$, $I_0^{(1)}(i)$, $I_-^{(1)}(i)$, $I_+^{(2)}(i)$, $I_0^{(2)}(i)$, $I_-^{(2)}(i)$ so that x is in C_i is equivalent to :

$$\left\{\begin{array}{lcl}
\left\langle W_j^{(1)}, x \right\rangle + b_j^{(1)} < -1 & \Leftrightarrow & j \in I_-^{(1)}(i) \\
\left\langle W_j^{(1)}, x \right\rangle + b_j^{(1)} > 1 & \Leftrightarrow & j \in I_+^{(1)}(i) \\
-1 < \left\langle W_j^{(1)}, x \right\rangle + b_j^{(1)} < 1 & \Leftrightarrow & j \in I_0^{(1)}(i) \\
\left\langle W_k^{(2)} \bar{W}'^{(1)}(i), x \right\rangle + b_k^{(2)} + W_k'^{(2)} \bar{b}^{(1)}(i) < -1 & \Leftrightarrow & k \in I_-^{(2)}(i) \\
\left\langle W_k^{(2)} \bar{W}'^{(1)}(i), x \right\rangle + b_k^{(2)} + W_k'^{(2)} \bar{b}^{(1)}(i) > 1 & \Leftrightarrow & k \in I_+^{(2)}(i) \\
-1 < \left\langle W_k^{(2)} \bar{W}'^{(1)}(i), x \right\rangle + b_k^{(2)} + W_k'^{(2)} \bar{b}^{(1)}(i) < 1 & \Leftrightarrow & k \in I_0^{(2)}(i)
\end{array}\right. \tag{2}$$

With $W_m^{(l)}$ is the mth line of $W^{(l)}$ and $\bar{W}^{(1)}(i)$ the matrix where $\bar{W}^{(1)}(i)_l$ is $W_l^{(1)}$ if l is in $I_0^{(1)}(i)$ else is a line of 0. $\bar{b}^{(1)}(i)$ is the vector which coordinate l is $b_l^{(1)}$ if l is in $I_0^{(1)}(i)$, 1 if l is in $I_+^{(1)}(i)$, and -1 if l is in $I_-^{(1)}(i)$.

Lemma 1 (Border matrices) *Given a PAP parametrized by $(W^{(1)}, b^{(1)}, W^{(2)}, b^{(2)})$ and generating the polyhedral partition $(C_i)_{i\in I}$, it exists matrices $(G_i)_{i\in I}$, $(D_i)_{i\in I}$, and $(F_i)_{i\in I}$ called border matrices of the PAP so that :*

$$\left\{\begin{array}{lll}
G_i x + D_i \geq 0 & ; & \text{for } x \in C_i \\
F_i \begin{pmatrix} x \\ 1 \end{pmatrix} = F_j \begin{pmatrix} x \\ 1 \end{pmatrix} & ; & \text{for } x \in C_i \cap C_j
\end{array}\right. \tag{3}$$

This comes easily from (2). From here we use the notations $E_i = \begin{bmatrix} G_i & D_i \\ 0 & 1 \end{bmatrix}$ for each i in I and $||u||_{P_k}^2 = u' P_k u$ for any vector u and $||M||_{P_i}^2 = \sup_{x'P_ix=1} x'M'P_iMx$ for any matrix M.

Result for continuous-time systems

The stability of a system like :

$$\dot{x} = f(x, \Psi_W(x)) \tag{4}$$

where Ψ_W is a PAP can be approximated on each polyhedral cells $(C_i)_{i\in I}$ generated by Ψ_W by :

$$\dot{x} = A_i x + B_i, \text{ when } x \in C_i \tag{5}$$

so that :

$$||f(x) - A_i x - B_i||_2 \leq \varepsilon_i \, ||x||_2 \, , \text{ when } x \in C_i \tag{6}$$

Using the theorem 2 of Johansson and Rantzer (1997) we have this result :

Theorem 1 (Continuous time stability within approximation range) *If (4) is approximated on each cell of the Piecewise Affine Perceptron Ψ_W by (10) such that (6) hold, then if there exist numbers $\gamma_i > 0$, symmetric matrices U_i and W_i with non-negative entries, and a symmetric positive definite matrix T such that :*

$$\forall i \in I, \; P_i = F_i' \, T \, F_i \tag{7}$$

satisfy

$$\forall i \in I, \begin{cases} E_i' \, U_i \, E_i < P_i < \gamma_i \, Id \\ -2\varepsilon_i \gamma_i \, Id > A_i' \, P_i + P_i \, A_i + E_i' \, W_i \, E_i \end{cases} \tag{8}$$

then every trajectory $x(t)$ of (4) tends to zero exponentially.

The proof comes easily from Johansson and Rantzer (1997).

Result for discrete-time systems

The system studied in this part is :

$$x_{n+1} = f(x_n, \Psi_W(x_n)) \tag{9}$$

where Ψ_W is a PAP can be approximated on each polyhedral cells $(C_i)_{i \in I}$ generated by Ψ_W by :

$$x_{n+1} = A_i x_n + B_i, \text{ when } x_n \in C_i \tag{10}$$

Then, if the **assumptions** are made *(A1)* that for all points x in a given cell C_i, then $A_i x + B_i$ is a cell sharing a common border with C_i and *(A2)* that the matrices $\mathcal{A}_i = \begin{bmatrix} A_i & B_i \\ 0 & 1 \end{bmatrix}$ are invertibles (in most pratical cases, *(A1)* implies *(A2)*), we have the following result (with the notation $\mathcal{C}_i$ for the set of indices of cells sharing a common border with C_i) :

Theorem 2 (Stability of a piecewise affine system controlled by a PAP) *If there exists symmetric matrices $(U_i)_{i \in I}$, and $(W_i)_{i \in I}$ such that $(U_i)_{i \in I}$ and $(W_i)_{i \in I}$ have non-negative entries, and a symmetric positive definite matrix T while*

$$\forall i \in I, \; P_i = F_i' \, T \, F_i \tag{11}$$

satisfy :

$$\forall i \in I, \begin{cases} P_i - E_i' \, W_i \, E_i & > \quad 0 \\ \mathcal{A}_i' \, P_i \, \mathcal{A}_i - E_i' \, U_i \, E_i & < \quad \mathcal{K}(j) \inf \left(P_i, \, \mathcal{A}_j^{-1\prime} \, P_i \, \mathcal{A}_j^{-1} \right), \\ & \qquad \forall j \in \mathcal{C}_i \end{cases} \tag{12}$$

where $\mathcal{K}(j) = \inf \left(1, \left|\left|\mathcal{A}_j^{-1}\right|\right|_{P_j}^{-2} \right)$.

Then $V(x_n) = (x_n, 1) \, P_i \, (x_n, 1)'$ when x_{n-1} in C_i is a Lyapunov function for system (10).

To prove this, the essential point is to compute $V(x_{n+1}) - V(x_n)$. Note that the notation of $V(x_n)$ (with the use of the cell of x_{n-1} and not of x_n) is only used for notation facilities. When x_n and x_{n-1} are in the same cell, it is easy to see that $V(x_{n+1}) - V(x_n) < 0$. When it's not the case, the second condition of (12) has to be used noticing that if $x_n \in C_i$ and $x_{n-1} \in C_j$, then $c \in [0, 1[$ exists, so that $\tilde{x}_n = c \, x_{n-1} + (1 - c) \, x_n \in C_i \cap C_j$. Using this composite point which verifies $(\tilde{x}_n, 1) \, P_i \, (\tilde{x}_n, 1)' = (\tilde{x}_n, 1) \, P_j \, (\tilde{x}_n, 1)'$ (because of the construction of P_i by (11)) conducts to the solution. □

Methodology to use these results

It is now possible to establish a framework to design Piecewise Affine Perceptrons to control nonlinear systems :
1. The PAP Ψ_W has to be initialized to control locally the system (4) or (9) using results in Lehalle and Azencott (1998).
2. The parameters W of the PAP has to be tuned (for instance through a gradient method) to minimize a given criterion like (for discrete-time systems and optimal control, ν_{x_0} are positive real numbers, Q and R symetric positive matrices, see Sussmann (1989)) :

$$\mathcal{I}_N(\mathcal{X}_0) = \sum_{x_0 \in \mathcal{X}_0} \nu_{x_0} \left(\sum_{k=0}^{N} \underbrace{x'_k Q x_k + u'_k R x_k}_{\mathcal{J}_k} \right) \tag{13}$$

3. After tuning a given PAP is obtained. Because the criterion minimized is $\mathcal{I}_N(\mathcal{X}_0)$ and not $\mathcal{I}_\infty(\mathcal{X}_0)$, it is always possible that the obtained PAP does not emulate a stable control.

So to ensure this, theorem 2 or 1 can be used to choose between several tuned neural networks, the one that will emulate a stable control.

Illustration: control of engine combustion

The illustration here comes from the design of a PAP to control a simple engine combustion model :

$$\sigma \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \log((1-\varphi_2)\cdot\exp(x_1) + \varphi_2 \cdot \varphi_1 \cdot \exp(u_1)) \\ x_1 + \log(\eta_r(R)) + u_2 \\ x_1 + \log(\eta_r(R)) + \log(\eta_a(A)) \end{pmatrix} \tag{14}$$

The state space considered is : $x_t = (x_1(t) - x_1^*, x_2(t) - x_2^*, x_3(t) - x_3^*, \sum_{\tau=0}^{t}(x_2(\tau) - x_2^*(\tau)), \sum_{\tau=0}^{t}(x_3(\tau) - x_3^*(\tau)), x_2^*, x_3^*, x_1^*)$ (where the variables with star (like x^*) are target values ; φ_1, φ_2, R, and A are constants ; and σ is the operator : $\sigma x_t = x_{t+1}$) and the control space is $u_t = (u_1(t), u_2(t))$. The criterion used in equation (13) is :

$$\mathcal{J}_n = x'_n \, diag(0, 10, 10, 100, 10, 0, 0, 0) \, x_n + u'_n \, diag(100000, 100000) \, u_n$$

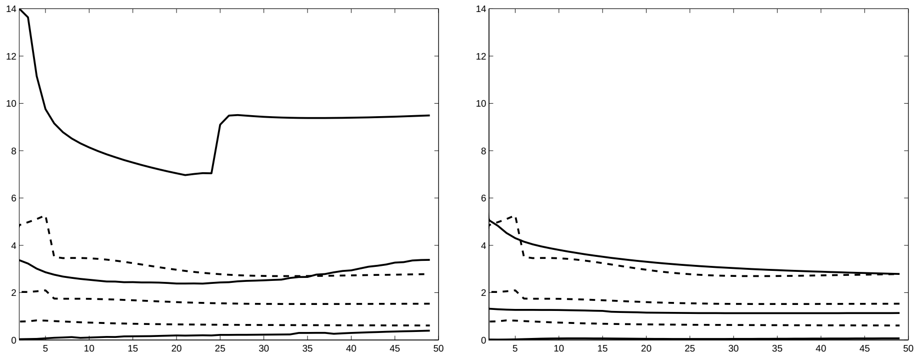
The PAP has 8 hidden units and has been initialized to emulate an linear optimal controller ; the dotted lines of graphs of figure 1 shows the mean, the minimum and the maximum of evolution of $\mathcal{J}_n$ through time computed for a set of points chosen at random in a given subspace X_0 of $\mathbb{R}^8$, different of the one used to tune the PAP.

A gradient algorithm has been used to tune the PAP using different sets of initial points and different time length. The two graphs of figure 1 show two situations for tuned PAPs. The first one conducts to a stable control (b) and not the second one (c).

This is verified by computing the solutions $(P_i)_{i \in I}$ of theorem 2.

Conclusion

One of the main purpose of this study is to tune a nonlinear control based on Piecewise Affine Perceptrons (PAP) to design control for nonlinear engine combustion. The illustration given here is based on a simple model of engine torque. The PAP determines the commands to apply to actuators to drive to a torque minimizing a given criterion.



(a) (b)

Figure 1: values of mean, minimum and maximum of $\mathcal{J}_n$ through time for : the initialized PAP (dotted), and two tuned PAPs one of them being stable (b) and the other not (a).

The results in this paper and previous results developed in Lehalle and Azencott (1998) conduct to a generic methodology to tune a PAP to control : first deduce a set of optimal linear controls from local linearizations of the nonlinear system to control, then construct a PAP that emulates locally each of these controls, after that the PAP can be tuned by a gradient method and finally the stability of the control can be predict using the theorems 2 and 1 of this paper. Beside we have shown in Lehalle and Azencott (1998) that PAP are as generic as classical perceptrons.

A converse version of the theorem 2 and the study of the robustness of the tuning (ie what conditions have to be set to ensure that a PAP tuned with a noisy model can control the original system) are ongoing research in collaboration with the Research Center of Renault.

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