Fractal Noise Synthesis from Fuzzy Splines

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ABSTRACT: This paper describes a new algorithm for the synthesis of constrained fractal curves. More precisely the algorithm synthesizes curves that *approximate* a set of sparse data and with prescribed roughness. The method works in successive phases: in the first stage a fuzzy spline modeling the data distribution is computed; successively the algorithm samples from this distribution a fractal curve. Roughness is hence adjusted applying a suitable non-linear weighted smoothing filter. The weights are chosen using the fuzzy spline computed in the earlier stage. The new method automatically takes into account the local variability of the data and behaves well in presence of fast variations.

KEYWORDS: fractal curves, fuzzy splines, BADD distribution.

INTRODUCTION

Simulation and rendering of natural phenomena require the synthesis of natural looking noise. Since B.B. Mandelbrot in 1976 observed that the geometry of several natural surfaces and curves have fractal dimension, the synthesis of fractal-like noise has received a great deal of attention. Impressive results have been obtained following Mandelbrot's ideas (Voss 1985, Pentland 1984, Fournier 1982, Perlin 1985, Musgrave 1989).

Most of the works, however, focus on synthesizing completely artificial curves and surfaces with a natural look. In 1989 R.Szeliski and D.Terzopoulous (Szeliski 1987, Szeliski 1989) posed the following variation of the basic problem: “How to obtain a natural looking curve/surface approximating a collection of some sparse data?”.

The solution of this problem claims for the conciliation of randomization and deterministic approximation. Szeliski [1989] uses energy constrained splines, Gaussian noise and Gibbs sampling (Geman 1984). In short, a variational spline is evolved towards a low energy configuration, while, at the same time it is perturbed with the addition and the local diffusion of Gaussian noise. Modulation of the noise and the choice of suitable parameters for the diffusion allow local control of the roughness of the resulting surface. The resulting surface has indeed fractal dimension (Szeliski 1987). Gibbs sampling is computationally demanding and several iterations are required for the convergence to the desired solution. For this reason several optimization and/or parallel version of Selinski-Terzopoulos idea have been hence introduced (Arakawa 1996, Vemuri 1997).

In this paper we propose a new alternative solution to the synthesis of constrained fractal curves based on the pre-modeling of the known sparse data with a possibilistic distribution, i.e. with a fuzzy spline (Anile 1998, Gallo 1998). The possibilistic distribution is, in turn, translated into a probabilistic distribution (Yager 1991). The method produces a natural looking curve through sampling from such a distribution and successively smoothing with a non-linear adaptive mask to correlate in a fractal fashion the resulting noise. A single, user selected, parameter controls the roughness of the surface. The new method automatically takes into account the local variability of the data and, moreover, behaves well in presence of sharp variations as shown in the reported examples.

The structure of this paper is as follows: Section 2 introduces the concept of possibilistic distributions and of fuzzy spline and their relations with probability theory. Section 3 describes the details of the proposed algorithm. Examples are reported in Section 4. Conclusions and future research related with the ideas reported in the paper are shortly summarized in Section 5.

2. STOCASTIC MODELING WITH FUZZY SPLINES

This section shortly introduces the concept of Fuzzy B-Spline (FBS). FBS with fuzzy numbers are the basic mathematical tools for modeling the distribution of the elevation over the reconstruction domain. For notations and definitions not explicitly reported here the reader is referred to Zimmermann 1991. Translations of fuzzy (possibilistic) distribution into a probabilistic distribution are also discussed (Yager 1991).

2.1 FUZZY B-SPLINE

The fuzzy B-spline may be naturally introduced extending the traditional definition of B-spline (Barsky 1987). A FBS is a convex combination of fuzzy number: $F(t)=\sum_{i=0}^{k+h-1} F_i B_{i,h}(t)$ where F_i the control coefficients, are fuzzy numbers and the $B_{i,h}(t)$ are the crisp B-spline basis function of order h.

Observe that this definition is consistent with the definition of fuzzy number. In other words, for every t, $F(t)$ is a genuine fuzzy number, i.e. it verifies the convexity property. This is guaranteed by the convex hull property of the B-spline together with the positiveness of the B-spline basis functions. Fig.1 shows some α-levels of a fuzzy B-spline.

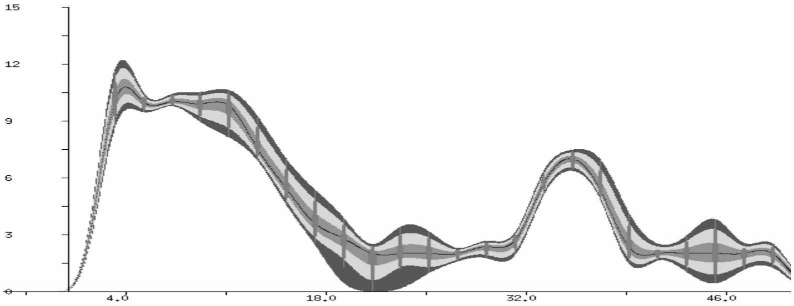


Figure 1: An example of FBS: several levels are depicted in different colours.

2.2 FROM POSSIBILITY TO PROBABILITY

In many applications it is important to be able to combine the powerful and century tested tools of probability theory with the flexibility of fuzzy numbers. The process of transforming from a possibility to a probability distribution defined on the same universe of discourse is called possibility-probability transformation. There are infinite ways to operate such transformation: in the algorithm presented in this paper we use the Basic Defuzzification Distribution transformations (in short, BADD transformations) introduced by Yager in 1991.

BADD transformations constitute a family of transformations parameterized by a value $b \in [0, +\infty[$ as follows. For a fixed value b a p.d.f. p_b is obtained from the membership function of the fuzzy number as follows. For every t in the universe of discourse:$p_\beta(t)=\mu(t)^\beta/\int_{I_0}\mu(s)^\beta ds$.

In Yager 1991 it is shown that indeed p_b is a p.d.f. and that the entropy of p_b is maximal for $b=0$ and decreases to a minimum for $b \rightarrow \infty$. This makes precise the intuitive idea that smaller values for the b parameter lead too closer to uniform p.d.f. Conversely high values for b lead to distributions that are very peaked around the mean. A consequence of this is that the variance of the p.d.f. p_b decreases as b gets larger. The graph of a distribution p_b per some b values is reported in Figure 2.

In the proposed reconstruction technique, the user is asked to provide a suitable value for β: higher values allow smaller variances and hence smoother surfaces, conversely smaller values for β lead to rougher terrain. The precise relation between b and the roughness degree (or the fractal dimension) of the reconstructed surface is discussed in a later section.

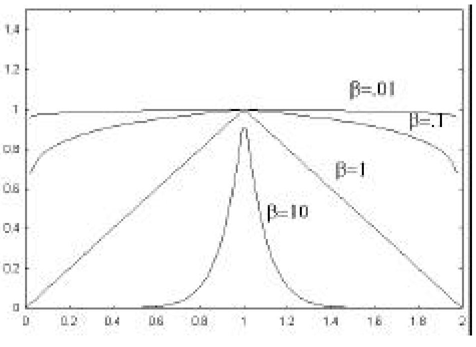


Figure 2: A symmetric fuzzy number and the BADD transformed distribution for several b values.

3. SYNTHESIS OF CONSTRAINED FRACTAL CURVES

As observed by Mandelbrot, fractal curves have a natural look: hence the interest for any simulation to synthesize fractal noise. Just "adding" to a cubic spline white noise at low energy hardly produces a convincing illusion of a natural profile: realism requires a certain degree of correlation in the perturbing noise. Fractal noise, indeed, and in particular Brownian noise, can be shortly characterized as a statistically correlated noise.
Selinsky and Terzopoulos have presented an algorithm to synthesize constrained fractal curves (Szeliski 1989). Essentially their algorithm adds a controlled amount of white noise to a spline approximating the assigned data while, at the same time, it provides an iterative mechanism (through Gibbs sampling) to correlate the noise over the domain. Approximation and randomization, in this way, play simultaneously in a unique computational mechanism.
The approach that we describe in detail in this section follows the same basic idea but changes the strategy: once a suitable simplified probabilistic model (i.e. a fuzzy B-spline) of the elevation distribution is available, it is possible to obtain a correlated noise with a much faster convergence rate than the one observed with Gibbs sampling. This leads to a more efficient reconstruction technique. In our approach, approximation and randomization are, hence, realized in successive steps.
Observe that it is also possible, using a fuzzy spline model of the data, to generate a fractal constrained curve in a sequential fashion as shown in (Gallo 1999).
The technique proposed here obtains a curves of controlled roughness, approximating an assigned sparse set S of data, over a domain D of dimension H. The user has also to specify a resolution, i.e., the number N of discrete points to sample over D. These points are taken as points of a uniform lattice whose step is $H/(N-1)$.
The output of the algorithm is a vector $h(i)$ of crisp values, with $i=0,\ldots, N-1$. Another parameter that should be provided to the procedure is a real number $b \in [0, +\infty[$. This parameter determines the transformation of a fuzzy number into a probability distribution according to the ideas described in Section 2.2. b controls the roughness of the sampled curve; higher values of b produce a smoother curve.
The proposed technique can be decomposed into three main stages: *Preprocessing, Sampling, Non-linear weighted smoothing*.

Preprocessing: the set S is given as input to a procedure that computes a fuzzy B-Spline M approximating the data. More precisely the fuzzy number $M(x)$ describes a possibilistic distribution of data values at the location $x \in D$. The details of this construction are beyond the scope of this paper. See Anile 1998, Gallo 1998.

Sampling: For every point x of the domain, a crisp value $B(x)$ is selected according to the probability distribution $p_b(M(x))$;

Smoothing: let $M^{(a)}(x)$ the α-*level* of the fuzzy number $M(x)$,

$$h(x)=M^{(1)}(x)+[E(x)-M^{(1)}(x)]|S|/N \qquad \text{where:}$$

$$E(x)=\frac{\sum_{i=x-2}^{x+2} w_i^{(x)} B(i)}{\sum_{i=x-2}^{x+2} w_i^{(x)}}$$ is the weighted mean of x's neighbors.

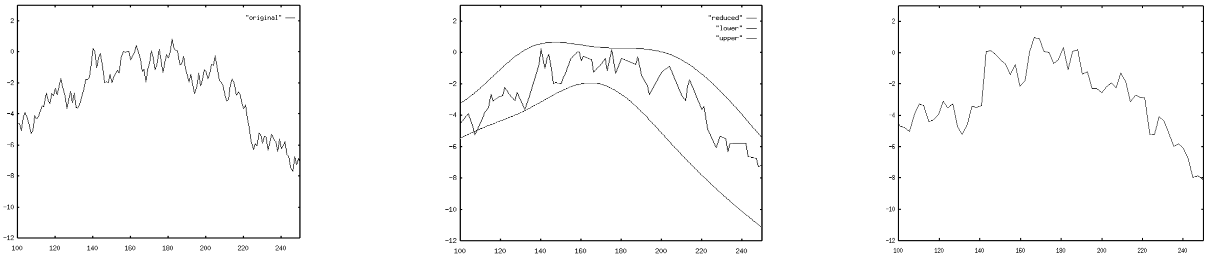
The weights are selected as follows: $w_i^{(x)}=p(i)*A(i-x+3)$.
A(j) is an appropriate smoothing kernel of width 5, centered in the processed datum. Experiments have shown that slight variations of the binomial smoothing filter are a good choice in the implementation. The number $p(i)$ is the probability density value of B(i) in the $p_b(M(i))$ distribution.
The **output** of the algorithm is hence the array of the h(x) values.

As it is promptly seen from the above description of the algorithm, the choices performed at every location are initially independent. Ideally to obtain a correlated noise the deviation from the expected value at a location x should be conditioned by the deviations from the mean at the neighboring locations. Since the fuzzy spline provides a complete distribution of the data in the neighboring location, the procedure of adaptive smoothing described above is able to introduce such a correlation.
Next Section reports the experimental results obtained: they show that the algorithm produces, indeed, a fractal curve.

4 EXPERIMENTAL RESULTS

This Section reports the results obtained using the algorithm described above. Measuring the fractal dimension of a curve is a difficult task and truly robust algorithms to this aim have not yet found (Penn 1997). Yokoya 1989 and Pentland 1984 have proposed estimation algorithms that use dense data and work well only under the assumption that the curve under observation is a Brownian fractal. Arakawa 1996 has extended Yokoya's algorithm to estimate the fractal dimension of a curve to the case when only a sparse distribution of points is known. It seems wiser, given the notorious sensitivity of all these algorithms, to consider the value that they provide for fractal simply as a way to quantify objectively the degree of roughness of a curve. For this reason in the discussion of the experiments we will freely use the term "degree of roughness" instead of "fractal dimension".

A first set of experiments has been realized on both synthetic and natural profiles. Figure 3(a) shows the trace of a Brownian motion in 1D. A random subset of about 40% of the whole profile has been selected as shown in (b). A fuzzy B-spline has been obtained from this data: upper and lower curves of the model are also shown (b). Figure.3(c) shows the reconstruction obtained with β=0.4. The fractal dimension of this reconstruction is 1.48.



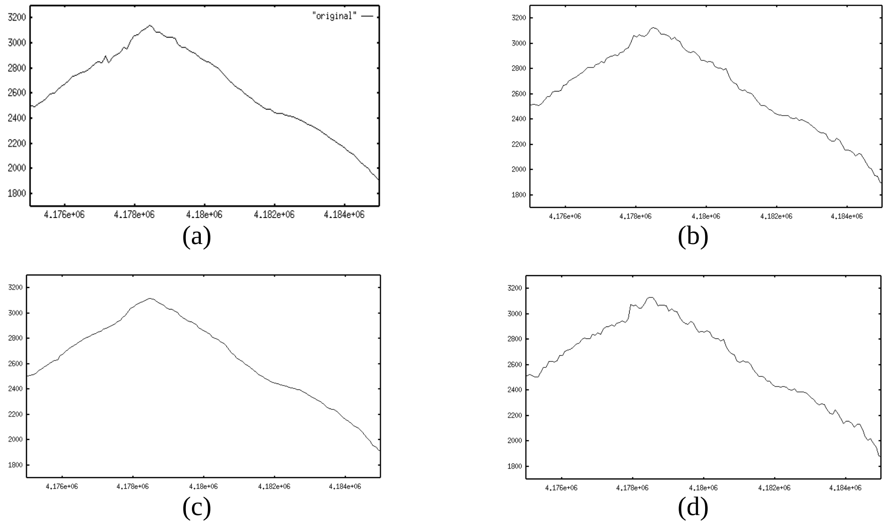
(a) (b) (c)

Figure 3: (a) Trace of a classical Brownian motion (fractal dimension 1.5); (b) A sample of 40% of the signal in (a), the upper and lower boundaries, at presumption level 0, of the fuzzy model for these data are also shown; (c) reconstructed signal with roughness parameter to 0.4.

Similar experiments have been realized using as input data the elevations of Mount Etna taken along a meridian at an average distance of 100 meters. Figure 4.(a) shows a portion of a linear interpolation of the original data, (b) shows the reconstruction obtained with value β=1, (c) shows the reconstruction obtained with value β=10 and (d) shows the reconstruction obtained with value β= 0.1. For all the reconstruction the roughness degree is around the value 1.21.

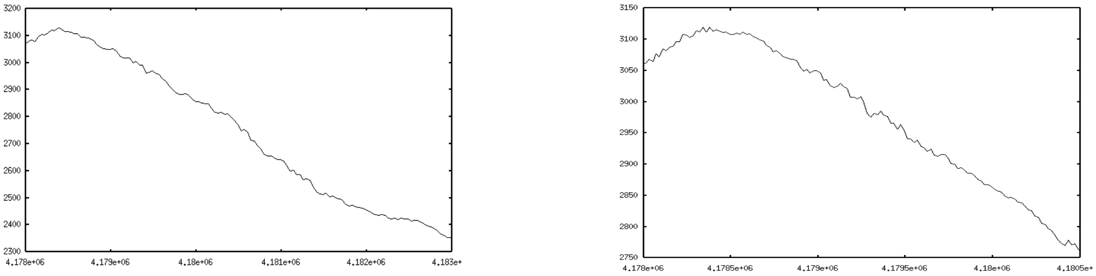
Zooming in, i.e., sampling on a progressively refined grid will add details that reproduce, at lower scales, the same statistical fractal behavior observed at larger scales as shown in Figure 5 (a) and (b).

To observe how the algorithm performs in presence of a sharp variations in the data we have performed a reconstruction as shown in Figure 6.



(a) (b) (c) (d)

Figure 4: (a) A portion of the profile of Mount Etna (500 sampled points); (b) a fractal reconstruction, with the same resolution of (a) with *b*=1; (c) a fractal reconstruction with the same resolution of (a) with *b*=10; (d) a fractal reconstruction, with the same resolution of (a) with *b*=0.1.



(a) (b)

Figure 5: (a) 1000 sampled points; (b) 2000 sampled points.

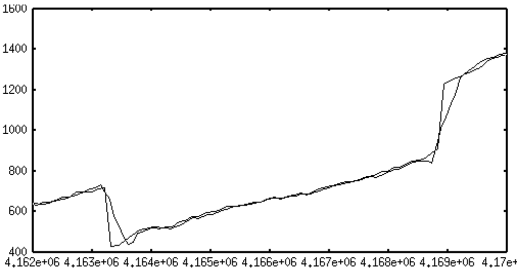


Figure 6: An artificially exaggerated crack has been introduced: the quality of the reconstruction is not strongly affected.

5 CONCLUSIONS

In this paper we have presented a new algorithm for the synthesis of fractal curves subjected to approximation constraints. The algorithm allows the control of the roughness of the resulting curve and realizes the model avoiding the long iterative process of previously published methods. The proposed method uncouples the approximation phase and the randomization phase. Using a fuzzy B-spline, noise correlation can be achieved efficiently. The generalization to higher dimensions is completely straightforward.

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