Extended Triangular Norms in Type 2 Fuzzy Logic

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ABSTRACT: This work focuses on type 2 fuzzy logic which is a logical system of fuzzy truth values equipped with disjunction, conjunction, t-conorm, t-norm, *R*-implication and strong negation. Assuming the normality, convexity and upper semicontinuity of fuzzy truth values, a t-norm, a t-conorm and a strong negation of type 2 are defined in an axiomatic way, and an *R*-implication is derived as the right residual of a t-norm. The expressions of these operations except for an *R*-implication can be obtained implicitly by extending those of the conventional (i.e. type 1) fuzzy logic by means of the extension principle. As the main result of this report, it is shown that the type 2 fuzzy logic forms a complete lattice-ordered monoid as well as the case of type 1; that is, it is a concrete model of L-fuzzy logic.

KEYWORDS: type 2 fuzzy logic, fuzzy truth values, the extension principle, t-norms, t-conorms, *R*-implications, strong negations, De Morgan triplet

INTRODUCTION

Zadeh (1975) has introduced the concept of a fuzzy set of type 2, in which membership grade of each element is a fuzzy subset in the unit interval $[0,1]$ (it is called a fuzzy grade). Mizumoto et al. (1976), (1981) have investigated in detail the algebraic properties of fuzzy sets of type 2. Following these theoretical works, many researchers have paid attention to its advantage; that is, the partial order structure of fuzzy grades effectively fit to human thinking and judgment, and they have applied a fuzzy set of type 2 to several methods to process information with uncertainty (e.g. Miyakoshi et al. (1980), Izumi et al. (1986), John (1998), Karnik et al. (1998), etc.).

As it is well known, a pair of a t-norm and a t-conorm is defined as a kind of product and sum in $[0,1]$ and plays an important role of a conjunctive operator and a disjunctive operator, respectively, in fuzzy logic. In this research, the authors extend the concept of a t-norm and a t-conorm to that for fuzzy grades (hereafter, we call them fuzzy truth values) and construct fuzzy logic of type 2 equipped with a t-norm and a t-conorm on the assumption that fuzzy truth values are normal, convex and upper semicontinuous. In the same manner as conventional (i.e. type 1) fuzzy logic, an implicative operator of type 2 is defined via residuation of a t-norm. Also, we define a strong negation of type 2 and De Morgan triplet of type 2 which is a combination of a t-norm, a t-conorm and a strong negation satisfying De Morgan law. We can construct a t-norm, a t-conorm and a strong negation of type 2 by extending those of type 1 by means of the extension principle (Zadeh (1975), but an implication of type 2 can not be realized by the extension of *R*-implication. As the main result of our work, it is shown that the type 2 fuzzy logic with a t-norm forms a complete lattice-ordered monoid; in other words, it is a concrete model of L-fuzzy logic originated by Goguen (1979).

FUZZY TRUTH VALUES

A fuzzy truth value is defined as a fuzzy subset of $[0,1]$. Let $F([0,1])$ denote the set of all fuzzy truth values, and $f_A:[0,1]\to[0,1]$ be the membership function of a fuzzy truth value A.

The logical operators for disjunction $\sqcup$, conjunction $\sqcap$ and negation $\tilde{\neg}$ of fuzzy truth values are defined by applying the extension principle (Zadeh (1975)) to the logical operators for the conventional fuzzy logic ($\vee=\max$, $\wedge=\min$, $\neg=1-$) as follows:

$$f_{A\sqcup B}(z) = \sup_{z=x\vee y} \min\{f_A(x),\ f_B(y)\},$$
$$f_{A\sqcap B}(z) = \sup_{z=x\wedge y} \min\{f_A(x),\ f_B(y)\},$$
$$f_{\tilde{\neg}A}(z) = \sup_{z=1-x} f_A(x) \qquad \big(A, B \in F([0,1])\big).$$

Through this paper, we assume that fuzzy truth values are normal, convex and upper semicontinuous (briefly, USC), and call the logic with such fuzzy truth values 'fuzzy logic of type 2' or simply 'type 2 fuzzy logic.' Let V denote the set of all normal, convex and USC fuzzy truth values. The α -level sets of $A \in V$

$$A_\alpha = \begin{cases} \{x \in [0,1] \| f_A(x) \geq \alpha\} & (0 < \alpha \leq 1) \\ \mathrm{cl}\{x \in [0,1] \| f_A(x) > \alpha\} & (\alpha = 0) \end{cases}$$

are closed intervals in $[0,1]$.

Mizumoto et al. (1976) have shown that the set of normal and convex fuzzy truth values forms De Morgan algebra under disjunction $\sqcup$, conjunction $\sqcap$ and negation $\tilde{\neg}$ with the greatest element **1** $\big(f_{\mathbf{1}}(x) = 1\,(x=1),\ 0\,(\text{otherwise})\big)$ and the least element **0** $\big(f_{\mathbf{0}}(x) = 1\,(x=0),\ 0\,(\text{otherwise})\big)$. Moreover, they have defined the partial order $\sqsubseteq$ as

$$\begin{aligned} A \sqsubseteq B \quad &\Leftrightarrow \quad A \sqcup B = B \\ &\Leftrightarrow \quad A \sqcap B = A \end{aligned}$$

for normal and convex fuzzy truth values A and B. It should be noted that $A \sqcup B = B$ is not necessarily equivalent to $A \sqcap B = A$ for $A, B \in F([0,1])$.

Theorem 1. The set of all normal, convex and USC fuzzy truth values V forms a complete Brouwerian lattice under disjunction $\sqcup$ and conjunction $\sqcap$ i.e. the following statement holds:

$$A \sqcap B \sqsubseteq C \quad \Leftrightarrow \quad B \sqsubseteq A \tilde{\to} C \qquad (A, B, C \in V),$$

where $A \tilde{\to} B = \sqcup\{X \in V | A \sqcap X \sqsubseteq B\}$.

By applying the result of Nguyen (1979) to disjunction $\sqcup$; normal, convex and USC fuzzy truth values $A, B \in V$ and a family $\{A_i\}$ of the elements of V, we can obtain the following properties:

$$(A \sqcup B)_\alpha = A_\alpha \vee B_\alpha,$$
$$\Big(\sqcup_{i\in I} A_i\Big)_\alpha = \vee_{i\in I} A_{i\alpha},$$

where the operation $\vee$ is interpreted by means of interval analysis as $A_\alpha \vee B_\alpha = \{a \vee b | a \in A_\alpha, b \in B_\alpha\}$. Analogously,
the same properties concerning conjunction $\sqcap$ hold.

t-NORMS AND t-CONORMS OF TYPE 2

Mizumoto et al. (1981) have extended algebraic product and algebraic sum to the case of type 2 and have shown that the set of normal and convex fuzzy truth values forms a lattice-ordered monoid under $\sqcup$, $\sqcap$ and the extended algebraic product. In this section, the authors extend a t-norm and a t-conorm to the case of type 2 according to their approach, and investigate the algebraic properties of type 2 fuzzy logic equipped with $\sqcup$, $\sqcap$ and the extended t-norm.

As it is well known, a t-norm $T:[0,1]^2 \to [0,1]$ is defined as a commutative, associative and non-decreasing binary operation with the unit element 1. On the other hand, a t-conorm $S:[0,1]^2 \to [0,1]$ is defined as that with the unit element 0. Now, the authors give the definitions of a t-norm and a t-conorm of type 2 by using V instead of $[0,1]$.

Definition 1. A t-norm of type 2 $T^*:V^2 \to V$ and a t-conorm of type 2 $S^*:V^2 \to V$ are defined as binary operations satisfying the following properties (T1)-(T4) and (S1)-(S4) for $\forall A,B,C \in V$, respectively:

(T1) unit element: $T^*(A,\mathbf{1}) = A$, (S1) $S^*(A,\mathbf{0}) = A$,

(T2) commutativity: $T^*(A,B) = T^*(B,A)$, (S2) $S^*(A,B) = S^*(B,A)$,

(T3) associativity: $T^*(T^*(A,B),C) = T^*(A,T^*(B,C))$, (S3) $S^*(S^*(A,B),C) = S^*(A,S^*(B,C))$,

(T4) monotonicity: $B \sqsubseteq C \Rightarrow T^*(A,B) \sqsubseteq T^*(A,C)$; (S4) $B \sqsubseteq C \Rightarrow S^*(A,B) \sqsubseteq S^*(A,C)$.

On the other hand, a t-norm $T:[0,1]^2 \to [0,1]$ and a t-conorm $S:[0,1]^2 \to [0,1]$ are extended by using the extension principle to $\tilde{T}:F([0,1])^2 \to F([0,1])$ and $\tilde{S}:F([0,1])^2 \to F([0,1])$, respectively, as follows:

$$f_{\tilde{T}(A,B)}(z) = \sup_{z=T(x,y)} \min\{f_A(x), f_B(y)\},$$
$$f_{\tilde{S}(A,B)}(z) = \sup_{z=S(x,y)} \min\{f_A(x), f_B(y)\} \quad \text{for } \forall A,B \in F([0,1]).$$

Theorem 2. Let A and B be normal, convex and USC fuzzy truth values i.e. $A,B \in V$, and a t-norm T and a t-conorm S be continuous. Then, $\tilde{T}(A,B)$ and $\tilde{S}(A,B)$ are also normal, convex and USC.

Proof. It is easy to verify the normality and the convexity. Since T is continuous and $A,B \in V$, we can apply the result of Nguyen (1979) and obtain $\left(\tilde{T}(A,B)\right)_\alpha = T(A_\alpha, B_\alpha)$, which means $\left(\tilde{T}(A,B)\right)_\alpha$ is a closed interval for $\forall \alpha \in [0,1]$. We can obtain the same result concerning $\tilde{S}(A,B)$. Therefore, $\tilde{T}(A,B)$ and $\tilde{S}(A,B)$ are USC. *Q.E.D.*

Theorem 3. Let a t-norm T and a t-conorm S be continuous. Then, $\tilde{T}:V^2 \to V$ is a t-norm of type 2, and $\tilde{S}:V^2 \to V$ is a t-conorm of type 2.

Proof. The existence of the unit element, commutativity, associativity and monotonicity of $\tilde{T}$ can be proved as follows:

(T1)
$$\begin{aligned} f_{\tilde{T}(A,\mathbf{1})}(z) &= \sup_{z=T(x,y)} \min\{f_A(x), f_{\mathbf{1}}(y)\} \\ &= \sup_{z=T(x,1)=x} f_A(x) = f_A(z) \qquad \therefore \tilde{T}(A,\mathbf{1}) = A, \end{aligned}$$

(T2)

$$\begin{aligned} f_{\tilde{T}(A,B)}(z) &= \sup_{z=T(x,y)} \min\{ f_A(x),\ f_B(y)\} \\ &= \sup_{z=T(y,x)} \min\{ f_B(y),\ f_A(x)\} \\ &= f_{\tilde{T}(B,A)}(z) \qquad\qquad \therefore \tilde{T}(A,B)=\tilde{T}(B,A), \end{aligned}$$

(T3)

$$\begin{aligned} f_{\tilde{T}(\tilde{T}(A,B),C)}(x') &= \sup_{x'=T(z',z)} \min\left\{ \sup_{z'=T(x,y)} \min\{ f_A(x),\ f_B(y)\},\ f_C(z)\right\} \\ &= \sup_{x'=T(T(x,y),z)} \min\{\min\{ f_A(x),\ f_B(y)\},\ f_C(z)\} \\ &= \sup_{x'=T(x,T(y,z))} \min\{ f_A(x), \min\{ f_B(y),\ f_C(z)\}\} \\ &= \sup_{x'=T(x,z')} \min\left\{ f_A(x),\ \sup_{z'=T(y,z)} \min\{ f_B(y),\ f_C(z)\}\right\} \\ &= f_{\tilde{T}(A,\tilde{T}(B,C))}(x') \qquad \therefore \tilde{T}(\tilde{T}(A,B),C)=\tilde{T}(A,\tilde{T}(B,C)). \end{aligned}$$

(T4) The distributivity of T on $\vee$ i.e. $T(a,b)\vee T(a,c) = T(a,b\vee c)$ $(\forall a,b,c \in [0,1])$ can be extended to the case of interval analysis as

$$\begin{aligned} T([a_1,a_2],[b_1,b_2])\vee T([a_1,a_2],[c_1,c_2]) &= [T(a_1,b_1),\ T(a_2,b_2)]\vee[T(a_1,c_1),\ T(a_2,c_2)] \\ &= [T(a_1,b_1)\vee T(a_1,c_1),\ T(a_2,b_2)\vee T(a_2,c_2)] \\ &= [T(a_1,b_1\vee c_1),\ T(a_2,b_2\vee c_2)] \\ &= T([a_1,a_2],[b_1\vee c_1,b_2\vee c_2]) \\ &= T([a_1,a_2],[b_1,b_2]\vee[c_1,c_2]). \end{aligned}$$

Assuming $B \sqsubseteq C$ i.e. $B \sqcup C = C$, we get for any $\alpha \in [0,1]$

$$\begin{aligned} \left(\tilde{T}(A,B)\sqcup\tilde{T}(A,C)\right)_\alpha &= T(A_\alpha,B_\alpha)\vee T(A_\alpha,C_\alpha) \\ &= T(A_\alpha,\ B_\alpha\vee C_\alpha) \\ &= \left(\tilde{T}(A,\ B\sqcup C)\right)_\alpha \\ &= \left(\tilde{T}(A,C)\right)_\alpha \qquad\qquad \therefore B\sqsubseteq C \Rightarrow \tilde{T}(A,B)\sqsubseteq\tilde{T}(A,C). \end{aligned}$$

In the same way as the case of $\tilde{T}$, we can obtain $\tilde{S}(A,\mathbf{0})=A$, $\tilde{S}(A,B)=\tilde{S}(B,A)$, $\tilde{S}(\tilde{S}(A,B),C)=\tilde{S}(A,\tilde{S}(B,C))$, $B\sqsubseteq C\Rightarrow\tilde{S}(A,B)\sqsubseteq\tilde{S}(A,C)$. *Q.E.D.*

Theorem 4. Let a t-norm T be continuous. Then, the algebraic system $(V,\sqcup,\sqcap,\tilde{T})$ forms a complete lattice-ordered monoid.

Proof. $(V,\sqcup,\sqcap)$ forms a complete distributive lattice (Theorem 1) and $\tilde{T}$ is a commutative monoid in V (Theorem 3). Thus, it is enough to show the infinite distributive law:

$$\sqcup_{i\in I}\tilde{T}(A,B_i)=\tilde{T}\left(A,\ \sqcup_{i\in I} B_i\right)$$

for any $A \in V$ and any family $\{B_i\}_{i \in I}$ of the elements of V. Since T is continuous, $\underset{i \in I}{\vee} T(a, b_i) = T\left(a, \underset{i \in I}{\vee} b_i\right)$ holds for any $a \in [0,1]$ and any family $\{b_i\}_{i \in I}$ of the elements of $[0,1]$. Then, we obtain

$$\left[\underset{i \in I}{\sqcup} \tilde{T}(A, B_i)\right]_\alpha = \underset{i \in I}{\vee} T(A_\alpha, B_{i\alpha}) = T\left(A_\alpha, \underset{i \in I}{\vee} B_{i\alpha}\right).$$

Therefore, we have $\underset{i \in I}{\sqcup} \tilde{T}(A, B_i) = \tilde{T}\left(A, \underset{i \in I}{\sqcup} B_i\right)$. *Q.E.D.*

Definition 2. An implication of type 2 is defined as the right residual of a t-norm of type 2 $T^*: V^2 \to V$ and denoted by $I(T^*): V^2 \to V$, that is,

$$I(T^*)(A, B) \overset{def.}{=} \sqcup\left\{X \in V \middle| T^*(A, X) \sqsubseteq B\right\} \quad \text{for } A, B \in V.$$

As a corollary of Theorem 4, we can obtain the following residuation (Izumi et al.(1983) called it ‘adjointness’) between a t-norm of type 2 $\tilde{T}: V^2 \to V$ and its right residual $I(\tilde{T})$:

$$\tilde{T}(A, B) \sqsubseteq C \quad \Leftrightarrow \quad B \sqsubseteq I(\tilde{T})(A, C)$$

for any continuous t-norm T.

Remark 1. Let $I(T)$ denote the right residual of a continuous t-norm T. $I(T): [0,1]^2 \to [0,1]$ can be extended by means of the extension principle to a binary operation $\tilde{I}(T)$. It should be noted that $\tilde{I}(T)$ does not coincide with $I(\tilde{T})$.

A STRONG NEGATION AND DE MORGAN TRIPLET OF TYPE 2

In this section, let us cope with the extended operation of a strong negation $n: [0,1] \to [0,1]$ which is an involutive and order-reversing operation satisfying $n(1) = 0$. The authors define a strong negation of type 2.

Definition 3. A strong negation of type 2 $n^*: V \to V$ is defined as an operation satisfying the following properties:

(N1) $n^*(\mathbf{1}) = \mathbf{0}$,

(N2) $n^*(n^*(A)) = A$ for $\forall A \in V$,

(N3) $A \sqsubseteq B \to n^*(B) \sqsubseteq n^*(A)$.

In the same manner as the cases of a t-norm and a t-conorm, a strong negation $n: [0,1] \to [0,1]$ is also extended to an operation $\tilde{n}: F([0,1])^2 \to F([0,1])$ as

$$f_{\tilde{n}(A)}(z) = \sup_{z = n(x)} f_A(x) \qquad \text{for } \forall A \in F([0,1]).$$

Theorem 5. If $A \in V$, then $\tilde{n}(A) \in V$.

Theorem 6. An extended negation $\tilde{n}: V \to V$ is a negation of type 2.

Proof.

(N1)

$$\begin{aligned} f_{\tilde{n}(\mathbf{1})}(z) &= \sup_{z=n(x)} f_{\mathbf{1}}(x) \\ &= \sup_{z=n(x)} \begin{cases} 1 & (x=1) \\ 0 & (\text{otherwise}) \end{cases} \\ &= \begin{cases} 1 & (z=n(1)=0) \\ 0 & (\text{otherwise}) \end{cases} \\ &= f_{\mathbf{0}}(z) \qquad \therefore \tilde{n}(\mathbf{1}) = \mathbf{0} \end{aligned}$$

(N2)

$$\begin{aligned} f_{\tilde{n}(\tilde{n}(A))}(z) &= \sup_{z=n(n(x))=x} f_A(x) \\ &= f_A(z) \qquad \therefore \tilde{n}(\tilde{n}(A)) = A \end{aligned}$$

(N3)

$$\begin{aligned} f_{\tilde{n}(A)\sqcup\tilde{n}(B)}(z) &= \sup_{z=x\vee y} \min\left\{ \sup_{x=n(x')} f_A(x'),\ \sup_{y=n(y')} f_B(y') \right\} \\ &= \sup_{\substack{z=n(x')\vee n(y') \\ =n(x'\wedge y')}} \min\{ f_A(x'),\ f_B(y') \} \\ &= f_{\tilde{n}(A\sqcap B)}(z) \end{aligned}$$

If $A \sqsubseteq B$ i.e. $A \sqcap B = A$, then we have $\tilde{n}(A) \sqcup \tilde{n}(B) = \tilde{n}(A)$. Therefore, $A \sqsubseteq B \to \tilde{n}(B) \sqsubseteq \tilde{n}(A)$. *Q.E.D.*

Theorem 7. Let a t-norm T be continuous. For a t-norm of type 2 $\tilde{T}: V^2 \to V$ and a negation of type 2 $\tilde{n}: V \to V$, $\tilde{n}\left(\tilde{T}(\tilde{n}(A), \tilde{n}(B))\right)$ is a t-conorm in type 2.

Proof. It is clear from Theorem 2 and Theorem 5 that $\tilde{n}\left(\tilde{T}(\tilde{n}(A), \tilde{n}(B))\right) \in V$ for $\forall A, B \in V$. Let S_{n-T} be the n-dual t-conorm of a t-norm T i.e. $S_{n-T}(x, y) = n(T(n(x), n(y)))$. Then, we have

$$\begin{aligned} f_{\tilde{n}(\tilde{T}(\tilde{n}(A),\, \tilde{n}(B)))}(z) &= \sup_{z=n(T(n(x),\, n(y)))} \min\{ f_A(x),\ f_B(y) \} \\ &= \sup_{z=S_{n-T}(x,y)} \min\{ f_A(x),\ f_B(y) \}. \end{aligned}$$

That is, $\tilde{n}\left(\tilde{T}(\tilde{n}(A), \tilde{n}(B))\right)$ coincides with the extended operation of $S_{n-T}: [0,1]^2 \to [0,1]$. Therefore, from Theorem 3, $\tilde{n}\left(\tilde{T}(\tilde{n}(A), \tilde{n}(B))\right)$ is a t-conorm of type 2. *Q.E.D.*

Remark 2. Let $\tilde{S}_{n-T}$ denote the extended operation of the n-dual t-conorm S_{n-T} of a continuous t-norm T i.e. $f_{\tilde{S}_{n-T}(A,B)}(z) = \sup_{z=S_{n-T}(x,y)} \min\{ f_A(x),\ f_B(y) \}$, and $S_{\tilde{n}-\tilde{T}}$ denote the $\tilde{n}$-dual t-conorm of a t-norm $\tilde{T}$ of type 2 i.e. $S_{\tilde{n}-\tilde{T}}(A, B) = \tilde{n}\left(\tilde{T}\left(\tilde{n}(A), \tilde{n}(\tilde{B})\right)\right)$. The proof of Theorem 7 suggests that $\tilde{S}_{n-T} = S_{\tilde{n}-\tilde{T}}$ for a continuous t-norm T. Clearly, a set $\{\tilde{T}, \tilde{S}_{n-T}, \tilde{n}\}$ can be regarded as De Morgan triplet of type 2 in the same way as a set $\{T, S_{n-T}, n\}$.

CONCLUDING REMARKS

Consequently, the authors have established type 2 fuzzy logic $\left(V, \sqcup, \sqcap, \tilde{T}, \tilde{S}_{n-T}, I\left(\tilde{T}\right), \tilde{n}\right)$ as a model of L-fuzzy logic under the restriction that the operations of type 2 $\tilde{T}$, $\tilde{S}_{n-T}$ and $I\left(\tilde{T}\right)$ are derived from a continuous t-norm T of type 1. The assumption through this approach of the normality, convexity and upper semicontinuity of fuzzy truth values is a sufficient condition for the convenient properties as for the completeness, etc. Recently, Emoto et al. (1998) have shown the necessary and sufficient condition for $\left(F([0,1]), \sqcup, \sqcap\right)$ to form De Morgan algebra.

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