Generated t-norms and the Archimedean property

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In this paper, we will discuss some properties of generated (in the sense of Definition 3) t-norms. First, we will investigate the relation between the Archimedean property and the diagonal property of generated t-norms, then we will discuss the relation between the Archimedean property and the continuity of generated t-norms and finally we will solve the problem if there is non-continuous generated t-norm with a continuous diagonal. Note that there are many non-continuous generated functions on $[0,1]^2$ with a continuous diagonal. However, as we will show in this paper, for a generated t-norm T its continuity is equivalent with the continuity of its diagonal. Let us recall some definitions.

Definition 1. A function $T : [0,1]^2 \to [0,1]$ which $\forall x, y, z \in [0,1]$ fulfils:

$$\begin{array}{ll} T(T(x,y),z) = T(x,T(y,z)) & \text{(associativity)} \\ x \leq z \Rightarrow T(x,y) \leq T(z,y) & \text{(monotonicity)} \\ T(x,y) = T(y,x) & \text{(commutativity)} \\ T(x,1) = x & \text{(boundary condition)} \end{array}$$

is called a *triangular norm* (a *t-norm* for short).

Definition 2. Let $f : [0,1] \to [0,\infty]$ be a non-increasing function. Then the function $f^{(-1)} : [0,\infty] \to [0,1]$ defined by

$$f^{(-1)}(y) = \sup\{x \in [0,1] \mid f(x) > y\}$$

is called the *pseudo-inverse of function f*.

Definition 3. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function, $f(1) = 0$ and let the function $T : [0,1]^2 \to [0,1]$ be given by formula

$$T(x,y) = f^{(-1)}(f(x) + f(y)) \quad \forall x, y \in [0,1] \tag{1}$$

where $f^{(-1)}$ is the pseudo-inverse of the function f. Then the function f is called a *conjunctive additive generator of the function* T.

A generated function T always posses the next properties.

Theorem 1. *Let $f : [0,1] \to [0,\infty]$ be a conjunctive additive generator of T. Then the function T is a commutative, non-decreasing and fulfils the boundary condition.*

From Theorem 1 and Definition 1 we immediately obtain that a generated function T is a t-norm if and only if T is associative. Some conditions ensuring the associativity of generated functions can be found, e.g., in [4], [6], [9].

Let T be a t-norm then

$$x_T^{(n)} = \begin{cases} 1 & \text{if } n = 0 \\ T(x_T^{(n-1)}, x) & \text{if } n \in \{1, 2 \ldots\}. \end{cases}$$

Definition 4. Let $T : [0,1]^2 \to [0,1]$ be a t-norm. We will say that

(i) T is *Archimedean* if $\lim_{n\to\infty} x_T^{(n)} = 0$ for all $x \in (0,1)$.

(ii) T has *diagonal property* if $T(x,x) < x$ for all $x \in (0,1)$.

Recall the following well known result, see [2].

Lemma 1. *Let f be a conjunctive additive generator of T. If f is a continuous function on $(0,1]$ then T is a continuous Archimedean t-norm.*

Let

$$\mathcal{F} = \{f : [0,1] \to [0,\infty] \mid f \text{ is strictly decreasing and } f(1) = 0 \},$$

$$\mathcal{T} = \{T : [0,1]^2 \to [0,1] \mid T \text{ is a t-norm and } \exists f \in \mathcal{F} \text{ such that } T \text{ fulfils (1) } \}.$$

Denote

$$\lim_{x\to a_-} f(x) = f(a_-), \quad \lim_{x\to a_+} f(x) = f(a_+),$$

$$D_f = \{a \in (0,1) \mid f(a_-) > f(a_+)\},$$

$$D_f^0 = \{a \in (0,1) \mid a \in D_f \text{ and } 2f(a) \le f(a_-)\},$$

$I_T = \{x \in (0,1) \mid T(x,x) = x\}$ (non-trivial idempotent elements).

We will need the following two lemmas.

Lemma 2. [10] *Let f be some conjunctive additive generator of T.*

(i) T is continuous if and only if f is continuous on (0,1].

(ii) T is continuous at point (1,1) if and only if f is left continuous at point 1.

Lemma 3. [10] *Let f be some additive generator of t-norm T. Then $I_T = D_f^0$.*

Remark 1.

(i) We know that there are no continuous non-Archimedean generated t-norms. This assertion is a consequence of Lemma 2 (i) and Lemma 1.

(ii) The examples of non-continuous Archimedean t-norms can be found in [3] (there are examples of non-continuous additive generators with a relatively closed range under the addition which means that the corresponding generated functions are non-continuous Archimedean t-norms).

(iii) The examples of non-continuous non-Archimedean generated t-norms can be found in [8], [9].

We know, that the Archimedean property of t-norm T implies the diagonal property of t-norm T and that the reverse is not true in general. The following Theorem 2 discusses the case of generated t-norms.

Theorem 2. *Let $T \in \mathcal{T}$. Then T is Archimedean if and only if $T(x,x) < x$ for all $x \in (0,1)$.*

Now, we know, that the Archimedean proprety and the diagonal property are equivalent in the set of all generated t-norms. Using Lemma 3, we can reformulate assertion of Theorem 2 in the following way.

Corollary 1. *Let f be a conjunctive additive generator of t-norm T. Then T is Archimedean if and only if $D_f^0 = \emptyset$.*

Theorem 3. *Let f be a conjunctive additive generator of T, $f(1_-) = 0$ and $D_f^0 = \emptyset$. If T is a t-norm then $D_f = \emptyset$.*

This important result makes possible to formulate the following claims.

Theorem 4. *Let $T \in \mathcal{T}$. Then the next assertions are equivalent:*

(i) T is continuous,

(ii) T is Archimedean and continuous at point (1,1),

(iii) $T(x, x) < x$ for all $x \in (0, 1)$ and T is continuous at point (1,1).

Proof First, the equivalence of an assertions (ii) and (iii) is a consequense of Theorem 2.

If T is a continuous generated t-norm then Lemma 2 (i) and Lemma 1 imply that T is an Archimedean t-norm. It means that (i) implies (ii).

Now we will prove that (ii) implies (i). If T is an Archimedean generated t-norm which is continuous at point (1,1), then there exists some conjunctive additive generator f of T and the Lemma 2 (ii) and Corollary 1 imply that $f(1_-) = 0$ and $D_f^0 = \emptyset$, respectively. Because of Theorem 3 we have that $D_f = \emptyset$. Thus f is continuous on $(0, 1]$ and the assertion (i) of Lemma 2 implies the continuity of T. We have just proved that (i) and (ii) are equivalent and the proof is complete.

We know that the continuity of a generated t-norm T implies the continuity of the diagonal of T. The question is if the reverse is true. The answer to this question is given in Theorem 5. Let $T : [0, 1]^2 \to [0, 1]$ and $\Delta = \{(x, x) \mid x \in [0, 1]\}$. Then $T_\Delta : [0, 1] \to [0, 1]$ defined by $T_\Delta(x) = T(x, x)$ is a function of one variable. Now, we will study the relation between continuity of generated t-norm T and continuity of T_Δ.

Theorem 5. *Let $T \in \mathcal{T}$. Then $T_\Delta(x) : [0, 1] \to [0, 1]$ is continuous if and only if T is continuous.*

The Theorem 5 shows that there is no non-continuous generated t-norm with a continuous diagonal.

Corollary 2. *Let $T \in \mathcal{T}$. If T_Δ is continuous then T is a continuous Archimedean t-norm.*

Remark 2.

(i) Recall the famous open problem of Schweizer and Sklar [6] whether the continuity of the T_Δ of an Archimedean t-norm T implies the continuity of T. Due to Corollary 2, the answer would be affirmative if each Archimedean t-norm is generated. However, it is still possible that there are non-generated Archimedean t-norms leading to a negative answer of above mentioned problem.

(ii) As an example of a non-continuous generated function T with a continuous diagonal T_Δ, put

$$f(x) = \begin{cases} \frac{3}{2} - x & \text{if } x \in [0, \frac{1}{4}), \\ 1 - x & \text{if } x \in [\frac{1}{4}, 1]. \end{cases}$$

Then T_Δ is continuous and T is non-continuous. However, T is not a t-norm.

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