LIMIT THEOREMS FOR QUASI-ARITHMETIC MEANS [1]

ANNA KOLESÁROVÁ

Faculty of Mechanical Engineering, Slovak Technical University,
812 31 Bratislava, SK
kolesarova@dekan.sjf.stuba.sk

Abstract. The limit properties of a class $\{M_{g^\lambda}\}_{\lambda\in(0,\infty)}$ of all quasi-arithmetic means generated by λ-powers of a generator g are studied. We work with special types of generators that uniquely correspond to the additive generators of continuous Archimedean t-norms or t-conorms. It is shown that for $\lambda \to \infty$, the situation is similar to that for t-norms and t-conorms. For $\lambda \to 0^+$, the limit operators are quasi-geometric means. Moreover, we also investigate the limit properties of a class $\{M_{g_\alpha}\}_{\alpha\in(0,\infty)}$ of all quasi-arithmetic means generated by functions g_α, $g_\alpha(x) = g(x^\alpha)$.

Keywords. Aggregation operator, arithmetic mean, quasi-arithmetic mean, triangular norm.

1 Introduction

A quasi-arithmetic mean is defined as a transformation of the standard arithmetic mean $M : \bigcup\limits_{n\in N} \bar{R}^n \to \bar{R},\ M(x_1, ..., x_n) = \frac{1}{n}\sum_{i=1}^n x_i$.
Without any loss of generality we restrict ourselves to the operators with input values from the interval $[0, 1]$.
Let $f : [0, 1] \to [-\infty, \infty]$ be a continuous strictly monotone function such that $Ran\ f \neq [-\infty, \infty]$. Let f^{-1} be the inverse function of f.
An operator $M_f : \bigcup\limits_{n\in N} [0, 1]^n \to [0, 1]$ is a quasi-arithmetic mean if it is defined by

$$M_f(x_1, .., x_n) = f^{-1}\left(M\left(f(x_1), ..., f(x_n)\right)\right). \tag{1}$$

Eq.(1) can also be written in the form $M_f(x_1, .., x_n) = f^{-1}\left(\frac{1}{n}\sum\limits_{i=1}^n f(x_i)\right)$. We call f the generator of the quasi-arithmetic mean M_f.

[1]The research was supported by the grants VEGA 1/4297/97 and 2/6087/99.

It can be easily shown that for each a, $b \in R$, $a \neq 0$ the property $M_{af+b} = M_f$ holds. Therefore, for any quasi-arithmetic mean M_f there exists a continuous strictly monotone function $g : [0,1] \rightarrow [0,\infty]$ such that $g(\frac{1}{2}) = 1$, $0 \in Ran\ g$ and $M_g = M_f$, which means that the class of all quasi-arithmetic means can be characterized by means of non-negative generators crossing the point $(\frac{1}{2}, 1)$ and having 0 in their range. From the properties of g it follows that $g^{-1}(0) \in \{0,1\}$.

The set of all mentioned functions g will be denoted by $\mathcal{G}$.

The property $g(\frac{1}{2}) = 1$ could be dropped, we will use these generators only because of uniqeness.

Definition 1. An element $a \in [0,1]$ is said to be an annihilator of an aggregation operator A if $a \in \{x_1, ..., x_n\}$ implies $A(x_1, ..., x_n) = a$.

It is clear that any aggregation operator has at most one annihilator. If M_f is a quasi-arithmetic mean with generator f, then either M_f has no annihilator (if $Ran\ f \subset R$) or $a \in \{0,1\}$, see [6].
For any quasi-arithmetic mean the following properties can be easily proved.

1. If a quasi-arithmetic mean M_f has no annihilator then there exist just two generators g^+, $g^- \in \mathcal{G}$ such that g^+ is an increasing function, g^- decreasing and $M_f = M_{g+} = M_{g-}$. Moreover, for g^+ and g^- we have
$$g^-(x) = \frac{g^+(1) - g^+(x)}{g^+(1) - 1} \quad \textit{and} \quad g^+(x) = \frac{g^-(0) - g^-(x)}{g^-(0) - 1}.$$
2. If a quasi-arithmetic mean M_f has an annihilator $a = 1$, then there exists a unique generator $g^+ \in \mathcal{G}$ that is an increasing bijection ($g^+(1) = \infty$) and $M_f = M_{g^+}$.
3. If a quasi-arithmetic mean M_f has an annihilator $a = 0$, then there exists a unique generator $g^- \in \mathcal{G}$ that is a decreasing bijection ($g^-(0) = \infty$) and $M_f = M_{g^-}$.

The set $\mathcal{G}$ is exactly the set of additive generators of continuous Archimedean triangular norms and conorms (t-norms and t-conorms for short). There exist one-to-one correspondeces between continuous Archimedean t-norms and decreasing generators $g^- \in \mathcal{G}$ and between continuous Archimedean t-conorms and increasing generators $g^+ \in \mathcal{G}$ [3, 7, 10].

2 Limit theorems

Let $\lambda \in (0, \infty)$ be an arbitrary constant and T (S) be an arbitrary t-norm (t-conorm) generated by an additive generator g. Let us denote by T_λ (S_λ) a t-norm (t-conorm) generated by g^λ. It is known [5] that

$$\lim_{\lambda \to \infty} T_\lambda = T_M \quad \textit{and} \quad \lim_{\lambda \to \infty} S_\lambda = S_M,$$

where $T_M(x, y) = \min(x, y)$ and $S_M(x, y) = \max(x, y)$, $(x, y) \in [0, 1]^2$ and both convergences are uniform.
Next,

$$\lim_{\lambda \to 0^+} T_\lambda = T_W \quad \textit{and} \quad \lim_{\lambda \to 0^+} S_\lambda = S_W,$$

where T_W and S_W are the drastic product and drastic sum. Note that the limit functions are also t-norms (t-conorms) and the results do not depend on the generator g.

Consider a function $g \in \mathcal{G}$ and a constant $\lambda \in (0, \infty)$. Then $g^\lambda \in \mathcal{G}$ and for the class $\{M_{g^\lambda}\}$ of all quasi-arithmetic means generated by λ-powers of a generator g, the following theorem can be proved.

Theorem 1. The class of operators $\{M_{g^\lambda}\}_{\lambda \in (0,\infty)}$ is monotone for each $g \in \mathcal{G}$. If g is an increasing generator then $M_{g^\lambda} \not\leq M_{g^\mu}$ if and only if $\lambda < \mu$ and if g is a decreasing generator then $M_{g^\lambda} \not\leq M_{g^\mu}$ if and only if $\lambda > \mu$.

In other words, the type of the monotonicity in parameter of the class $\{M_{g^\lambda}\}_{\lambda \in (0,\infty)}$ is the same as the type of the monotonicity of the generator g.

In the next part we prove the limit theorems for quasi-arithmetic means. To this end, we need the followig lemmas.

Lemma 1. Let $u_1, ..., u_n \in [0, \infty]$. Then

$$\lim_{\lambda \to \infty} \left(\frac{1}{n} \sum_{i=1}^{n} u_i^\lambda \right)^{1/\lambda} = \max_{1 \leq i \leq n} u_i.$$

Lemma 2. Let $u_1, ..., u_n \in [0, \infty]$. Then

$$\lim_{\lambda \to 0^+} \left(\frac{1}{n} \sum_{i=1}^{n} u_i^\lambda \right)^{1/\lambda} = \sqrt[n]{\prod_{i=1}^{n} u_i} = G(u_1, ..., u_n),$$

using the convention $0.\infty = \infty$, where G is the geometric mean.

Theorem 2. Let $g^+ \in \mathcal{G}$ be an increasing generator of a quasi-arithmetic mean M_{g^+}. Then

$$\lim_{\lambda\to\infty} M_{(g^+)^\lambda} = Max \qquad and \qquad \lim_{\lambda\to 0^+} M_{(g^+)^\lambda} = G_{g^+},$$

where

$$G_{g^+}(x_1, ..., x_n) = (g^+)^{-1}\left(\sqrt[n]{\prod_{i=1}^{n} g^+(x_i)}\right) = (g^+)^{-1}\left(G\left(g^+(x_1), ..., g^+(x_n)\right)\right).$$

Proof. The first part of the assertion follows from Lemma 1, the second can be proved by Lemma 2.

Theorem 3. Let $g^- \in \mathcal{G}$ be a decreasing generator of a quasi-arithmetic mean M_{g^-}. Then

$$\lim_{\lambda\to\infty} M_{(g^-)^\lambda} = Min \qquad \text{and} \qquad \lim_{\lambda\to 0^+} M_{(g^-)^\lambda} = G_{g^-},$$

where G_{g^-} is an operator defined in the same way as G_{g^+} in Theorem 2.

The limit operators G_{g^+}, G_{g^-} will be called quasi-geometric means.

As we can see, if $\lambda \to \infty$, the situation for quasi-arithmetic means is similar to that for t-norms and t-conorms. The limit operator is always the same, independent of the starting generator $g \in \mathcal{G}$ (up to the monotonicity). However, Max and Min are not quasi-arithmetic means. Since all considered operators are continuous and monotone, the convergences are for any $g \in \mathcal{G}$ uniform. Hence the operators Max and Min can be uniformly approximated by means of quasi-arithmetic means. Note that for any M_g we have $Min \nleq M_g \nleq Max$.

The situation for $\lambda \to 0^+$ is completely different from that for t-norms and t-conorms. The limit operator is dependent on the generator g. If $Ran\ g \neq [0, \infty]$, the convergences are also uniform. However, if $Ran\ g = [0, \infty]$, the uniform convergence is violated.

The limit operators G_g are quasi-arithmetic means only if $\sup\ g(x) = u < \infty$. In that case $G_g = M_h$, where $h = \log g$ (or any linear transformation of $\log g$). If $Ran\ g = [0, \infty]$, then G_g is a generalized quasi-arithmetic mean generated by the bijection $h\ :\ [0, 1] \to [-\infty, \infty]$, $h = -\log g$, if the conventions $\infty.0 = \infty$ and $-\infty + \infty = -\infty$ are used. The corresponding generalized quasi-arithmetic mean M_h has an annihilator (0 or 1) and also a weak annihilator (1 or 0), see [6].

Example 1. (i) The standard arithmetic mean is generated by the generator $g^+(x) = 2x$ as well as by $g^-(x) = 2 - 2x$ (if we use generators from the class $\mathcal{G}$).
For instance, $M_{(g^+)^2}$ is the quadratic mean and, by Theorem 2, $\lim\limits_{\lambda\to 0^+} M_{(g^+)^\lambda}$ is exactly the standard geometric mean G.
On the other hand, by Theorem 3, $\lim\limits_{\lambda\to 0^+} M_{(g^-)^\lambda} = G_{g^-}$, where

$$\begin{aligned} G_{g^-}(x_1,\ldots,x_n) &= 1 - \sqrt[n]{\prod_{i=1}^{n}(1-x_i)} = 1 - G(1-x_1,\ldots,1-x_n) = \\ &= G^d(x_1,\ldots,x_n), \end{aligned}$$

i.e., G_{g^-} is the dual operator G^d of the geometric mean G.

(ii) The harmonic mean $H(x_1,...,x_n) = \left(\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{x_i}\right)^{-1}$ is generated by the decreasing function $g^-(x) = \frac{1-x}{x}$, i.e., $H = M_{g^-}$. By Theorem 3, $\lim\limits_{\lambda\to 0^+} M_{(g^-)^\lambda} = G_{g^-}$, where

$$\begin{aligned} G_{g^-}(x_1,\ldots,x_n) &= \frac{\sqrt[n]{\prod\limits_{i=1}^{n} x_i}}{\sqrt[n]{\prod\limits_{i=1}^{n} x_i} + \sqrt[n]{\prod\limits_{i=1}^{n}(1-x_i)}} = \\ &= \frac{G(x_1,\ldots,x_n)}{G(x_1,\ldots,x_n) + G(1-x_1,\ldots,1-x_n)}, \end{aligned}$$

whereby $\frac{0}{0} = 0$.

Remark 1. The operator G_{g^-} from Example 1, (ii), is a generalized quasi-arithmetic mean generated by the bijection $h : [0,1] \to [-\infty,\infty]$, $h(x) = -\log\ g^-(x) = \log\frac{x}{1-x}$, which is an additive generator of an associative compensatory operator C introduced in Klement et al. [5], see also [6] .
Note that the operator G_{g^-}, $G_{g^-}(x_1,x_2) = \frac{\sqrt{x_1x_2}}{\sqrt{x_1x_2}+\sqrt{(1-x_1)(1-x_2)}}$ is not continuous in the points $(0,1)$ and $(1,0)$.

Finally, we formulate two limit theorems for quasi-arithmetic means M_{g_α} generated by functions $g_\alpha : [0,1] \to [0,\infty]$, $g_\alpha(x) = g(x^\alpha)$, where $g \in \mathcal{G}$ and $\alpha \in (0,\infty)$.

Theorem 4. Let $g \in \mathcal{G}$ be a function with the derivative $g'(1^-) \in R - \{0\}$. Then

$$\lim_{\alpha \to 0^+} M_{g_\alpha} = G$$

independently of the generator g.

Theorem 5. Let $g \in \mathcal{G}$ be a function with the derivative $g'(0^+) \in R - \{0\}$. Then

$$\lim_{\alpha \to +\infty} M_{g_\alpha} = Max$$

independently of the generator g.

Note that the assumptions in Theorems 4 and 5 concerning the derivatives are important and cannot be dropped in general.

Remark 2 The known root-power operators that are quasi-arithmetic means generated by functions $f_\alpha(x) = x^\alpha$, $\alpha \in R - \{0\}$, i.e., defined by

$$M_\alpha(x_1, ...x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^\alpha\right)^{1/\alpha}$$

can be seen for $\alpha \in R^+$ as operators studied in Theorems 4 and 5, for $g(x) = x$. The obtained general results agree with known results for root-power operators, compare for instance with [2].

3 Conclusion

We have studied the limit properties for quasi-arithmetic means that are generated by λ-powers , $\lambda \in (0, \infty)$, of a given genarator g. The limit operators for $\lambda \to \infty$ are Max or Min and depend only on the monotonicity of a generator g. However, the limit operators for $\lambda \to 0^+$ are g-transformations of the standard geometric mean (Theorems 2 and 3).

For quasi-arithmetic means generated by functions g_α, $\alpha \in (0, \infty)$, $g_\alpha(x) = x^\alpha$, the limit operators (under assumptions given in Theorems 4 and 5) are independent of the starting generator g. Namely , for $\alpha \to 0^+$ the limit operator is the geometric mean and for $\alpha \to \infty$ the operator Max.

References

[1] J.C. Fodor : *On bisymetric operations on the unit interval*, submitted.

[2] J.C. Fodor and M. Roubens : Fuzzy preference modelling and multicriteria decision support, Kluwer Academic Publishers, Dordrecht (1994).

[3] E.P. Klement and R. Mesiar : *Triangular norms*, Tatra Mountains Math. Publ., vol.13 (1997) 169-194.

[4] E.P. Klement, R. Mesiar and E. Pap : *On the relationship of associative compensatory operators to triangular norms and conorms*, J. Uncertainty, Fuzziness and Knowledge-Based Systems (1996) 129-144.

[5] E.P. Klement, R. Mesiar and E. Pap : Triangular norms, monograph in preparation.

[6] A.Kolesárová : *Comparison of quasi-arithmetic means*, Proceedings EUROFUSE SIC'99, Budapest, to appear.

[7] C.M. Ling : *Representation of associative functions*, Publ. Math. Debrecen, vol.12 (1965) 189-212.

[8] R. Mesiar and M. Komorníková : *Aggregation operators*, Proceedings XI. Conference on Applied Mathematics, eds. D. Herceg, K. Surla, Institute of Mathematics, Novi Sad, 193-211.

[9] R. Mesiar and M. Komorníková :*Triangular norm-based aggregation of evidence under fuzziness*, Aggregation and Fusion of Imperfect Information, ed. Bernardette Bouchon-Meunier, Physica-Verlag (1997) 11-35.

[10] B. Schweizer and A. Sklar : Probabilistic Metric Spaces, North-Holland, Amsterdam (1983).