ON SOME PROPERTIES OF S-DECOMPOSABLE MEASURES

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S-decomposable measures

As it is very well known by a triangular conorm S (t-conorm briefly) a function $S\colon [0,1]^2 \to [0,1]$ is understood such that it is commutative, associative, monotone and such that $S(x,0)=x$ for every x.

Let us mention some simple examples:

$$\begin{aligned} S_M(x,y) &= \max(x,y)\,, \\ S_P(x,y) &= x+y-xy\,, \\ S_L(x,y) &= \min(x+y,1)\,. \end{aligned}$$

An S-decomposable measure (with respect to a t-conorm S) is a function $\mu\colon \mathcal{R} \to [0,1]$ ($\mathcal{R}$ being an arbitrary ring of sets) such that $\mu(\emptyset)=0$ and

$$\mu(A\cup B) = S(\mu(A),\mu(B)),$$

whenever $A,B\in\mathcal{R}$, $A\cup B\in\mathcal{R}$ and $A\cap B=\emptyset$. The function is σ-S-decomposable, if

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mathop{\mathrm{S}}_{i=1}^{n} \mu(A_i)\,.$$

whenever A_i are pairwise disjoint sets from $\mathcal{R}$ and $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{R}$.

It is known ([3], Remark 11.11, [4], Remark 2) that if S is left continuous then μ is σ-S-decomposable if and only if μ is S-decomposable and continuous from below. Moreover, the following assertion holds.

Proposition. *If S is a continuous t-conorm, μ is S-decomposable measure on a ring and ($A_n \searrow \emptyset \Rightarrow \mu(A_n) \searrow 0$), then μ is continuous from below and continuous from above.*

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Alexandrov theorem

We shall consider t-conorms S satisfying the inequality $S \leq S_L$, where S_L is the Lukasiewicz conorm, i.e., $S(x,y) \leq \min(x+y,1)$.

To formulate the Alexandrov theorem we need moreover the following definitions. A family $\mathcal{C}$ of sets is said to be compact, if the following implication holds:

$$(C_n)_{n=1}^{\infty} \subset \mathcal{C}\,,\ \bigcap_{i=1}^{n} C_i \neq \emptyset \qquad \text{for any} \quad n \Rightarrow \bigcap_{i=1}^{\infty} C_i \in \mathcal{C}\,.$$

An S-decomposable measure $\mu\colon \mathcal{R} \to [0,1]$ is said to be compact if there exists a compact family $\mathcal{C}$ such that to any $A \in \mathcal{R}$ and any $\varepsilon > 0$ there exist $C \in \mathcal{C}$ and $B \in \mathcal{R}$ such $B \subset C \subset A$ and

$$\mu(A \setminus B) < \varepsilon\,.$$

Theorem 1. *Let S be a continuous t-conorm, $S \leq S_L$. Let $\mathcal{R}$ be a ring of sets and $\mu\colon \mathcal{R} \to [0,1]$ be a compact S-additive measure. Then μ is continuous from above.*

Proof. Let $A_n \searrow \emptyset$, $A_n \in \mathcal{R}$ $(n = 1,2,\dots)$. Since $(\mu(A_n))_{n=1}^{\infty}$ is decreasing, there exists $a = \lim\limits_{n\to\infty} \mu(A_n)$. We want to prove that $a = 0$. Assume $a > 0$. By the compactness of μ there exist $B_n \in \mathcal{R}$, $C_n \in \mathcal{C}$ such that $B_n \subset C_n \subset A_n$ and

$$\mu(A_n \setminus B_n) < \frac{a}{2^n}\,.$$

Put $D_n = \bigcap\limits_{i=1}^{n} C_i$. We want to prove that $D_n \neq \emptyset$ $(n = 1,2,\dots)$. In the opposite case there exists n such that $D_n = \emptyset$. Then

$$A_n = A_n \setminus D_n = \bigcup_{i=1}^{n} (A_n \setminus C_i) \subset \bigcup_{i=1}^{n} (A_i \setminus B_i)\,.$$

Therefore

$$a \leq \mu(A_n) \leq \mathop{\mathrm{S}}_{i=1}^{n} \mu(A_i \setminus B_i) \leq \sum_{i=1}^{n} \mu(A_i \setminus B_i) < a$$

what is a contradiction.

We have proved that $\bigcap\limits_{i=1}^{n} C_i \neq \emptyset$ for any n. Therefore $\bigcap\limits_{i=1}^{\infty} C_i \neq \emptyset$. Of course, $\bigcap\limits_{i=1}^{\infty} C_i \subset \bigcap_{i=1}^{\infty} A_i$, hence $\bigcap\limits_{i=1}^{\infty} A_i \neq \emptyset$, what is a contradiction. Therefore $0 = a = \lim\limits_{n\to\infty} \mu(A_n)$.

Remark. From the proof of Theorem 1 it is easy to see that instead of the inequality $S \leq S_L$ we can assume the following implication:

$$\forall a > 0 \exists (a_i)_{i=1}^{\infty}\,,\ a_i > 0 \quad \forall n\colon \mathop{\mathrm{S}}_{i=1}^{n} a_i < a\,.$$

Moreover, this property holds for any continuous t-conorm S (see [2]). Therefore we obtain the following generalizatrion of Theorem 1.

Theorem 2. *Let S be a continuous t-conorm . Let $\mathcal{R}$ be a ring of sets and $\mu\colon \mathcal{R} \to [0,1]$ be a compact S-decomposable measure. Then μ is continuous from above.*

Corollary. *Let S be a continuous t-conorm. Let $\mathcal{R}$ be a ring of sets and $\mu\colon \mathcal{R} \to [0,1]$ be a compact S-decomposable measure. Then μ is σ-S-decomposable measure.*

Product of S-decomposable measures

Let $\mu\colon \mathcal{S} \to [0,1]$, $\nu\colon \mathcal{T} \to [0,1]$ be two S-decomposable measures defined on some rings. Of course the product of μ and ν should be considered with respect to a t-norm T. Recall that a t-norm is a mapping $T\colon [0,1]^2 \to [0,1]$ such that it is commutative, associative, monotone and such that $T(x,1) = x$ for any x.

If S is a t-conorm, then by the de Morgan rule a dual t-norm T can be obtained:

$$T(x,y) = S(1-(1-x)\,(1-y)).$$

So we obtain

$$\begin{aligned} T_M(x,y) &= \min(x,y)\,, \\ T_P(x,y) &= xy\,, \\ T_L(x,y) &= \max(x+y-1,0)\,. \end{aligned}$$

Of course we need not a dual pair, but a couple satisfying a distributive law ([3], def. 4.4.1):

T is said to be conditionally distributive over S, if for all $x, y, z \in [0,1]$

$$T(x, S(y,z)) = S(T(x,y), T(x,z))$$

whenever $S(y,z) < 1$.

A simple example of the distributive law are the t-norm T_P and the t-norm S_L. The problem of conditional distributivity for continuous T, S is completely solved in [3] Theorem 4.4.3.

Theorem 3. *Let T be a t-norm conditionally distributed over a t-conorm S. Let $\mu\colon \mathcal{S} \to [0,1)$, $\nu\colon \mathcal{T} \to [0,1)$ be S-decomposable measures defined on the rings $\mathcal{S}$ and $\mathcal{T}$, respectively. Let $\mathcal{R}$ be the ring generated by the family $\{A \times B; A \in \mathcal{S}, B \in \mathcal{T}\}$. Then there exists exactly one S-decomposable measure $\lambda\colon \mathcal{R} \to [0,1]$ such that $\lambda(A \times B) = T(\mu(A), \nu(B))$. If S, T are moreover continuous, μ, ν are continuous from above and $\mathcal{S} = \mathcal{T} = \mathcal{B}(R)$, then λ is continuous from above, too.*

Remark. The conditions $\mu(A) < 1$, $\nu(B) < 1$ for all A and B can be substituted by S-faithfullness of μ and ν (see [1] and [4]).

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