DISCRETE T-NORMS VERSUS DISCRETIZATIONS OF T-NORMS

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Abstract

In this paper, we study the relationships between discrete t-norms, i.e. t-norms on a finite chain, and t-norms on the unit interval. Firstly, we investigate when and how a discrete t-norm can be extended to a (continuous) t-norm on the unit interval. Secondly, we investigate when a discretization of a t-norm on the unit interval yields a discrete t-norm.

Keywords: discretization, finite chain, t-norm.

1 T-norms on the unit interval

Triangular norms (t-norms) have been introduced in the sixties by Schweizer and Sklar [7] as commutative, associative and increasing $[0,1]^2 \to [0,1]$ mappings with neutral element 1. Continuous t-norms have been completely characterized as ordinal sums with continuous Archimedean summands [5, 7]. Recall that continuous Archimedean t-norms are characterized by the diagonal inequality $(\forall x \in]0,1[)(\mathcal{T}(x,x) < x)$.

Continuous Archimedean t-norms are representable by means of additive generators [5]: a t-norm $\mathcal{T}$ is continuous and Archimedean if and only if there exists a continuous, strictly decreasing $[0,1] \to [0,\infty]$ mapping f with $f(1) = 0$ (called an additive generator of $\mathcal{T}$) such that $\mathcal{T}(x,y) = f^{-1}(\min(f(0), f(x) + f(y)))$, for any $(x,y) \in [0,1]^2$. Note that an additive generator f of a continuous Archimedean t-norm $\mathcal{T}$ is unique up to a positive multiplicative constant.

Any continuous Archimedean t-norm $\mathcal{T}$ is either a strict t-norm (i.e. continuous and strictly increasing on $]0,1]^2$) or a nilpotent (i.e. non-strict) t-norm. Strict t-norms are characterized by unbounded additive generators ($f(0) = +\infty$) and are isomorphic to the product t-norm $T_P(x,y) = xy$. On the other hand, nilpotent t-norms are characterized by bounded additive generators ($f(0) < +\infty$) and are isomorphic to the Łukasiewicz t-norm $T_L(x,y) = \max(0, x + y \Leftrightarrow 1)$.

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2 Discrete t-norms

Applications in various fields such as probabilistic metric spaces, fuzzy logic, fuzzy control, generalized measure theory, multi-criteria decision making, etc., have increased the interest in t-norms. Practical applications supported by computer implementations are often based on arguments taken from a finite scale, i.e. a finite subchain $\{x_1, \ldots, x_n\}$ of $[0,1]$, where $x_1 = 0 < x_2 < \ldots < x_n = 1$. Note that any other scale of length n can be transformed into a scale of the above type. Since there is no problem with introducing the concept of a t-norm on an arbitrary bounded poset [3] (for an in-depth study, in particular on product lattices, see [2]), we can introduce t-norms on a finite chain as well.

Definition 1 *Consider a finite chain $C_n = \{x_1, \ldots, x_n\}$, $n \in \mathbb{N}$, with $x_1 < x_2 < \ldots < x_n$. A ${C_n}^2 \to C_n$ mapping $\mathcal{D}$ is called a discrete t-norm (on C_n) if it is commutative, associative, increasing and has x_n as neutral element, i.e. $(\forall i \in \{1, \ldots, n\})(\mathcal{D}(x_i, x_n) = x_i)$.*

All algebraic notions, such as the Archimedean property, strict monotonicity, nilpotent elements, etc., used for t-norms on $[0,1]$ can be introduced for discrete t-norms in a straightforward way. The particular structure of a finite chain, however, leads to additional observations [2]: the Archimedean property is equivalent to the diagonal inequality, there exists no strictly increasing discrete t-norm, etc.

The number of discrete t-norms on a finite chain C_n is known only for $n \le 14$ [1]:

n	number of t-norms	number of smooth t-norms
1	1	1
2	1	1
3	2	2
4	6	4
5	22	8
6	94	16
7	451	32
8	2.386	64
9	13.775	128
10	86.417	256
11	590.489	512
12	4.446.029	1.024
13	37.869.449	2.048
14	382.549.464	4.096

An important subclass is the class of smooth discrete t-norms characterized by Mayor and Torrens as a counterpart of continuous t-norms on $[0,1]$ [6]. In fact, the continuity of a t-norm $\mathcal{T}$ on $[0,1]$ is equivalent with

$$(\forall (x,y) \in [0,1]^2)(x \le y \Leftrightarrow (\exists z \in [0,1])(x = \mathcal{T}(y,z))).$$

However, for a discrete t-norm $\mathcal{D}$ on a finite chain C_n, the above property is equivalent with the following: for any $(i,j) \in \{2, \ldots, n\}^2$ it holds that if $\mathcal{D}(x_i, x_j) = x_r$, then $\mathcal{D}(x_{i-1}, x_j) = x_p$ and $\mathcal{D}(x_i, x_{j-1}) = x_q$ with $r \Leftrightarrow 1 \le p, q \le r$. This property is called the smoothness property by Godo and Sierra [4]; it can be seen as some kind of Lipschitz condition.

The class of smooth discrete t-norms on a finite chain C_n has been characterized completely by Mayor and Torrens [6]. Firstly, there exists a unique smooth Archimedean discrete t-norm D_L

on it, defined by $D_L(x_i, x_j) = x_{\max(1,i+j-n)}$. Secondly, for any given subset I of $\{x_2, \ldots, x_{n-1}\}$, there exists a unique smooth discrete t-norm that has I as set of non-trivial idempotent elements. As a consequence, smooth discrete t-norms show an ordinal sum structure similar to that of continuous t-norms on $[0,1]$. It then also follows that there exist 2^{n-2} smooth discrete t-norms on C_n.

3 Extensions of discrete t-norms

An interesting problem with possible practical consequences is that of the extension of a discrete t-norm $\mathcal{D}$ on a finite subchain $C_n = \{x_1, \ldots, x_n\}$ of $[0,1]$, where $x_1 = 0 < x_2 < \ldots < x_n = 1$, to a t-norm $\mathcal{T}$ on $[0,1]$ such that $\mathcal{T}|_{{C_n}^2} = \mathcal{D}$. The following results hold.

(1) A right-continuous extension of $\mathcal{D}$ is always possible. Indeed, consider the $[0,1] \to C_n$ mapping q defined by $q(x) = \max\{x_i \in C_n \mid x_i \leq x\}$. Then the $[0,1]^2 \to [0,1]$ mapping $\mathcal{T}_\mathcal{D}$ defined by $\mathcal{T}_\mathcal{D}(x,y) = \mathcal{D}(q(x), q(y))$ is a right-continuous extension of $\mathcal{D}$.

(2) A continuous extension of $\mathcal{D}$ is not always possible. In fact, there exist Archimedean discrete t-norms without continuous extension. Hence, Archimedean discrete t-norms are not necessarily representable by means of a continuous additive generator.

(3) If $\mathcal{D}$ is smooth, then it can always be extended to a continuous t-norm $\mathcal{T}$ that has the same idempotent elements as $\mathcal{D}$.

(4) There exist non-smooth discrete t-norms with a continuous extension. This is the case for the weakest discrete t-norm D_W (defined by $D_W(x_i, x_j) = 0$ whenever $\max(x_i, x_j) < 1$).

Obviously, the problem of characterizing all discrete t-norms admitting a continuous extension is still open!

4 Discretizations of t-norms

On the other hand, we may wonder when a discretization of a t-norm on $[0,1]$ yields a discrete t-norm, i.e. given a t-norm $\mathcal{T}$ on $[0,1]$, determine the finite subchains C of $[0,1]$ for which $\mathcal{D} = \mathcal{T}|_{C^2}$ is a discrete t-norm. The main problem is, of course, that the discretization should yield an operation which is internal on the selected subchain. The following results hold.

(1) The discretization works for any finite subchain C if and only if the t-norm $\mathcal{T}$ satisfies $(\forall (x,y) \in [0,1]^2)(\mathcal{T}(x,y) = 0 \vee \mathcal{T}(x,y) = \min(x,y))$, i.e. if and only if there exists a symmetric down-set (order-ideal) K of $([0,1[^2, \leq)$, i.e. $(x,y) \in K \Rightarrow (y,x) \in K$, such that

$$\mathcal{T}(x,y) = \begin{cases} 0 & \text{, if } (x,y) \in K \\ \min(x,y) & \text{, elsewhere} \end{cases} .$$

Well-known examples of such t-norms are T_M (the minimum operator), the weakest t-norm T_W (defined by $T_W(x,y) = 0$ whenever $\max(x,y) < 1$), and the nilpotent minimum T^{nM} defined by

$$T^{nM}(x,y) = \begin{cases} 0 & \text{, if } x + y \leq 1 \\ \min(x,y) & \text{, elsewhere} \end{cases} .$$

(2) For a strict t-norm $\mathcal{T}$, only the trivial subchain $C = \{0, 1\}$ is acceptable and then, of course, $\mathcal{D}$ is nothing else but the Boolean conjunction.

(3) For a nilpotent t-norm $\mathcal{T}$ with additive generator f, it holds that $\mathcal{D}$ is a discrete t-norm if and only if $f(C) = \{f(x) \mid x \in C\}$ is relatively closed under addition, i.e. for any $(x, y) \in C^2$ there either exists $z \in C$ such that $f(x) + f(y) = f(z)$ or $f(x) + f(y) > f(0)$.

(4) Finally, we consider the case of a general continuous t-norm $\mathcal{T}$ with ordinal sum representation $(\langle a_k, b_k, \mathcal{T}_k \rangle)_{k \in K}$. In this case, C can contain any of the idempotent elements of $\mathcal{T}$ and $C_k = C \cap]a_k, b_k[$ can be non-empty if and only if $\mathcal{T}_k$ is nilpotent. In that case, if f_k is an additive generator of $\mathcal{T}_k$, for any $k \in K$, a necessary and sufficient condition is again that $f_k(C_k)$ is relatively closed under addition.

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