Aggregation Operator and Fuzzy Dynamic Equations: Different Dynamics for Different Operators. What is an optimal choice?

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Abstract

A reasonable approximation of the state space for many systems in physics, biology, economy, social and political science, etc. can be considered as a manifold. In the framework of fuzzy logic for an effective description of the state of these systems we have to define a membership function and an aggregation operator that is self-consistent for an any partition of the state space. In our previous works we have used fuzzy state space and possibility density for definition of the membership functions and max s-norm and algebraic sum as the aggregation operators. It was shown, that these s-norms and membership functions lead to different types of dynamics, but the question: "could other s-norm be used and how many types of fuzzy dynamics exist?" was remained open.

In this work we show that a common logic admits only three types of nontrivial fuzzy dynamics. These dynamics are well known in theoretical physics and its cover almost all dynamics equations successfully used in the physics up to the present time.

It is shown that there is a relation between local properties of the chosen s-norm and the admissible form of the membership functions

for a system state. In fact, the fuzzy dynamics deals with s-norm - membership function pairs rather with two separate objects.

A reasonable approximation of the state space for many systems in physics, biology, economy, social and political science, *etc.* can be considered as a manifold. In the framework of fuzzy logic for an effective description of the state of these systems we have to define a membership function and an aggregation operator that is self-consistent for an any partition of the state space. In the previous works (see [1],[2],[3] and the bibliography there) we have used *fuzzy state space* and *possibility density* for definition of the membership functions and *algebraic sum* and *max* s-norm as the aggregation operators. It was shown, that these s-norm and membership functions lead to different types of the dynamics, but the question: "could other s-norm be used and how many types of fuzzy dynamics exist?" was remained open.

In this work we want to show that a common logic admits only three types of nontrivial fuzzy dynamics. We will discuss also of a general condition for s-norms and membership function which leads to these dynamics.

1 σ - decomposable *s-norms - membership function* pair

We will show here that for systems with a manifold as a state space the admissible forms of the membership functions depend on the chosen s-norm. The membership function in such cases must satisfy the following property. For any disjoint partition $P^k = \{V_j^k\}$ with of some domain V in the state space $V = \bigcup\limits_j V_j^k$ the possibility that the system is in V must coincide with the possibility calculated from the possibilities that the system is in any subdomains of P^k (by using given s-norm). Denote by $d\left(V_j^k\right)$ a natural (defined by the system properties) measure and by $\left|V_j^k\right|$ the diameter of V_j^k. For any point x in the manifold and for any partition P^k define $V_{j(x)}^k$ to be the element of P^k containing x.

Define the membership function $\mu\left(V_{j(x)}^k\right)$ as a *possibility* that the system is in $V_{j(x)}^k$ $(0 \leq \mu \leq 1)$. Let S be a σ*-decomposable* s-norm, then for given μ and any partition P^k with $\lim_{k\to\infty} \max\left|V_j^k\right|$ we should have:

$$\mu\left(V\right) = \lim_{k\to\infty} S(\mu\left(V_1^k\right), ..., \mu\left(V_n^k\right)). \qquad (1)$$

Here $\mu(V)$ is a possibility that the system is in the domain V. Note, that we don't require a global continuity of S, but only a local one near $\mu = 0$.

We will distinguish two situations:

a) $\forall x : \lim_{k\to\infty} \mu\left(V^k_{j(x)}\right) = 0$,

b) $\forall x : \lim_{k\to\infty} \mu\left(V^k_{j(x)}\right) > 0$.

In the first situation there is a function F such that $F\left(\mu\left(V^k_{j(x)}\right)\right) \approx F(\nu(x))\,dx$ for small enough $dx =_{def} d\left(V^k_{j(x)}\right)$, with:

$$\nu(x) = \lim_{k\to\infty} F^{-1}\left(\frac{F\left(\mu\left(V^k_{i(x)}\right)\right)}{d\left(V^k_{j(x)}\right)}\right).$$

. It can be shown, that in this case S is at least asymptotically archimedean near $\mu = 0$:

$$S(\mu\left(V^k_1\right), ..., \mu\left(V^k_n\right)) \approx \lim_{p\to\infty} \min\left\{F_p^{-1}\left(\sum_i^n F_p\left(\mu\left(V^k_i\right)\right)\right), 1\right\} \tag{2}$$

where $F_p(x)$ is a continuous near zero asymptotic sequence of the functions with $F_p(0) = 0$. For $n \to \infty$ we have:

$$\mu(V) = \min\left\{F^{-1}\left(\int_V F(\nu(x))\,dx\right), 1\right\}. \tag{3}$$

As an example, let us consider *fundamental s-norm* discussed in [4]:

$$S(\mu_1, \mu_2) =_{def} 1 - \log_s\left(1 + \frac{\left(s^{1-\mu_1} - 1\right)\left(s^{1-\mu_2} - 1\right)}{s-1}\right). \tag{4}$$

In this case:

$$\begin{aligned} F_s(x) &= -\ln\frac{s^{1-x}-1}{s-1}, \\ F_s^{-1}(x) &= 1 - \log_s\left(1 + (s-1)\exp(-x)\right), \end{aligned} \tag{5}$$

and

$$\mu(V) = 1 - \log_s\left(1 + (s-1)\exp - \int_V \zeta(x)\,dx\right). \tag{6}$$

with

$$\zeta(x) = -\left(\ln \frac{s^{1-\nu(x)} - 1}{s - 1}\right). \tag{7}$$

Correspondingly:

$$\mu\left(V_{j(x)}^{k}\right) = F_s^{-1}\left(F_s\left(\nu(x)\right) dx\right) \approx \frac{s-1}{s \ln s} \zeta(x)\, dx, \tag{8}$$

Note, that Eq.(3) includes *s-max* norm as a limiting case. To see this, let us consider an asymptotic $s \to 0$, then:

$$\begin{aligned}
\mu(V) &\approx 1 - \log_s\left(1 + (s-1)\exp -s \int\limits_V s^{\ln(1/s)\nu(x)} dx\right) \approx \\
&\approx \nu(x_m)\left(1 + o\left(\sqrt{\frac{2\pi}{\nu_m''}} \frac{1}{\ln 1/s}\right)\right),
\end{aligned}$$

where $\nu_m = \nu(x_m) = \max_x(\nu(x)) = \max_x F_s^{-1}\left(F_s\left(\mu\left(V_{i(x)}^{k}\right)\right)/dx\right)$. Thus:

$$\mu(V) = \max_{x \in V}(\mu_x). \tag{9}$$

In the situation *b)* $\mu\left(V_{i(x)}^{k}\right)$ can be nonzero only in a discrete set of the points $x = x_1, ..., x_N$. In this case we will say that $\left\{\mu\left(V_1^k\right), ..., \mu\left(V_N^k\right)\right\}$ determine a *fuzzy lattice* on the manifold.

2 Three types of the fuzzy dynamics

Consider, at first, the case *a)*. The master equation of fuzzy dynamics in this case (see [1] for detail) will be:

$$\mu(x, t|dx) = F^{-1}\left(\int\limits_{V_{x'}} F\left(T\left[P_\delta(x, x', t|dx), \mu(x', t-\delta|dx')\right]\right)\right), \tag{10}$$

where $P_\delta(x, x', t|dx)$ is a possibility of transition from the point x' to the $V_{i(x)}^{k}$ neighborhood of the point x during the time interval δ, $\mu(x, t|dx)$ is a possibility that the system is in $V_{j(x)}^{k}$ at the time t and $T(x, y)$ is some t-norm. For self-consistence of Eq.(10) for small $\left|V_{j(x)}^{k}\right|$ it should be:

$$F\left(T\left[P_\delta(x, x', t|dx), \mu(x', t-\delta|dx')\right]\right) \sim dx dx',$$

that is:

$$T\left[P_{\delta}\left(x,x',t|dx\right),\mu\left(x',t-\delta|dx'\right)\right]\approx F^{-1}\left(F\left(P_{\delta}\right)\cdot F\left(\mu\right)\right). \tag{11}$$

Thus the Eq.(10) leads to:

$$F\left(\nu\left(x,t\right)\right)=\int\limits_{V_{x'}}F\left(p_{\delta}\left(x,x',t\right)\right)F\left(\nu\left(x',t-\delta\right)\right)dx' \tag{12}$$

where

$$\begin{aligned} p_{\delta}\left(x,x',t\right) &= F^{-1}\left(\frac{F\left(P_{\delta}\left(x,x',t|dx\right)\right)}{dx}\right), \\ \nu\left(x,t\right) &= F^{-1}\left(\frac{F\left(\mu\left(x,t|dx\right)\right)}{dx}\right). \end{aligned}$$

Note, that neither $F\left(p_{\delta}\left(x,x',t\right)\right)$ nor $F\left(\nu\left(x,t\right)\right)$ can not to be considered, generally speaking, as probability densities, because all what we need is

$$\min\left\{F^{-1}\left(\int\limits_{V}F\left(\nu\left(x,t\right)\right)dx\right),1\right\}=1, \tag{13}$$

so, $F\left(\nu\left(x,t\right)\right)$ should not be normalized. The dynamics (12) has been considered in [3]. It was shown, that in general case the equation like (12) leads to Dirac or generalized Fokker-Plank equations for $F\left(\nu\left(x,t\right)\right)$.

The limiting case (9) corresponds to *s-max* fuzzy logic and leads to the Hamiltonian type of the dynamics. This case has been considered in detail in [2].

Consider the case *b)*. Denoting $\lim_{dx\to 0}\mu\left(V^{k}_{j(x)}\right)=\nu\left(x,t\right)$ and $x_{i_*}=x$ one obtain:

$$\nu\left(x,t\right)=S\left(T\left[P_{\delta}\left(x,x_1,t\right),\nu\left(x_1,t-\delta\right)\right],...,T\left[P_{\delta}\left(x,x_N,t\right),\nu\left(x_N,t-\delta\right)\right]\right). \tag{14}$$

Continuity of $\nu\left(x,t\right)$ requires that $\lim_{\delta\to 0}P_{\delta}\left(x,x',t\right)=\delta^{x'}_{x}$. In most real situations a system state can't to change drastically, so $P_{\delta}\left(x,x',t\right)$ has to quickly vanish for $\left|x-x'\right|\gg l\left(\delta\right),\, l\left(\delta\right)\to 0$. In this case $P_{\delta}\left(x,x',t\right)$ can be represented in the form:

$$P_{\delta}\left(x,x',t\right)\approx\delta^{x'}_{x}\left(1-c_1\left(\delta\right)p_1\left(x,t\right)\right)+\left(1-\delta^{x'}_{x}\right)c_2\left(\delta\right)p_2\left(\frac{x-x'}{l\left(\delta\right)},x,t\right), \tag{15}$$

where $c_1(\delta), c_2(\delta) \to 0$ together with δ and $p_2\left(\frac{x-x'}{l(\delta)}, x, t\right)$ quickly tends to zero for $|x-x'| \gg l(\delta)$. Denoting $x - x_i = u_i$ we have:

$$\begin{aligned}\nu(x,t) &= S\left(T\left[1-c_1(\delta)p_1(x,t), \nu(x,t-\delta)\right], T\left[c_2(\delta)p_2\left(\frac{u_1}{l(\delta)}, x, t\right),\right.\right.\\ &\left.\left.\nu(x_1,t-\delta)\right] ..., T\left[c_2(\delta)p_2\left(\frac{u_N}{l(\delta)}, x, t\right), \nu(x_N, t-\delta)\right]\right). \quad (16)\end{aligned}$$

Expanding of the right side of this equation for small δ one obtain:

$$\frac{\partial\nu(x,t)}{\partial t} + \Psi(x,t,\nu(x,t)) = 0, \quad (17)$$

where

$$\Psi(x,t,\nu(x,t)) = \Omega(\nu(x,t))\, p_1^n(x,t) - \Phi(\nu(x,t)),$$

with

$$\begin{aligned}\Omega(\nu(x,t)) &= \frac{1}{n!}\left.\frac{\partial^n T(w,\nu)}{\partial w^n}\right|_{w=1} \lim_{\delta\to 0}\frac{c_1^n(\delta)}{\delta},\\ \Phi(\nu(x,t)) &= \frac{1}{m!}\left.\frac{\partial^m S(\nu,w)}{\partial w^m}\right|_{w=0} \lim_{\delta\to 0}\sum_k \frac{\left(T\left[c_2(\delta)p_2(u_k/l(\delta), x, t), \nu\right]\right)^m}{\delta},\end{aligned}$$

where n and m are the first nonzero derivatives of T and S in the corresponding points. For example, for Lukasiewicz norms $n = m = 1$ and the functions Ω and Φ are the constants:

$$\Omega(\nu(x,t)) = \alpha, \qquad \Phi(\nu(x,t)) = 0,$$

while for Yager norms $n = m = q$ and these functions are :

$$\begin{aligned}\Omega(\nu(x,t)) &= \alpha(-1)^{q-1}\frac{1}{q}(1-\nu(x,t))^{1-q},\\ \Phi(\nu(x,t)) &= 0.\end{aligned}$$

3 Concluding remarks

In above presented results we have answered on the question: "how many different types of the fuzzy dynamics exist?". Note, that all of these dynamics are well known in theoretical physics. Amazing, that these dynamics cover almost all dynamics equations successfully used in the physics up to the present time. Correspondingly, we will call these types of the fuzzy dynamics as *DFP-f-dynamics* (Dirac-Fokker-Plank fuzzy-dynamics), *H-f-dynamics* (Hamiltonian) and *K-f-dynamics* (Kinetics).

It should be emphasized that in the fuzzy dynamics there is a deep relation between the local property of the chosen s-norm and admissible form of the membership function for a system state. In fact, fuzzy dynamics deals with *s-norm - membership function* pair rather with two separate objects.

Answer on the next question: "what is an optimal choice?" is not so unequivocal. What is understood now, that there are a few classes (only three ?) of the norm triples with substantially different features, while in each class the difference between the norms is rather quantitative then qualitative. It is important to study what are the crucial features of the norms for the fuzzy dynamics description of the problem. Note, that optimal choice of the norm is critical, when we deal with an iterative procedure (like a long term prediction or fuzzy dynamics) where errors accumulate with the iterations.

References

[1] Friedman Y. and Sandler U., 1996, "Evolution of Systems under Fuzzy Dynamic Law", Fuzzy Set and Systems, v84, p.61-74.

[2] Friedman Y. and Sandler U., 1997, "Dynamics of Fuzzy Systems", Inter.J. of Chaos Theory and Application, v2, No.3-4, p.3-23.

[3] Friedman Y. and Sandler U., 1999 "Fuzzy Dynamics as Alternative to Statistical Mechanics", Fuzzy Set and Systems, v106, p.61-74.

[4] Butnariu D., Klement E.P., Zafrany S., 1993, "On Triangular Norm Propositional Fuzzy Logic", Preprint.