CONSENSUS WITH ORDINAL DATA*

JEAN-LUC MARICHAL
University of Liège - Faculty of Economics
Sart Tilman - B31 - 4000 Liège, Belgium.
Phone: +32-4-3663105, Fax: +32-4-3662821
Email: jl.marichal@ulg.ac.be

MARC ROUBENS
University of Liège - Institute of Mathematics
Sart Tilman - B37 - 4000 Liège, Belgium.
Phone: +32-4-3669423, Fax: +32-4-3669547
Email: m.roubens@ulg.ac.be

Abstract

We present a model allowing to aggregate decision criteria when the available information is of qualitative nature. The use of the Sugeno integral in this model is justified by an axiomatic approach. An illustrative example is also provided.

Keywords: multicriteria decision making, ordinal scales, Sugeno integral.

1 Problem setting

Assume $A = \{a, b, c, \ldots\}$ is a finite set of potential *alternatives*, among which the decision maker must choose. Consider also a finite set of *criteria* $N = \{1, \ldots, n\}$ to be satisfied. Each criterion is represented by a mapping g_i from the set of alternatives A to a given finite ordinal scale $X_i = \{g_i^{(1)} < \cdots < g_i^{(\ell_i)}\}$. Such a scale can be included in the unit interval $[0, 1]$ without loss of generality. For each alternative $a \in A$ and each criterion $i \in N$, $g_i(a)$ then represents the evaluation of a along criterion i. We assume that all the mappings g_i are given beforehand.

Our central interest is the problem of constructing a single comprehensive criterion from the given criteria. Such a criterion, which is supposed to be a representative of the original criteria, is modeled by a mapping g from A to a given finite ordinal scale $X = \{0 = g^{(1)} < \cdots < g^{(\ell)} = 1\}$. The value $g(a)$ then represents the global evaluation of alternative a expressed in the scale X.

In order to aggregate properly the partial evaluations of $a \in A$, we will assume that there exist n nondecreasing mappings $U_i : X_i \to X$ $(i \in N)$ and an aggregation operator $M : X^n \to X$ such that

$$g(a) = M\big[U_1(g_1(a)), \ldots, U_n(g_n(a))\big], \qquad a \in A.$$

The mappings U_i, called *utility functions*, enable us to express all the partial evaluations in the common scale X, so that the operator M aggregates commensurable evaluations. We will also make the assumption that $U_i(g_i^{(1)}) = 0$ and $U_i(g_i^{(\ell_i)}) = 1$ for all $i \in N$.

The construction of g can be done in two steps. We first propose an axiomatic-based aggregation operator M, then we identify each mapping U_i by asking appropriate questions to the decision maker.

2 Meaningful aggregation operators

In this section we propose an axiomatic setting allowing to determine a suitable aggregation operator $M : X^n \to X$. Since the scale $X \subset [0, 1]$ is of ordinal nature, the numbers that constitute it are defined up to an increasing bijection φ from $[0, 1]$ to $[0, 1]$. A meaningful aggregation operator should then satisfy the following property:

*To be presented at EUFIT'99, Aachen, Germany, September 13–16, 1999.

Definition 2.1 *$M:[0,1]^n\to[0,1]$ is comparison meaningful for ordinal scales if for any increasing bijection $\varphi:[0,1]\to[0,1]$ and any n-tuples $x,x'\in[0,1]^n$, we have*

$$M(x)\leq M(x') \quad\Leftrightarrow\quad M(\varphi(x))\leq M(\varphi(x')),$$

where the notation $\varphi(x)$ means $(\varphi(x_1),\ldots,\varphi(x_n))$.

We will also assume that M is internal to the set of its arguments, that is,

$$\bigwedge_{i\in N} x_i \leq M(x_1,\ldots,x_n)\leq \bigvee_{i\in N} x_i,\qquad x\in[0,1]^n,$$

where $\wedge$ and $\vee$ denote the minimum and maximum operations, respectively. This implies that M is *idempotent*, i.e., $M(x,\ldots,x)=x$ for all $x\in[0,1]$. Moreover, it has been shown in [3, Theorem 4.1] that any idempotent aggregation operator $M:[0,1]^n\to[0,1]$ satisfying the comparison meaningful property above is such that

$$M(x_1,\ldots,x_n)\in\{x_1,\ldots,x_n\},\qquad x\in[0,1]^n.$$

Finally, it seems natural to assume that any $E\subseteq[0,1]^n$ is a closed subset whenever its image $\{M(x)\,|\,x\in E\}$ is a closed subset of $[0,1]$. This condition simply expresses that M is a continuous function.

We then have the following axiomatic characterization, see [1, Theorem 3.4.12].

Theorem 2.1 *$M:[0,1]^n\to[0,1]$ is continuous, idempotent, and comparison meaningful for ordinal scales if and only if there exists a monotone set function $\mu:2^N\to\{0,1\}$ fulfilling $\mu(\emptyset)=0$ and $\mu(N)=1$ such that*

$$M(x)=\bigvee_{\substack{T\subseteq N\\ \mu(T)=1}}\bigwedge_{i\in T} x_i,\qquad x\in[0,1]^n.$$

Theorem 2.1 provides the general form of operators $M:X^n\to X$ that are appropriate to aggregate the given criteria. However, all these operators present the following major drawback: If e_S represents the characteristic vector in $\{0,1\}^n$ of a given subset of criteria $S\subseteq N$ then we have

$$M(e_S)=\mu(S)\in\{0,1\}.$$

This means that the global evaluation of an alternative that fully satisfies criteria S and totally fails to satisfy the other criteria is always an extreme value of X, that is, 0 or 1. In particular, the compensation effects are not allowed. As we will see in the next section, this indesirable phenomenon can be overcome by the use of a fuzzy measure and the corresponding Sugeno integral.

3 The Sugeno integral as an aggregation operator

The observation above shows that it is necessary to enrich the aggregation model to authorize compensation effects. Whatever the operator M considered, it seems natural to interpret $M(e_S)$ as the importance of the combination S of criteria. This importance should be expressed in X and not restricted to the extreme values.

It is clear that any mapping $\mu:2^N\to X$ that represents the importance of each combination of criteria should be a fuzzy measure, a concept introduced by Sugeno [4].

Definition 3.1 *A fuzzy measure on N is a monotone set function $\mu:2^N\to[0,1]$ such that $\mu(\emptyset)=0$ and $\mu(N)=1$. Monotonicity means that $\mu(S)\leq\mu(T)$ whenever $S\subseteq T$.*

Now, a suitable aggregation operator should take into consideration the importance of each combination of criteria. However, since both the partial evaluations and the importances are expressed in the same scale X, it is natural to consider as new aggregation operator a function $M:X^{n+2^n-2}\to X$ in which the first n arguments are related to the partial evaluations $U_i(g_i)$ and the others to the importance coefficients $\mu(S)$ for $S\neq\emptyset,N$ (assuming a fixed order in 2^N).

By definition of importance, we will impose the following condition:

$$M(e_S;\mu)=\mu(S),$$

for any $S\subseteq N$ and any fuzzy measure μ on N. As in the previous section, we will also ask the operator M to be continuous and comparison meaningful for ordinal scales. Finally, we will assume that M is idempotent in the first n arguments, i.e., $M(x,\dots,x;\mu)=x$ for any $x\in[0,1]$ and any fuzzy measure μ on N. In particular, this implies that M is idempotent and we have

$$M(x;\mu)\in\{x_1,\dots,x_n\}\cup\{\mu(S)\,|\,S\subset N \text{ and } S\neq\emptyset,N\}$$

for any $x\in[0,1]^n$ and any fuzzy measure μ on N.

The Sugeno integral is a typical example of such an aggregation operator. Its definition is the following, see [4].

Definition 3.2 *Let μ be a fuzzy measure on N. The Sugeno integral of $x\in[0,1]^n$ with respect to μ is defined by*

$$\mathcal{S}_\mu(x):=\bigvee_{i=1}^{n}\big[x_{(i)}\wedge\mu(\{(i),\dots,(n)\})\big],$$

where $(\cdot)$ indicates a permutation of N such that $x_{(1)}\leq\cdots\leq x_{(n)}$.

In fact, it can be proved that the Sugeno integrals on $[0,1]^n$, viewed as operators having $(n+2^n-2)$ arguments, are exactly those operators that satisfy the four properties mentioned above. The statement of this result is the following.

Theorem 3.1 *Let $M:[0,1]^{n+2^n-2}\to[0,1]$ be such that the last (2^n-2) arguments are related to importance coefficients $\mu(S)$ for $S\neq\emptyset,N$. This operator*
- *fulfils $M(e_S;\mu)=\mu(S)$ for any $S\subseteq N$ and any fuzzy measure μ on N,*
- *is continuous,*
- *is comparison meaningful for ordinal scales,*
- *is idempotent in the first n arguments,*

if and only if $M(x;\mu)=\mathcal{S}_\mu(x)$ for any $x\in[0,1]^n$ and any fuzzy measure μ on N.

Theorem 3.1 provides a rather good motivation for the use of the Sugeno integral as an appropriate operator to aggregate the criteria. We shall thus adopt this operator as we continue.

The following result shows that the Sugeno integral can take different interesting forms, see e.g. [1, 2]. In particular, it can be expressed as the median of $(2n-1)$ values.

Proposition 3.1 *Let $x\in[0,1]^n$ and μ be a fuzzy measure on N. Then we have*

$$\mathcal{S}_\mu(x)=\bigvee_{T\subseteq N}\Big[\mu(T)\wedge(\bigwedge_{i\in T}x_i)\Big],$$

and

$$\mathcal{S}_\mu(x)=\text{median}\big(x_1,\dots,x_n,\mu(\{(2),\dots,(n)\}),\mu(\{(3),\dots,(n)\}),\dots,\mu(\{(n)\})\big),$$

where $(\cdot)$ indicates a permutation of N such that $x_{(1)}\leq\cdots\leq x_{(n)}$.

The first formula in Proposition 3.1 shows that the operators described in Theorem 2.1 are nothing but Sugeno integrals with respect to fuzzy measures taking their values in $\{0,1\}$.

4 The aggregation model

Now that the Sugeno integral has been chosen for the aggregation process, we have yet to appraise not only the utility functions U_i $(i\in N)$ but also the importance of each combination of criteria, that is, the fuzzy measure μ.

The importance coefficients $\mu(S)$ can be provided directly by the decision maker. Of course, this consists of (2^n-2) questions. However, in practical problems the total violation of at least two criteria often lead to the lowest global evaluation. Combining this with the monotonicity of the fuzzy measure, the number of coefficients to appraise can be reduced significantly.

Now, the evaluation of the utility functions is done as follows. For each fixed $i\in N$, the decision maker is asked to appraise in X the following global evaluations

$$\mathcal{S}_\mu\big(U_i(g_i^{(r)})\,e_i+e_{N\setminus\{i\}}\big),\qquad r\in\{1,\dots,\ell_i\}.$$

Clearly, all these evaluations are comprised between $\mathcal{S}_\mu(e_{N\setminus\{i\}}) = \mu(N \setminus \{i\})$ and $\mathcal{S}_\mu(e_N) = 1$. More exactly, one can easily show that

$$\mathcal{S}_\mu\big(U_i(g_i^{(r)})\, e_i + e_{N\setminus\{i\}}\big) = U_i(g_i^{(r)}) \vee \mu(N \setminus \{i\}), \qquad r \in \{1, \ldots, \ell_i\}.$$

Therefore, for each fixed $r \in \{1, \ldots, \ell_i\}$, we have two exclusive cases:

- If $\mathcal{S}_\mu\big(U_i(g_i^{(r)})\, e_i + e_{N\setminus\{i\}}\big) = \mu(N \setminus \{i\})$ then we have an upper bound for $U_i(g_i^{(j)})$:

$$U_i(g_i^{(r)}) \leq \mathcal{S}_\mu\big(U_i(g_i^{(r)})\, e_i + e_{N\setminus\{i\}}\big).$$

- If $\mathcal{S}_\mu\big(U_i(g_i^{(r)})\, e_i + e_{N\setminus\{i\}}\big) > \mu(N \setminus \{i\})$ then $U_i(g_i^{(r)})$ is uniquely determined:

$$U_i(g_i^{(r)}) = \mathcal{S}_\mu\big(U_i(g_i^{(r)})\, e_i + e_{N\setminus\{i\}}\big).$$

If all the utility functions are not completely determined, one could go further and ask the decision maker to appraise the following global evaluations

$$\mathcal{S}_\mu\big(U_i(g_i^{(r)})\, e_i + e_S\big), \qquad S \subseteq N \setminus \{i\},\ r \in \{1, \ldots, \ell_i\},$$

for which one can show that

$$\mathcal{S}_\mu\big(U_i(g_i^{(r)})\, e_i + e_S\big) = \mathrm{median}\big(U_i(g_i^{(r)}), \mu(S), \mu(S \cup \{i\})\big), \qquad S \subseteq N \setminus \{i\},\ r \in \{1, \ldots, \ell_i\}.$$

However, appraising all these evaluations can become a very difficult task for the decision maker. Moreover, most of these evaluations can be very low and bring no information about the utility functions.

5 An illustrative example

Consider the problem of ranking candidates that apply for a definitive position in a given university. The evaluations are done on three criteria: 1) Scientific value, 2) Teaching effectiveness, and 3) Interview by evaluation committee. The ordinal scales are given as follows:

Scientific value	:	Exc. > Very Good > Good > Satisfactory > Weak
Teaching effectiveness	:	Exc. > Very Good > Satisfactory > Weak > Very Weak
Interview	:	$+ > 0 > -$
Global evaluation	:	$A_1 > A_2 > B > C$

The decision maker gives the following global evaluations:

$$\begin{array}{llll} \mu(123) = A_1 & \mu(12) = A_2 & \mu(1) = B & \mu(\emptyset) = C \\ & \mu(13) = B & \mu(2) = C & \\ & \mu(23) = C & \mu(3) = C & \end{array}$$

To determine U_1, he/she proposes the following evaluations:

$$\begin{aligned} \mathcal{S}_\mu\big(U_1(VG), 1, 1\big) &= A_1 \\ \mathcal{S}_\mu\big(U_1(G), 1, 1\big) &= A_2 \\ \mathcal{S}_\mu\big(U_1(S), 1, 1\big) &= B. \end{aligned}$$

Since $\mathcal{S}_\mu\big(U_1, 1, 1\big) = U_1 \vee \mu(23) = U_1$, these three evaluations determine completely U_1. We then have $U_1(W) = C$, $U_1(S) = B$, $U_1(G) = A_2$, $U_1(VG) = A_1$, $U_1(E) = A_1$.

For U_2, the following evaluations are proposed:

$$\begin{aligned} \mathcal{S}_\mu\big(1, U_2(VG), 1\big) &= A_1 \\ \mathcal{S}_\mu\big(1, U_2(S), 1\big) &= A_1 \\ \mathcal{S}_\mu\big(1, U_2(W), 1\big) &= A_2. \end{aligned}$$

Since $\mathcal{S}_\mu\big(1, U_2, 1\big) = U_2 \vee \mu(13) = U_2 \vee B$, these evaluations determine completely U_2. We then have $U_2(VW) = C$, $U_2(W) = A_2$, $U_2(S) = A_1$, $U_2(VG) = A_1$, $U_2(E) = A_1$.

Finally, the decision maker gives:

$$\mathcal{S}_\mu\big(1, 1, U_3(0)\big) = A_2.$$

Since $\mathcal{S}_\mu\big(1, 1, U_3\big) = U_3 \vee \mu(12) = U_3 \vee A_2$, this evaluation only indicates that $U_3(0) \leq A_2$. We then have: $U_3(-) = C$, $U_3(0) \in \{C, B, A_2\}$, $U_3(+) = A_1$.

Although $U_3(0)$ is not known, the Sugeno integral is completely determined. To see this, let us use the first formula in Proposition 3.1. We then have

$$\begin{aligned}
\mathcal{S}_\mu\big(U_1, U_2, U_3(0)\big) &= C \vee \big(B \wedge U_1\big) \vee \big(C \wedge U_2\big) \vee \big(C \wedge U_3(0)\big) \vee \big(A_2 \wedge U_1 \wedge U_2\big) \\
&\quad \vee \big(B \wedge U_1 \wedge U_3(0)\big) \vee \big(C \wedge U_2 \wedge U_3(0)\big) \vee \big(A_1 \wedge U_1 \wedge U_2 \wedge U_3(0)\big) \\
&= (B \wedge U_1) \vee (A_2 \wedge U_1 \wedge U_2).
\end{aligned}$$

For instance, suppose that a candidate presents the profile $(E, S, 0)$. The global evaluation of this candidate will then be given by

$$\mathcal{S}_\mu\big(U_1(E), U_2(S), U_3(0)\big) = (B \wedge A_1) \vee (A_2 \wedge A_1 \wedge A_1) = A_2.$$

References

[1] J.-L. Marichal (1998), *Aggregation operators for multicriteria decision aid*, Ph.D. Thesis, University of Liège, Liège, Belgium.

[2] J.-L. Marichal (1999), On Sugeno integral as an aggregation function, *Fuzzy Sets and Systems*, in press.

[3] S. Ovchinnikov (1996), Means on ordered sets, *Mathematical Social Sciences* **32**, 39–56.

[4] M. Sugeno (1974), *Theory of fuzzy integrals and its applications*, Ph.D. Thesis, Tokyo Institute of Technology, Tokyo.