On interdependent biobjective decision problems

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Abstract

We consider biobjective decision problems with interdependent objectives. First we state the biobjective decision problem with independent objectives and then by introducing additive linear interdependences between the objective functions we explain the behavior of compromise solutions.

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1 Biobjective interdependent decision problems

A biobjective independent decision problem in a (normalized) criterion space can be defined as follows

$$\max\{f_1, f_2\};\ \text{subject to } 0 \le f_1, f_2 \le 1. \qquad (1)$$

As f_1 and f_2 take their values independently of each others, we can first maximize f_1 subject to $f_1 \in [0,1]$ then f_2 subject to $f_2 \in [0,1]$ and take the (feasible) *ideal point*, $(1,1)$, as the unique solution to problem (1). A typical two-dimensional independent problem is

$$\max\{x_1, x_2\};\ \text{subject to } 0 \le x_1, x_2 \le 1,$$

that is, $f_1(x_1,x_2) = x_1$ and $f_2(x_1,x_2) = x_2$.

More often than not, the ideal point is not feasible, that is, we can not improve one objective without deteriorating the other one. If we happen to know the exact values of the objectives in all conceivable cases then a good compromise solution can be defined as a Pareto-optimal solution satisfying some additional requirements specified by the decision maker [5, 6].

Suppose we do not know exactly the values of the objectives, but we are able to identify some (linguistic) interdependences between them, such as "f_1 *distracts* f_2 *if the value of* f_1 *is big*" or "f_2 *supports* f_1 *if the value of* f_2 *is medium*", where the linguistic terms are represented by fuzzy numbers. The family of fuzzy numbers will be denoted by $\mathcal{F}$.

Suppose further that the interdependent values f_1' and f_2' of f_1 and f_2, respectively, are observed to satisfy the following equations

$$f_1' = f_1 + \alpha_{12}(f_2)f_2, \quad f_2' = f_2 + \alpha_{21}(f_1)f_1$$

where $\alpha_{ij} : \mathcal{F} \to \mathbb{R}$ is interpreted as the grade of interdependency [2, 3] between f_i and f_j, which may vary depending on the linguistic value of f_j.

If $\alpha_{ij}(f_j) = \alpha_{ij} \in \mathbb{R}$ is constant then we say that "f_j supports f_i" or "f_j hinders f_i" *globally*. In this case, depending on the value of α_{ij}, we can have the following simple additive linear interdependences between the objectives

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- if $\alpha_{12} = 0$ then we say that f_1 is independent from f_2;
- if $\alpha_{12} > 0$ then we say that f_2 unilaterally supports f_1;
- if if $\alpha_{12} < 0$ then we say that f_2 hinders f_1;
- if $\alpha_{12} > 0$ and $\alpha_{21} > 0$ then we say that f_1 and f_2 mutually support each others;
- if $\alpha_{12} < 0$ and $\alpha_{21} < 0$ then we say that f_1 and f_2 are conflicting;
- if $\alpha_{12} + \alpha_{21} = 0$ then we say that f_1 are f_2 are in a trade-off relation;

It is clear, for example, that if f_2 unilaterally and globally supports f_1 then the larger the improvement in f_2 (supporting objective function) the more significant is its contribution to f_1 (supported objective function).

The following two theorems characterize the behavior of max-min compromise solutions of biobjective interdependent decision problems with conflicting and supporting objective functions.

Theorem 1.1. *If f_1 hinders f_2 globally and f_2 hinders f_1 globally with the same degree of hindering $0 \leq \alpha \leq 1$ then*

$$\max\min\{f_1', f_2'\} \leq (1-\alpha)\max\min\{f_1, f_2\}; \text{ subject to } 0 \leq f_1, f_2 \leq 1.$$

Proof. From the relationships

$$f_1' = f_1 - \alpha f_2, \quad f_2' = f_2 - \alpha f_1$$

and

$$(1-\alpha)f_1 \geq f_1 - \alpha f_2,$$

if $f_2 \geq f_1$, and

$$(1-\alpha)f_2 \geq f_2 - \alpha f_1,$$

if $f_1 \geq f_2$, we get

$$(1-\alpha)\min\{f_1, f_2\} \geq \min\{f_1 - \alpha f_2, f_2 - \alpha f_1\},$$

which ends the proof. □

In other words, Theorem 1.1 shows an upper bound for the max-min compromise solution, which can interpreted as ”the bigger the conflict between the objective functions the less are the criteria satisfactions”. For example, in the extremal case, $\alpha = 1$ we end up with totally opposite objective functions

$$f_1' = f_1 - f_2, \quad f_2' = f_2 - f_1,$$

and, therefore, we get

$$\max\min\{f_1', f_2'\} = \max\min\{f_1 - f_2, f_2 - f_1\} \leq (1-\alpha)\max\min\{f_1, f_2\} = 0.$$

On the other hand, if $\alpha = 0$ then the objective functions remain independent, that is,

$$\max\min\{f_1', f_2'\} \leq (1-\alpha)\max\min\{f_1, f_2\} = \max\min\{f_1, f_2\}.$$

Theorem 1.2. *If f_1 supports f_2 globally with degree $\alpha \geq 1$ then*

$$\max\min\{f_1', f_2'\} = \max f_1; \text{ subject to } 0 \leq f_1, 0 \leq f_2.$$

Proof. From $\alpha \geq 1$ we get

$$f_1 \leq f_2 + \alpha f_1$$

and therefore,

$$\max\min\{f_1, f_2 + \alpha f_1\} = \max f_1,$$

which ends the proof. □

Theorem 1.2 demonstrates that if one objective supports the other one globally with a degree of support bigger than one, then the independent maximum of the supporting objective gives the max-min compromise solution to the interdependent problem. In other words, in the case of unilateral support, it is enough to maximize the supporting objective function to reach a max-min compromise solution.

It should be noted that Theorem 1.2 remains valid for non-negative objective functions (that is, the constraints $f_1 \leq 1$ and $f_2 \leq 1$ can be omitted). If, however f_1 does satisfy the constraint $f_1 \leq 1$ then the ideal point $(1, 1)$ will be the unique max-min solution.

2 Fuzzy reasoning for interdependent biobjective decision problems

Suppose now that $\alpha_{ij}(f_j)$ may vary depending on the linguistic value of f_j, for example "f_1 supports f_2 when f_1 is small" but "f_1 hinders f_2 when f_1 is big". Then we use fuzzy rules for knowledge representation and the interdependent values of the objectives will be calculated by simplified fuzzy reasoning schemes [7].

For example, if we have two linguistic values *small* and *big* for f_1 and f_2 then the interdependencies between the objective functions can be described by the following fuzzy rule base

$\Re_1$: if f_1 is small		then		$f_2' = f_2 + \alpha_{21} f_1$
$\Re_2$: if f_1 is big		then		$f_2' = f_2 - \beta_{21} f_1$
$\Re_3$: if	f_2 is small	then	$f_1' = f_1 - \beta_{12} f_2$	
$\Re_4$: if	f_2 is big	then	$f_1' = f_1 + \alpha_{12} f_2$	

where the rules are interpreted as

$\Re_1$: f_1 supports f_2 if f_1 is small, where $\alpha_{21} > 0$ is the (linear) feedback from f_1 to f_2

$\Re_2$: f_1 hinders f_2 if f_1 is big, where $-\beta_{21} < 0$ is the (linear) feedback from f_1 to f_2

$\Re_3$: f_2 supports f_1 if f_2 is big, where $-\beta_{12} < 0$ is the (linear) feedback from f_2 to f_1

$\Re_4$: f_2 hinders f_1 if f_2 is small, where $\alpha_{12} > 0$ is the (linear) feedback from f_2 to f_1

For example, if $\mathrm{small}(t) = 1 - t$ and $\mathrm{big}(t) = t$ then the interdependent values f_1' and f_2' are obtained from the rule base $\Re = \{\Re_1, \dots, \Re_4\}$ as

$$f_1' = (1 - f_2)(f_1 - \beta_{12} f_2) + f_2(f_1 + \alpha_{12} f_2).$$

and

$$f_2' = (1 - f_1)(f_2 + \alpha_{21} f_1) + f_1(f_2 - \beta_{21} f_1),$$

In this way the original independent problem turns into

$$\max\min\{f_1 - \beta_{12} f_2 + (\alpha_{12} + \beta_{12}) f_2^2,\ f_2 + \alpha_{21} f_1 - (\alpha_{21} + \beta_{21}) f_1^2\}$$

$$\text{subject to } 0 \leq f_1, f_2 \leq 1.$$

3 Concluding remarks

We have considered biobjective decision problems (in the criterion space), which have been derived from the original independent problem by introducing some observed interdependencies, such as conflict and support, between the objective functions. We have explained how the max-min compromise solution changes for globally conflicting and supporting objective functions. We also showed a fuzzy-reasoning-based approach to decision problems with degree of interdependency varying according to the values of the objective functions. The extension of the biobjective model to multiple objective problems will be the subject of our future research.

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