On Enumeration of Possibly Optimal Extreme Points in Linear Programming Problems with Interactive Possibilistic Variables

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ABSTRACT: In this paper, we deal with a linear programming problem whose objective function coefficient vector is unknown but known in a range given by a polytope. The possibly optimal solution set is defined as a set of least rational solutions. The properties of the problem and its possibly optimal solutions are discussed. An enumeration method of all possibly optimal extreme points (basic solutions) is developed. In several cases, obtaining a superset of the possibly optimal extreme point set is sufficient for further discussion of the final solution. From this point of view, in order to see the efficiency of the proposed method, we compare it with an enumeration of elements of such a superset by Steuer's method.

INTRODUCTION

So far, non-interactive possibilistic variables have been treated in linear programming problems with uncertain parameters whose range is given by a fuzzy set or a crisp set. Owing to the non-interaction, the fuzzy/crisp range of uncertain parameter vector is represented by a Cartesian product of the projections and hence, the reduced problem is very tractable. However, in several cases, the assumption of non-interaction will not be proper representation of the real situation of the problem and sometimes leads to a strange result (see Inuiguchi and Tanino 1999). Recently, authors have tried to introduce a certain interaction between uncertain parameters with a small loss of tractability (see Inuiguchi and Tanino 1998).

In this paper, we treat a linear programming problem with an uncertain objective function coefficient vector. We assume that the range of the objective function coefficient vector is given by a polytope. Whereas the range has been assumed to be a box set so far, it is assumed to be only a polytope. By this generalization, we can treat interaction among uncertain parameters (possibilistic variables). A possibly optimal solution set for our problem is defined as a set of least reasonable solutions (see Inuiguchi and Sakawa 1994). Under the assumption of the boundedness of the feasible area, the possibly optimal extreme point set is finite. Enumeration of possibly optimal extreme points is important for estimating the optimal solution range and for computations of a minimax regret solution (Inuiguchi and Sakawa 1995) as well as a maximin achievement rate solution (Inuiguchi and Sakawa 1997). Given a possibly optimal basic solution, a necessary and sufficient condition for an adjacent basic solution to be possibly optimal is shown. Based on this condition, an enumeration method for all possibly optimal extreme points is designed as an extension of Steuer's method (Steuer 1981) proposed in the box set case.

In several cases, it is sufficient for further discussion of the final solution to obtain a superset of the possibly optimal extreme point set. For example, calculations of a minimax regret solution (Inuiguchi and Sakawa 1995) and a maximin achievement rate solution (Inuiguchi and Sakawa 1997) require only such a superset. In those cases, a direct application of Steuer's method combined with an outer approximation of the polytope by a box set is conceivable for obtaining a superset. From this point of view, the proposed enumeration method is compared with this outer approximation approach for the superset generation. The results show that the proposed method is more efficient in computation time and at the same time yields the exact possibly optimal extreme point set.

LINEAR PROGRAMMING PROBLEM WITH UNCERTAIN COEFFICIENTS

In this paper, we treat the following linear programming problem with uncertain objective function coefficient vector:

$$\begin{array}{rl} \text{maximize} & \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} \end{array} \tag{1}$$

where A is an $m \times n$ matrix, $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)^{\mathrm{T}}$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_m)^{\mathrm{T}}$. $\boldsymbol{\gamma}$ is a possibilistic variable vector restricted by the following polytope Γ:

$$\Gamma = \{\boldsymbol{c} = (c_1, c_2, \dots, c_n)^{\mathrm{T}} \mid D\boldsymbol{c} \leq \boldsymbol{g}\} \tag{2}$$

where D is a $p \times n$ matrix and $\boldsymbol{g} = (g_1, g_2, \dots, g_p)^{\mathrm{T}}$. We assume that the feasible area X of Problem 1 is bounded.

So far, Γ has been assumed to be a box set, i.e., $D = (I, -I)^{\mathrm{T}}$ and $\boldsymbol{g}^{\mathrm{T}} = (\boldsymbol{c}^{\mathrm{R}^{\mathrm{T}}}, -\boldsymbol{c}^{\mathrm{L}^{\mathrm{T}}})$ with $\boldsymbol{c}^{\mathrm{L}^{\mathrm{T}}} \leq \boldsymbol{c}^{\mathrm{R}^{\mathrm{T}}}$. However, this assumption is too strong to represent a real situation. For example, when one uncertain parameter $\alpha \in [0, 1]$ affects on two coefficients γ_1 and γ_2 of the objective function, e.g., $\gamma_1 = \alpha + 3$ and $\gamma_2 = 5\alpha - 1$, we cannot express the interaction between those coefficients by a box set. Moreover, when the decision variable may take a negative value as well as a non-negative value, the box set representation cannot allow such a free variable. Namely, under the box set assumption, each decision variable should be non-negative or non-positive. Under the polytope assumption, we can express both cases suitably: for the former case, the range of $(\gamma_1, \gamma_2)^{\mathrm{T}}$ can be expressed by (2) with

$$D = \begin{pmatrix} 5 & -1 \\ -5 & 1 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 16 \\ -16 \\ 4 \\ -3 \end{pmatrix}.$$

It should be noted that we cannot express non-linear interaction between two possibilistic variables by (2) but linear interaction. For the latter case, let us consider the following linear programming problem with uncertain objective function coefficients and free variables:

$$\begin{array}{rl} \text{maximize} & \boldsymbol{\gamma}'^{\mathrm{T}}\boldsymbol{x}' \\ \text{subject to} & A'\boldsymbol{x}' = \boldsymbol{b}. \end{array} \tag{3}$$

where the range Γ' of $\boldsymbol{\gamma}$ is given by (2) with replacement of Γ, D and $\boldsymbol{g}$ by Γ', D' and $\boldsymbol{g}'$, respectively. As is usually done in linear programming literature, let us introduce non-negative variables $\boldsymbol{x}^+ \geq \boldsymbol{0}$ and $\boldsymbol{x}^- \geq \boldsymbol{0}$ such that $\boldsymbol{x}' = \boldsymbol{x}^+ - \boldsymbol{x}^-$. Problem (3) is rewritten as

$$\begin{array}{rl} \text{maximize} & \boldsymbol{\gamma}'^{\mathrm{T}}\boldsymbol{x}^+ - \boldsymbol{\gamma}'^{\mathrm{T}}\boldsymbol{x}^- \\ \text{subject to} & A'\boldsymbol{x}^+ - A'\boldsymbol{x}^- = \boldsymbol{b},\ \boldsymbol{x}^-,\ \boldsymbol{x}^+ \geq \boldsymbol{0}. \end{array} \tag{4}$$

Let $\boldsymbol{\gamma}^{\mathrm{T}} = (\boldsymbol{\gamma}'^{\mathrm{T}}, -\boldsymbol{\gamma}'^{\mathrm{T}})$, $\boldsymbol{x}^{\mathrm{T}} = (\boldsymbol{x}^{+\mathrm{T}}, \boldsymbol{x}^{-\mathrm{T}})$, $A = (A', -A')$,

$$D = \begin{pmatrix} D' & -D' \\ I & -I \\ -I & I \end{pmatrix}, \quad \text{and} \quad \boldsymbol{g} = \begin{pmatrix} \boldsymbol{g}' \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix},$$

where I is an identity matrix. Then Problem (3) is reduced to Problem (1) where the range of γ is given by Γ in (2). Thus, the polytope assumption still requires the linearity of Γ but it is much more flexible than the box set assumption.

POSSIBLY OPTIMAL SOLUTIONS

Let $Opt(\boldsymbol{c})$ be the optimal solution set with respect to a linear programming problem with objective function coefficient vector $\boldsymbol{c}$,

$$Opt(\boldsymbol{c}) = \left\{ \boldsymbol{x} \in X \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \max_{\boldsymbol{y} \in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} \right\}. \tag{5}$$

Since the objective function coefficient vector is not specified precisely but in a given range Γ in Problem (1), the following two kinds of optimal solution sets are defined (see Inuiguchi and Sakawa 1994):

$$\Pi S = \bigcup_{\boldsymbol{c} \in \Gamma} Opt(\boldsymbol{c}), \qquad NS = \bigcap_{\boldsymbol{c} \in \Gamma} Opt(\boldsymbol{c}). \tag{6}$$

An element of ΠS is a solution which is optimal for at least one realization of $\boldsymbol{\gamma}$ in Γ and called a possibly optimal solution. On the other hand, an element of NS is a solution which is optimal for any realization of $\boldsymbol{\gamma}$ in Γ and called a necessarily optimal solution. A necessarily optimal solution is the most reasonable solution but, unfortunately, does not exist in many cases. In this sense, possibly optimal solutions may be significant as candidates or reference points of the terminal solution.

Lemma 1 *Let $\boldsymbol{x}^1 \in X$ and $\boldsymbol{x}^2 \notin \Pi S$. Then $\boldsymbol{x} = \lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2$ is not a possibly optimal solution for any $\lambda \in [0, 1)$.*

Using Lemma 1, we can prove the following theorem.

Theorem 1 *A possibly optimal solution can be represented as a convex combination of possibly optimal extreme points.*

From Theorem 1, enumerating possibly optimal extreme points are important to know the range of possibly optimal solution set. Moreover, let $V(\Gamma) = \{\boldsymbol{c}^1, \boldsymbol{c}^2, \ldots, \boldsymbol{c}^q\}$ be a vertex set of the polytope Γ, we can prove the following theorem.

Theorem 2 *The possibly optimal solution set coincides with the weakly efficient solution set of the following multiple objective linear programming problem:*

$$\begin{array}{rl} \text{v-maximize} & \left(\boldsymbol{c}^{1\mathrm{T}}\boldsymbol{x}, \boldsymbol{c}^{2\mathrm{T}}\boldsymbol{x}, \ldots, \boldsymbol{c}^{q\mathrm{T}}\boldsymbol{x}\right) \\ \text{subject to} & \boldsymbol{x} \in X = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} \end{array} \tag{7}$$

where v-maximize stands for 'vector maximize'.

Theorem 2 implies that possibly optimal extreme points can be enumerated by an enumeration techniques for weak efficient extreme points developed in multiple objective linear programming if $V(\Gamma)$ is given. Of course, we can enumerate all elements of $V(\Gamma)$ by a certain techniques. However, this approach may require a lot of computation effort in enumeration of vertices of Γ as well as enumeration of efficient extreme points of Problem (7) since q must be a large number. Steuer (1981) proposed an enumeration method of possibly optimal extreme points without calculation of $V(\Gamma)$ when Γ is a box set. We extend Steuer's method to the polytope case.

In Steuer's method, we start from a possibly optimal basic solution (extreme point) and check the possible optimality of the adjacent basic solutions. A possibly optimal basic solution is easily obtained by taking $\boldsymbol{c} \in \Gamma$ and solve a linear programming problem with an objective function $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ and the constraint $\boldsymbol{x} \in X$. Thus we should discuss the possible optimality test of the adjacent basic solutions. The following theorem gives the possible optimality test.

Theorem 3 *Without loss of generality, we assume that first m variables are basic variables at the current basic solution. Thus, A can be represented as $(B\ A^{\mathrm{N}})$, where B is the basic matrix. The adjacent basic solution obtained by introducing the kth non-basic variable to the basis is possibly optimal if the optimal value of the following linear programming problem is zero:*

$$\begin{array}{rl} \text{minimize} & \boldsymbol{g}^{\mathrm{T}}\boldsymbol{v} \\ \text{subject to} & Y\boldsymbol{u} - \boldsymbol{y}_k w - D^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{0} \\ & \boldsymbol{u} \geq \boldsymbol{0},\ w \geq 0,\ \boldsymbol{v} \geq \boldsymbol{0} \end{array} \tag{8}$$

where $Y = ((B^{-1}A^{\mathrm{N}})^{\mathrm{T}}, -I)^{\mathrm{T}}$ and $\boldsymbol{y}_k$ is the kth column of Y.

Problem (8) always has a feasible solution and the optimal value is zero or unbounded. Thus, if Problem (8) has an unbounded solution then the adjacent basic solution is not possibly optimal, and otherwise the adjacent basic solution is possibly optimal. Based on Theorem 3, we can check the possible optimality of any adjacent basic solution to the current basic solution. Moreover, because of the equality constraints of Problem (8), we have

$$\boldsymbol{u} = \boldsymbol{e}_k w - D^{\mathrm{N}^{\mathrm{T}}} \boldsymbol{v}, \tag{9}$$

where D^{N} is a submatrix of D composed of columns corresponding to non-basic variables, i.e., D can be represented as $D = (D^{\mathrm{K}} \;\; D^{\mathrm{N}})$. $\boldsymbol{e}_k$ is a unit vector whose kth element is the unity. Hence, Problem (8) can be reduced to the following linear programming problem with a smaller number of constraints:

$$\begin{array}{rl} \text{maximize} & \boldsymbol{g}^{\mathrm{T}} \boldsymbol{v} \\ \text{subject to} & (B^{-1} A^{\mathrm{N}} D^{\mathrm{N}^{\mathrm{T}}} + D^{\mathrm{K}^{\mathrm{T}}}) \boldsymbol{v} + B^{-1} (\boldsymbol{a}_k^{\mathrm{N}} - A^{\mathrm{N}} \boldsymbol{e}_k) w = 0 \end{array} \tag{10}$$

where $\boldsymbol{a}_k^{\mathrm{N}}$ is the kth column of A^{N}. However, it is required a matrix calculation to obtain the coefficients of Problem (10).

As the result, all possibly optimal extreme points are enumerated by the following algorithm:

[Algorithm]

Step 1. Chose $\boldsymbol{c} \in \Gamma$. Solve a linear programming problem, $\max_{\boldsymbol{x} \in X} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}$. By this procedure, we obtain a possibly optimal basic solution $\boldsymbol{x}^0$ and the basis β_0.

Step 2. Let $\mathcal{E}$ be a set of explored bases, $\mathcal{U}$ a set of unexplored bases, ΠB a set of found possibly extreme points. Let $\mathcal{E} = \emptyset$, $\mathcal{U} = \emptyset$ and $\Pi B = \{\boldsymbol{x}^0\}$. Set $i = 0$.

Step 3. For all non-basic variables, the following procedure is applied.

Step 3(a): Let $\mathcal{B}$ be a set of adjacent feasible bases obtained by introduction of the non-basic variable to the current basis. Note that $\mathcal{B}$ is singleton if the adjacent basic solution is not degenerate.

Step 3(b): If $\mathcal{B} \subseteq \mathcal{E} \cup \mathcal{U}$, return to Step 3(a) and consider the next non-basic variable. If no next non-basic variable exists, go to Step 4.

Step 3(c): Solve Problem (8). If the optimal value is zero, update $\mathcal{U} = \mathcal{U} \cup \mathcal{B}$. Otherwise, update $\mathcal{E} = \mathcal{E} \cup \mathcal{B}$.

Step 4. If $\mathcal{U} = \emptyset$, terminate the algorithm.

Step 5. Update $\mathcal{E} = \mathcal{E} \cup \{\beta_i\}$. Update $i = i + 1$, select a basis β_i from $\mathcal{U}$ and update $\mathcal{U} = \mathcal{U} \backslash \{\beta_i\}$. Calculate the corresponding basic solution $\boldsymbol{x}^i$ and update $\Pi B = \Pi B \cup \{\boldsymbol{x}^i\}$ Return to Step 3.

NUMERICAL EXPERIMENT

In several cases, a superset of the possibly optimal extreme point set is sufficient for the further discussion of the final solution. In such cases, it is conceivable to calculate a superset by Steuer's method together with an outer approximation of the polytope Γ by a box set $\hat{\Gamma}$. For the outer approximation, we should solve $2n$ linear programming problems,

$$c_i^{\mathrm{L}} = \min_{\boldsymbol{c} \in \Gamma} \; c_i, \qquad c_i^{\mathrm{R}} = \max_{\boldsymbol{c} \in \Gamma} \; c_i, \qquad i = 1, 2, \ldots, n. \tag{11}$$

Those $2n$ problems can be solved sequentially by a post-optimization technique of linear programming. For the box set case, the possibility optimality test corresponding to Problem (8) is easier since the problem has at least $(n - m)$ smaller number of constraints than Problem (8) or does not require the matrix calculation needed in Problem (10). Thus, this approach may be an alternative method of the exact enumeration of the possibly extreme point set. We compare the computation time of the proposed exact enumeration method to this superset enumeration method by a numerical experiment.

Using tangent hyperplanes of ellipsoids, we generate $\check{X} \subseteq \mathbf{R}_+^{n_1}$ and $\check{\Gamma} \subseteq \mathbf{R}^{n_1}$ in each trial. Namely, we have

$$\check{X} = \{\boldsymbol{z} \mid \check{A} \boldsymbol{z} \leq \boldsymbol{b}\} \qquad \check{\Gamma} = \{\boldsymbol{r} \mid \check{D} \boldsymbol{r} \leq \boldsymbol{g}\} \tag{12}$$

Table 1: Results of the numerical experiment

(n,m,p)	superset (A) average (sec.)	proposed (B) average (sec.)	ratio (A/B) average	superset (C) average card.	proposed (D) average card.	ratio (C/D) average
(15,10,10)	0.0825	0.0324	2.8044	49.4	14	4.1003
(15,10,15)	0.0870	0.0528	1.7680	48.7	20.2	2.7066
(15,10,20)	0.0824	0.0457	2.0697	41	14.3	3.9542
(20,15,10)	0.1768	0.0605	3.1973	79	22	3.9955
(20,15,15)	0.1526	0.0644	2.8597	66.6	21.6	3.8348
(20,15,20)	0.1721	0.0777	2.4363	74.6	22.4	3.8881
(25,20,10)	0.4013	0.0830	5.4510	133.7	25.5	5.7272
(25,20,15)	0.4512	0.1245	3.9348	147.1	33.9	4.8889
(25,20,20)	0.4019	0.1573	2.6780	129.4	38.4	3.5923
(30,20,20)	704.2847	31.352	57.5819	16700.2	1294.8	30.8150
(30,20,30)	670.4635	48.9237	19.5194	16418.1	1620.2	13.9287
(30,20,40)	677.2097	111.8198	8.6165	16498.4	2891.9	7.7136
(40,30,20)	4861.1364	72.4784	165.9479	49457.8	2546.8	46.1050

where $\check{A}$ and $\check{D}$ are $m \times n_1$ and $p \times n_1$ matrices. In order to obtain a problem in the form of Problem (1), we define

$$A = (\check{A}\ \ I), \qquad D = (\check{D}\ \ 0), \qquad \boldsymbol{x} = (\boldsymbol{z}^{\mathrm{T}}, \boldsymbol{s}^{\mathrm{T}})^{\mathrm{T}}, \qquad \boldsymbol{c} = (\boldsymbol{r}^{\mathrm{T}}, \boldsymbol{0}^{\mathrm{T}})^{\mathrm{T}} \tag{13}$$

and $n = n_1 + m$.

For each combination (m, n, p), we did 10 trials and measured the computation time (CPU time) in both approaches. We also count the number of enumerated extreme points (cardinality of the obtained superset and ΠB). The results are shown in Table 1. As shown in Table 1, the proposed method has a great advantage over the alternative approach of getting a superset. Indeed, we observed that the computation time of the proposed method is less than that of the alternative approach in all problems generated in our numerical experiment. From Table 1, the larger the problem size is, the more advantageous the proposed method is.

CONCLUSIONS

We extended Steuer's method for enumeration of possibly optimal extreme points in linear programming problems with an interval objective function to the case when the range of the objective function coefficient vector is represented as a polytope. We showed that the difference is only in the possible optimality test which is done by solving a linear programming problem. By a numerical experiment, we showed that the proposed enumeration method is more efficient than an alternative approach by enumerating a superset of the possibly optimal extreme point set.

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