CLASSIFICATION OF STATES FOR A POSSIBILISTIC MARKOV PROCESS

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Abstract. We consider the problem of classifying the states of a discrete possibilistic system for which the available information is specified by stationary one-step transition possibilities and initial possibilities. Therefore, we shall model this system by a possibilistic Markov process, i.e., a collection of variables with a common sample space satisfying a possibilistic counterpart of the Markov condition. We define the relation 'possibly leads to' and indicate how this relation can be used for dividing all states into disjoint 'possibilistic classes'. We then introduce the concept of 'possibilistic class property' and focus on two examples: the property of being possibilistically essential and the property of having a given possibilistic period. We furthermore define 'possibilistically recurrent' states. After deriving a number of alternative interpretations for the relation 'possibly leads to' and the possibilistic recurrence of a state, we indicate that finite possibilistically closed sets contain at least one possibilistically recurrent state.

1. Introduction

Possibility measures were proposed by Zadeh [18] for modelling information in natural language. Formally, a *possibility measure* Π on the power set of a nonempty set Ω is a supremum preserving set mapping on $\wp(\Omega)$ that assumes its values in the real unit interval $([0,1],\leq)$, that is, Π is a $\wp(\Omega)-[0,1]$-mapping such that

$$\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J}\Pi(A_j)$$

for any collection $(A_j \mid j\in J)$ of elements of $\wp(\Omega)$. The $\Omega-[0,1]$-mapping π such that $\pi(\omega)=\Pi(\{\omega\})$, $\forall\omega\in\Omega$ is called the *distribution* of Π. Obviously, Π is completely determined by π, since for any $E\in\wp(\Omega)$: $\Pi(E)=\sup_{\omega\in E}\pi(\omega)$. The distribution π is called *normal* if $\Pi(\Omega)=\sup_{\omega\in\Omega}\pi(\omega)=1$. A possibility measure with a normal distribution is also called a *normal* possibility measure. The triple $(\Omega,\wp(\Omega),\Pi_\Omega)$ is called a *possibility space*.

In this paper we will be concerned with discrete possibilistic systems, that is, systems for which the available information is given by possibility measures, or alternatively, by distributions of possibility measures, and that are having a countable time set. Provided that the observations lead to a consistent collection of distributions that are defined on finite Cartesian products, a possibilistic system can be modelled by a collection of variables with a common basic space. Using a possibilistic counterpart of the Daniell-Kolmogorov theorem [14, 13] all variables can be considered as *possibilistic variables* [5, 6], i.e., their behaviour is determined by a possibility measure. Similarly to a stochastic variable, a possibilistic variable has a *basic space* Ω and a *sample space* X. The available information is represented by a possibility measure Π_Ω on $(\Omega,\wp(\Omega))$. A $\Omega-X$-mapping f is called a possibilistic variable in X. The $X-[0,1]$-mapping π_f, given for any $x\in X$ by $\pi_f(x)=\Pi_\Omega(f^{-1}(\{x\}))$, is called the *possibility distribution function* of f. The *joint possibility distribution function* of a finite sequence $f_o,\dots,f_n$, $n\in\mathbb{N}$ of possibilistic variables, having basic space $(\Omega,\wp(\Omega),\Pi_\Omega)$ and sample spaces $X_o,\dots,X_n$, is given by:

$$\pi_{(f_o,\dots,f_n)}(x_o,\dots,x_n)=\Pi_\Omega(\cap_{i=0}^n f_i^{-1}(\{x_i\})),\quad \forall(x_o,\dots,x_n)\in X^{n+1}.$$

A collection of possibilistic variables having a common sample space X is called a *possibilistic process* in X.

In Section 2 we shall explain how a discrete possibilistic system that is specified by stationary one-step transition possibilities and initial possibilities can be represented by a possibilistic Markov process, i.e., a possibilistic process satisfying a possibilistic counterpart of the Markov condition. We show in Section 3 how the states of such a system can be divided into disjoint possibilistic classes. To achieve this, we introduce the relation 'possibly leads to'. We define possibilistically recurrent and possibilistically nonrecurrent states. By representing the available system information by a specific possibilistic Markov process, we shall derive a number

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of alternative interpretations for the previously mentioned notions. Finally, we indicate that finite possibilistic closed sets always contain a possibilistically recurrent state.

2. Discrete possibilistic systems

Consider a discrete possibilistic system having the set of all natural numbers $\mathbb{N}$ as its time set. It is furthermore assumed that $\mathbb{N}$ is ordered by the usual linear ordering $\leq$ of natural numbers.

Assume that the following information was gathered from the observations of the system:

- X is the set of all possible states for the system at any time $n \in \mathbb{N}$;
- *initial possibilities* $\overline{q}$, i.e., a $X - [0,1]$-mapping $\overline{q}$ such that $\overline{q}(x)$ is the possibility that the system is in state $x \in X$ at time 0;
- a possibilistic matrix $\overline{\boldsymbol{P}}$, i.e., a $X^2 - [0,1]$-mapping $\overline{\boldsymbol{P}}$ such that, for any couple $(x,y) \in X^2$, $\overline{\boldsymbol{P}}(x,y)$ denotes the *one-step stationary transition possibility* from state x at time $n \in \mathbb{N}$ to state y at time $n+1$, and that is normalised as follows:

$$\sup_{y \in X} \overline{\boldsymbol{P}}(x,y) = 1, \quad \forall x \in X. \tag{1}$$

Consequently, the partial mapping $\overline{\boldsymbol{P}}(x,\cdot)$ is the distribution of a unique, normal possibility measure on $(X, \wp(X))$ for every element $x \in X$. The mapping $\overline{q}$ can be taken as the distribution of a unique possibility measure $\overline{Q}$ on $(X, \wp(X))$.

Using the foregoing information we want to determine the possibility that the system visits a finite number of states $x_o, \ldots, x_n$, $n \in \mathbb{N}$ at the corresponding times $0, \ldots, n$. We are furthermore interested in determining the k-step transition possibilities of the system. First of all, we shall give a number of formulae for these possibilities, and then show how the formulae can be justified. To compute these values a triangular norm $\mathcal{T}$ on the real unit interval may be used. Recall that a triangular norm $\mathcal{T}$ is a nondecreasing, commutative and associative binary operator on the real unit interval $([0,1], \leq)$, with neutral element 1 and absorbing element 0 [16, 17].

Using $\mathcal{T}$ we define the k-step transition possibility from state $x \in X$ at time $n \in \mathbb{N}$ to state $y \in X$ at time $n+k$ as

$$\overline{\boldsymbol{P}}^k(x,y) = \begin{cases} \sup\limits_{\substack{(z_o,\ldots,z_k) \in X^{k+1} \\ z_o = x, z_k = y}} \mathcal{T}_{j=o}^{k-1} \overline{\boldsymbol{P}}(z_j, z_{j+1}) & \text{if } k \geq 2; \\ \overline{\boldsymbol{P}}(x,y) & \text{if } k = 1. \end{cases} \tag{2}$$

This means that the k-th power $\overline{\boldsymbol{P}}^k$ of the possibilistic matrix $\overline{\boldsymbol{P}}$ in the $(\sup, \mathcal{T})$-algebra is taken for the k-step transition possibilities of the system.

In a similar way we define the possibility $\pi_{\{0,\ldots,n\}}(x)$ that some 'joint state' $x = (x_o, \ldots, x_n) \in X^{n+1}$, $n \in \mathbb{N}$ is assumed by the system at the corresponding times $0, \ldots, n$ as the value:

$$\pi_{\{0,\ldots,n\}}(x) = \begin{cases} \mathcal{T}(\overline{q}(x_o), \mathcal{T}_{j=0}^{n-1} \overline{\boldsymbol{P}}(x_j, x_{j+1})) & \text{if } n \geq 1 \\ \overline{q}(x) & \text{if } n = 0. \end{cases} \tag{3}$$

Obviously, $\pi_{\{0,\ldots,n\}}$ can be considered as the distribution of a possibility measure on $(X^{n+1}, \wp(X^{n+1}))$.

If the triangular norm $\mathcal{T}$ that we are working with has left-continuous partial mappings, the resulting distributions $(\pi_{\{0,\ldots,n\}} \mid n \in (\mathbb{N}, \leq))$ satisfy the following natural consistency condition: if $(n,m) \in \mathbb{N}^2$ such that $n < m$, then

$$\pi_{\{0,\ldots,n\}}(x_o, \ldots, x_n) = \sup_{n+1 \leq j \leq m} \sup_{y_j \in X} \pi_{\{0,\ldots,m\}}(x_o, \ldots, x_n, y_{n+1}, \ldots, y_m) \quad \text{for all } (x_o, \ldots, x_n) \in X^{n+1}.$$

In that case, the possibilistic Daniell-Kolmogorov theorem [14, 13] guarantees the existence of a possibilistic process $(f_n \mid n \in (\mathbb{N}, \leq))$ in X such that

$$\pi_{(f_o,\ldots,f_n)} = \pi_{\{0,\ldots,n\}}, \quad \forall n \in \mathbb{N}. \tag{4}$$

To establish this, the following choices can be made:

- for f_n, $n \in \mathbb{N}$ take the projection operator $\mathbf{pr}_{\mathbb{N},n}$ that maps a sequence $x \in X^{\mathbb{N}}$ onto its $n+1$-th component x_n;

– for the basic space of $\mathbf{pr}_{\mathbb{N},n}$, $n \in \mathbb{N}$, take $(X^{\mathbb{N}}, \wp(X^{\mathbb{N}}), \Pi_{\mathbb{N},\overline{q}})$ where $\Pi_{\mathbb{N},\overline{q}}$ is the possibility measure with distribution $\pi_{\mathbb{N},\overline{q}}$ whose value in a sequence $x = (x_o, \dots, x_n, \dots) \in X^{\mathbb{N}}$ is given by

$$\pi_{\mathbb{N},\overline{q}}(x) = \inf_{n\in\mathbb{N}} \pi_{\{0,\dots,n\}}(x_o, \dots, x_n) = \inf_{n\in\mathbb{N}\setminus\{0\}} \mathcal{T}(\overline{q}(x_o), \mathcal{T}_{j=0}^{n-1}\overline{\boldsymbol{P}}(x_j, x_{j+1})).$$

Moreover, $\Pi_{\mathbb{N},\overline{q}}$ is the greatest (or least specific) possibility measure on the basic space $(X^{\mathbb{N}}, \wp(X^{\mathbb{N}}))$ for which the variables $\mathbf{pr}_{\mathbb{N},n}$, $n \in \mathbb{N}$ satisfy (4).

We now give a justification for the formulae (2)–(4). Any collection $(f_n \mid n \in (\mathbb{N}, \leq))$ of possibilistic variables representing the information $(\pi_{\{0,\dots,n\}} \mid n \in (\mathbb{N}, \leq))$ as expressed by (2)–(4) satisfies a possibilistic analogon of the Markov condition [9, 12]. To derive this result, we need to use the following notion of conditional possibility [3, 4, 5, 7]. Consider two possibilistic variables g_1, g_2 having sample spaces Y_1 and Y_2. Let $y_1 \in Y_1$. Then, for all $y_2 \in Y_2$, the *conditional possibility* $\pi_{g_1|g_2}(y_1 \mid y_2)$ that g_1 assumes the value y_1, given that g_2 assumes the value y_2, is a solution of the equation

$$\mathcal{T}(y, \pi_{g_2}(y_2)) = \pi_{(g_1,g_2)}(y_1, y_2) \text{ where } y \in [0,1]. \tag{5}$$

To ensure the existence of a solution of (5), it is sufficient that $\mathcal{T}$ have continuous partial mappings. The equation (5) may in general have more than one solution. In that case the least specific (or greatest) solution is often proposed for the conditional possibility $\pi_{g_1|g_2}(y_1 \mid y_2)$. Various authors [3, 4, 5] have argued that $\mathcal{T}$ should furthermore satisfy the cancellation law, or equivalently, should have strictly increasing partial mappings. A continuous triangular norm $\mathcal{T}$ having this additional property is also called a *strict* triangular norm and can be characterised as a ϕ-transform of the algebraic product [15]. More specifically this means: $\mathcal{T} = \mathcal{T}_\phi$, where ϕ is a continuous, strictly increasing – and therefore invertible – transformation of the real unit interval $[0,1]$ such that $\phi(0) = 0$ and $\phi(1) = 1$, and $\mathcal{T}_\phi$ is given by

$$\mathcal{T}_\phi(x, y) = \phi^{-1}(\phi(x)\phi(y)), \quad \forall(x, y) \in [0,1]^2.$$

Choosing a strict triangular norm for $\mathcal{T}$ ensures a unique solution of (5) provided that $\pi_{g_2}(y_2) > 0$.

Guided by this discussion, let us, for the sake of convenience, take the algebraic product for $\mathcal{T}$. Our discussion can be generalised, however, to more general, continuous, choices for $\mathcal{T}$. We obtain the following formulae for the conditional possibility:

$$_{DE}\pi_{g_1|g_2}(y_1 \mid y_2) = \begin{cases} \dfrac{\pi_{(g_1,g_2)}(y_1, y_2)}{\pi_{g_2}(y_2)} & \text{if } \pi_{g_2}(y_2) > 0; \\ 1 & \text{if } \pi_{g_2}(y_2) = 0; \end{cases}$$

where the least informative value is taken for $_{DE}\pi_{g_1|g_2}(y_1 \mid y_2)$ when $\pi_{g_2}(y_2) = 0$. As expected, Dempster's rule is recovered for conditioning the variables g_1 and g_2.

The following, obvious relation now holds between the transition possibilities and the conditional possibilities, formed with the possibilistic variables in the collection $(f_n \mid n \in (\mathbb{N}, \leq))$. Consider two natural numbers n and $k \neq 0$ and let $(x, y) \in X^2$, then it follows from (2)–(4) that

$$_{DE}\pi_{f_{n+k}|f_n}(y \mid x) = \overline{\boldsymbol{P}}^k(x, y) \quad \text{ if } \pi_{f_n}(x) > 0. \tag{6}$$

Also, the possibilistic variables $(f_n \mid n \in (\mathbb{N}, \leq))$ are furthermore conditionally independent in the following way. Consider a finite subset $\{n_i \mid i \in \{1, \dots, k\}\}$ of the time set $\mathbb{N}$ such that $k \in \mathbb{N} \setminus \{0\}$ and $n_1 < \dots < n_k$. Let $n \in \mathbb{N}$ such that $n_k < n$. If $x_{n_i} \in X$ for all $i \in \{1, \dots, k\}$ and $y \in X$, then

$$_{DE}\pi_{f_n|(f_{n_1},\dots,f_{n_k})}(y \mid (x_{n_1}, \dots, x_{n_k})) = {_{DE}\pi_{f_n|f_{n_k}}}(y \mid x_{n_k}) \quad \text{ if } \pi_{(f_{n_1},\dots,f_{n_k})}(x_{n_1}, \dots, x_{n_k}) > 0. \tag{M}$$

Condition (M) can be regarded as a possibilistic analogon of the Markov condition [9]. In [12] we used condition (M) as a starting point for developing a formal, measure-theoretic theory of possibilistic Markov families (processes), i.e., families of possibilistic variables satisfying property (M). This ends our justification of the formulae (2)–(4).

3. Classification of states

Let us reconsider the discrete possibilistic system introduced in the foregoing section, i.e., the discrete possibilistic system for which the following information is given:

– the partially ordered set of natural numbers $(\mathbb{N}, \leq)$ as time set;
– a nonempty set X as state space;
– a possibilistic matrix $\overline{\boldsymbol{P}}\colon X^2 - [0,1]$ specifying the stationary one-step transition possibilities of the system;
– initial possibilities $\overline{q}\colon X - [0,1]$.

In the sequel we shall denote by $\overline{\mathbb{N}}$ the set that is obtained by adding a top $+\infty$ to $(\mathbb{N}, \leq)$. The k-step transition possiblities of the system are again defined as the k-th power $\overline{\boldsymbol{P}}^k$ of the possibilistic matrix $\overline{\boldsymbol{P}}$ in the $(\sup, \mathcal{T})$-algebra, where a continuous triangular norm on $([0,1], \leq)$ is taken for the operator $\mathcal{T}$.

As explained in Section 2 it is possible to model the available system information by a possibilistic Markov process in X, allowing us to interpret the k-step transition possibilities as conditional possibilities. In this light the following definition of the binary relation 'possibly leads to' on the state space X is meaningful.

Definition 3.1. *Consider two states x and y in X, then*

- $x^{\rightarrow}y$, *i.e., x possibly leads to y if there exists a natural number $n > 0$ such that $\overline{\boldsymbol{P}}^n(x,y) > 0$;*
- $x^{\leftrightarrow}y$, *i.e., x possibly communicates with y if $x^{\rightarrow}y$ and $y^{\rightarrow}x$.*

A state x is called possibilistically essential if for any state y in X such that $x^{\rightarrow}y$ it follows that $y^{\rightarrow}x$, otherwise x is called possibilistically inessential.

Since $\overline{\boldsymbol{P}}$ is a possibilistic matrix satisfying condition (1), i.e.,

$$\sup_{y \in X} \overline{\boldsymbol{P}}(x,y) = 1, \quad \forall x \in X,$$

it is immediately clear that every state $x \in X$ possibly leads to some state $y \in X$. However, the relation $^{\rightarrow}$ is in general not reflexive nor symmetric. In the following proposition we derive a condition that is sufficient for the transitivity of $^{\rightarrow}$.

Proposition 3.2. *Assume that $\mathcal{T}$ has no zero divisors, i.e., $\neg(\exists(x,y) \in]0,1[^2)(\mathcal{T}(x,y) = 0)$. Then $^{\rightarrow}$ is transitive binary relation on X.*

If $\mathcal{T}$ has no zero divisors, then $^{\leftrightarrow}$ is symmetric and transitive. It is furthermore reflexive over the set of states that possibly communicate with some other state. Using $^{\leftrightarrow}$ we may divide the states into disjoint subsets called *possibilistic classes*. This goes as follows. Two states belong to the same possibilistic class if they possibly communicate. Each state that does not possibly communicate with any other state forms a possibilistic class by itself.

A property defined for all states is called a *possibilistic class property* if its possession by a state of a possibilistic class implies that it shared by all states of that possibilistic class. In that case the possibilistic class is said to have the given property. Obviously, the negation of a possibilistic class property is also a possibilistic class property.

The following proposition gives a first example of a possibilistic class property.

Proposition 3.3. *Assume that $\mathcal{T}$ has no zero divisors. Then a possibilistically essential state cannot lead to a possibilistically inessential state. In particular, the property of being possibilistically essential is a possibilistic class property. Moreover, $^{\rightarrow}$ is an equivalence relation on the possibilistically essential states in X.*

It can be shown to hold that the property of having a *possibilistic period* equal to $d \in \mathbb{N} \setminus \{0\}$ is also a possibilistic class property. More precisely, the possibilic period of a state $x \in X$ for which $x^{\rightarrow}x$ is defined as

$$\overline{d}(x) = \gcd\{n \mid n \in \mathbb{N} \setminus \{0\} \text{ such that } \overline{\boldsymbol{P}}^n(x,x) > 0\},$$

i.e., the greatest common divisor of the set of natural numbers $n \in \mathbb{N} \setminus \{0\}$ for which $\overline{\boldsymbol{P}}^n(x,x) > 0$. To achieve this result, it is once again sufficient to take a triangular norm $\mathcal{T}$ that has no zero divisors.

To define *possibilistically recurrent* and *possibilistically nonrecurrent* states, we need the following mappings. For any state $a \in X$, we denote by $\boldsymbol{T}_a$ the *hitting time* of a. Formally, $\boldsymbol{T}_a$ is the $X^{\mathbb{N}} - \overline{\mathbb{N}}$-mapping given for any sequence $x = (x_o, \dots, x_n, \dots) \in X^{\mathbb{N}}$ by

$$\boldsymbol{T}_a(x) = \begin{cases} \inf\{n \mid n \in \mathbb{N} \setminus \{0\} \text{ such that } x_n = a\} & \text{if } \exists n \in \mathbb{N} \setminus \{0\} \text{ such that } x_n = a; \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

We now want to determine for every couple of states $(a,b) \in X^2$ the possibility $\overline{P}_{ab}$ that the system will be in b at least once, given that it starts from a, or equivalently, the possibility of the event $\{\boldsymbol{T}_b < +\infty\}$, given that the system starts from a. To compute these values we need to represent the available system information by a possibilistic Markov process. As explained in Section 2 we may therefore take the projection operators $(\mathbf{pr}_{\mathbb{N},n} \mid n \in \mathbb{N})$, provided that the behaviour of these variables is determined by the possibility measure $\Pi_{\mathbb{N},\overline{q}}$ on $(X^{\mathbb{N}}, \wp(X^{\mathbb{N}}))$ with distribution $\pi_{\mathbb{N},\overline{q}}$ given by

$$\pi_{\mathbb{N},\overline{q}}(x) = \inf_{n \in \mathbb{N} \setminus \{0\}} \mathcal{T}(\overline{q}(x_o), \mathcal{T}_{j=0}^{n-1} \overline{\boldsymbol{P}}(x_j, x_{j+1})) \quad \text{for all } x = (x_o, \dots, x_n, \dots) \in X^{\mathbb{N}}.$$

Since $\mathbf{pr}_{\mathbb{N},n}$, $n \in \mathbb{N}$ is the projection operator on $X^{\mathbb{N}}$ that maps any sequence $x = (x_o, \ldots, x_n, \ldots) \in X^{\mathbb{N}}$ onto its $n+1$-component x_n, (7) can be rewritten as:

$$\boldsymbol{T}_a(x) = \begin{cases} \inf\{n \mid n \in \mathbb{N} \setminus \{0\} \text{ such that } \mathbf{pr}_{\mathbb{N},n}(x) = a\} & \text{if } \exists n \in \mathbb{N} \setminus \{0\} \text{ such that } \mathbf{pr}_{\mathbb{N},n}(x) = a; \\ +\infty & \text{otherwise.} \end{cases}$$

The hitting time $\boldsymbol{T}_b$ of b can be considered as a possibilistic variable in $\overline{\mathbb{N}}$ that is dependent of the possibilistic variables $\mathbf{pr}_{\mathbb{N},n}$, $n \in \mathbb{N} \setminus \{0\}$. Consequently, its behaviour is also determined by $\Pi_{\mathbb{N},\overline{q}}$. In this way we obtain the following formula:

$$\overline{P}_{ab} = \Pi_{\mathbb{N},\overline{q}_a}(\{\boldsymbol{T}_b < +\infty\}),$$

where the initial possibilities $\overline{q}_a$ are given by

$$\overline{q}_a(x) = \begin{cases} 1 & \text{if } x = a; \\ 0 & \text{if } x \in X \setminus \{a\}. \end{cases}$$

Obviously, the possibility measure associated with $\overline{q}_a$ is a probability measure telling us that is completely sure that system starts from a.

By a short computation it turns out that $\overline{P}_{ab}$ is equal to

$$\overline{P}_{ab} = \sup_{n \geq 1} \overline{\boldsymbol{P}}^n(a, b).$$

This brings us to the following characterisation of the relation $^{\rightarrow}$.

Proposition 3.4. *For every couple of states $(a, b) \in X^2$ we have:*

$$a^{\rightarrow}b \quad \textit{iff} \quad \overline{P}_{ab} > 0.$$

We now call a state $a \in X$ *possibilistically recurrent* when it is completely possible that the system will eventually return to a, given that it starts from a.

Definition 3.5. *A state $a \in X$ is called possibilistically recurrent if $\overline{P}_{aa} = 1$, otherwise, it is called possibilistically nonrecurrent.*

To derive a second alternative definition of the relation $^{\rightarrow}$, we introduce the following variables. Consider a state $b \in X$. Then $\boldsymbol{N}_b$ denotes the $X^{\mathbb{N}} - \overline{\mathbb{N}}$-mapping $\boldsymbol{N}(b)$ that indicates for every sequence of states $x = (x_o, \ldots, x_n, \ldots) \in X^{\mathbb{N}}$ the number of times $n \in \mathbb{N} \setminus \{0\}$ that the system is in b, i.e.,

$$\boldsymbol{N}(b)(x) = |\{n \mid n \in \mathbb{N} \setminus \{0\} \text{ and } x_n = b\}|.$$

We naturally want to determine the 'expected' number of visits to b, given that the system starts from some state $a \in X$. As before, we consider $\boldsymbol{N}(b)$ as a possibilistic variable in $\overline{\mathbb{N}}$ with basic space $(X^{\mathbb{N}}, \wp(X^{\mathbb{N}}), \Pi_{\mathbb{N},\overline{q}_a})$. To calculate the requested expected value we may for instance use the Choquet-integral (C) $\int \boldsymbol{N}(b) \mathrm{d}\Pi_{\mathbb{N},\overline{q}_a}$ of $\boldsymbol{N}(b)$ with respect to the possibility measure $\Pi_{\mathbb{N},\overline{q}_a}$, which is equal to the indefinite Riemann-integral [1, 8]

$$\text{(C)} \int \boldsymbol{N}(b) \mathrm{d}\Pi_{\mathbb{N},\overline{q}_a} = \int_0^{+\infty} \Pi_{\mathbb{N},\overline{q}_a}(\{\boldsymbol{N}(b) \geq t\}) \mathrm{d}t. \tag{8}$$

For every natural number $n \in \mathbb{N} \setminus \{0\}$ the possibility that the system will be in b at least n times, given that it starts from a, is equal to

$$\Pi_{\mathbb{N},\overline{q}_a}(\{\boldsymbol{N}(b) \geq n\}) = \begin{cases} \overline{P}_{ab} & \text{if } n = 1; \\ \mathcal{T}(\overline{P}_{ab}, \mathop{\mathcal{T}}_{j=1}^{n-1} \overline{P}_{bb}) & \text{if } n > 1. \end{cases} \tag{9}$$

Taking (8) and (9) into account it is readily seen that

$$\text{(C)} \int \boldsymbol{N}(b) \mathrm{d}\Pi_{\mathbb{N},\overline{q}_a} = \sum_{n \geq 1} \mathcal{T}(\overline{P}_{ab}, \mathcal{T}_{j=1}^{n-1} \overline{P}_{bb}), \tag{10}$$

Using the result of Proposition 3.4 we may restate the definition of $^{\rightarrow}$ as follows.

Proposition 3.6. *For every couple of states $(a, b) \in X^2$ we have:*

$$a^{\rightarrow}b \quad \textit{iff} \quad \text{(C)} \int \boldsymbol{N}(b) \mathrm{d}\Pi_{\mathbb{N},\overline{q}_a} > 0.$$

Example 3.7.

– If b is a possibilistically recurrent state, then we find for any state $a \in X$:

$$(\mathrm{C})\int \boldsymbol{N}(b)\mathrm{d}\Pi_{\mathcal{T},\overline{q}_a} = \begin{cases} 0 & \text{if } \overline{P}_{ab} = 0; \\ +\infty & \text{if } \overline{P}_{ab} > 0. \end{cases}$$

– Suppose that b is a possibilistically nonrecurrent state.
  - If $\mathcal{T}$ is the algebraic product on $([0,1],\leq)$, then we find for any state $a \in X$:

$$(\mathrm{C})\int \boldsymbol{N}(b)\mathrm{d}\Pi_{\mathcal{T},\overline{q}_a} = \frac{\overline{P}_{ab}}{1-\overline{P}_{bb}}.$$

  - If $\mathcal{T}$ is the minimum operator on $([0,1],\leq)$, then we find for any state $a \in X$:

$$(\mathrm{C})\int \boldsymbol{N}(b)\mathrm{d}\Pi_{\mathbb{N},\overline{q}_a} = \begin{cases} 0 & \text{if } \overline{P}_{ab} = 0; \\ \overline{P}_{ab} & \text{if } \overline{P}_{ab} > 0 = \overline{P}_{bb}; \\ +\infty & \text{if } \min(\overline{P}_{ab},\overline{P}_{bb}) > 0. \end{cases}$$

$\triangle$

For every couple of states $(a,b) \in X^2$ we now determine the possibility that the system will be at infinitely many times $n \in \mathbb{N}\setminus\{0\}$ in b, given that it starts from a. The calculation of these possibilities, carried out in the next proposition, leads us to the following, alternative characterisation of possibilistic recurrence.

Proposition 3.8. *For every couple of states $(a,b) \in X^2$ we have:*

– *if $\mathcal{T}$ is the mininum operator on $([0,1],\leq)$, then*

$$\Pi_{\mathbb{N},\overline{q}_a}(\{\boldsymbol{N}(b) = +\infty\}) = \overline{P}_{ab} \wedge \overline{P}_{bb};$$

– *if $\mathcal{T}$ is a continuous Archimedean triangular norm on $([0,1],\leq)$, i.e., a continuous triangular norm $\mathcal{T}$ on $([0,1],\leq)$ such that $\mathcal{T}(x,x) < x$, $\forall x \in]0,1[$, then*

$$\Pi_{\mathbb{N},\overline{q}_a}(\{\boldsymbol{N}(b) = +\infty\}) = \begin{cases} \overline{P}_{ab} & \text{if } \overline{P}_{bb} = 1 \\ 0 & \text{if } \overline{P}_{bb} < 1. \end{cases}$$

In both cases we have: $a \in X$ is a possibilistically recurrent state iff $\Pi_{\mathbb{N},\overline{q}_a}(\{\boldsymbol{N}(a) = +\infty\})$.

From the foregoing proposition it is immediately clear that, when a continuous Archimedean triangular norm – for instance the algebraic product – is taken for $\mathcal{T}$, the possibility $\Pi_{\mathbb{N},\overline{q}_a}(\{\boldsymbol{N}(a) = +\infty\})$, $a \in X$ is either 1 or 0 according as a is possibilistically recurrent or not.

Let us now introduce the notion of a possibilistically closed set.

Definition 3.9. *A nonempty subset C of X is called possibilistically closed if no state belonging to C possibly leads to a state outside C, i.e., if $x \in C$ and $y \in X \setminus C$, then $\neg(x^{\rightarrow}y)$.*

A well-known result of the theory of stochastic Markov processes is recovered [2, 10].

Proposition 3.10. *A non-empty, finite, possibilistically closed subset C of X contains at least one possibilistically recurrent state.*

4. Conclusion

We have described a method of classifying the states of a discrete possibilistic system specified by stationary one-step transition possibilities and initial possibilities. To justify the formulae and the notions required for this classification we have used a specific possibilistic Markov process for representing the the available system information. In this way we have recovered a number of results that are well-known in the theory of stochastic Markov processes.

To formulate our notion of a possibilistic Markov process we have made use of one particular definition of conditional possibility introduced in the ordinal version of possibility theory. In a recent paper we have shown that the initial possibilities, transition possibilities, etc., may be given a behavioral interpretation as marginal betting rates against events, when the 'conditional possibilities' are calculated with Dempster's conditioning rule [11]. In a forthcoming paper we shall give a more detailed discussion including the proofs of the results.

References

[1] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953–1954.

[2] Kai Lai Chung. *Markov Chains*. Springer Verlag, New York, 1967.

[3] B. De Baets, E. Tsiporkova, and R. Mesiar. Conditioning in possibility theory with strict order norms. *Fuzzy Sets and Systems*. submitted for publication.

[4] B. De Baets, E. Tsiporkova, and R. Mesiar. The surprising possibilistic nature of the algebraic product. In H.-J. Zimmermann, editor, *Proceedings of EUFIT '96*, volume 1, pages 549–553, Aachen, 1996.

[5] G. de Cooman. *Evaluatieverzamelingen en -afbeeldingen - Een ordetheoretische benadering van vaagheid en onzekerheid [Evaluation Sets and Mappings - An Order-Theoretic Approach to Vagueness and Uncertainty]*. PhD thesis, Universiteit Gent, Ghent, 1993.

[6] G. de Cooman. Possibility theory I: the measure- and integral-theoretic groundwork. *International Journal of General Systems*, 25:291–323, 1997.

[7] G. de Cooman. Possibility theory II: conditional possibility. *International Journal of General Systems*, 25:325–351, 1997.

[8] D. Denneberg. *Non-Additive Measure and Integral*. Kluwer, Dordrecht, 1994.

[9] J. L. Doob. *Stochastic Processes*. John Wiley & Sons, New York, 1967.

[10] David Freedman. *Markov Chains*. Holden-Day, San Francisco, 1971.

[11] H. J. Janssen, G. de Cooman, and E. E. Kerre. Coherent models for discrete possibilistic systems. In *Proceedings of the First International Symposium on Imprecise Probabilities and Their Applications (ISIPTA '99, Ghent, Belgium, 29 June - 2 July, 1999)*. accepted for publication.

[12] H. J. Janssen, G. de Cooman, and E. E. Kerre. First results for a mathematical theory of possibilistic markov processes. In *Proceedings of IPMU '96 (Information Processing and Management of Uncertainty in Knowledge Based Systems, Granada, Spain, July 1-5, 1996)*, volume III, pages 1425–1431, 1996.

[13] H. J. Janssen, G. de Cooman, and E. E. Kerre. Some remarks on stationary possibilistic processes. In *Prooceedings of the 3rd International FLINS Workshop (Antwerp, Belgium, September 14-16, 1998)*, pages 52–60, 1998.

[14] H. J. Janssen, G. de Cooman, and E. E. Kerre. A possibilistic Daniell-Kolmogorov theorem for supremum preserving upper probabilities. *Fuzzy Sets and Systems*, 102, 1999.

[15] P. E. Klement and R. Mesiar. Triangular norms. *Tatra Mountains Math. Publ.*, 13:169–193, 1997.

[16] B. Schweizer and A. Sklar. Associative functions and abstract semi-groups. *Publ. Math. Debrecen*, 10:69–84, 1963.

[17] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces*. Elsevier Science Publishing Company, New York, 1983.

[18] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.

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