

Different Approaches towards Fuzzy Mathematical Morphology

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Abstract

Fuzzy mathematical morphology is an alternative extension of binary morphology to gray-scale morphology, using techniques from fuzzy set theory. In this paper we will discuss several approaches and show how they are connected, not only to each other, but also to binary and classical gray-scale morphology. We review the general logical framework using the notions of conjunctor and implicator, we discuss the model of Bloch and Maître, we discuss an approach starting from a fuzzification of the Minkowski addition, and we introduce some approaches starting from different fuzzifications of set inclusion. In particular, the fuzzified set inclusions of Zadeh, Sinha & Dougherty, Kitainik and Bandler & Kohout will be applied.

KEYWORDS: Image processing, Fuzzy mathematical morphology, Fuzzified set inclusion.

1 Introduction

Mathematical morphology plays an important role in image processing, and is used to analyse the shape of n -dimensional images. These images can be binary images (represented as subsets of \mathbf{R}^n) or gray-scale images (represented as \mathbf{R}^n - $[0, 1]$ functions). A morphological operation P transforms an image A by means of a structuring element B into a new object $P(A, B)$. The basic morphological operations are dilation, erosion, closing and opening. Here, we will restrict ourselves to dilation and erosion. For detailed comments and proofs we refer to Nachtegaele & Kerre (1998).

The paper is organized as follows: in section 2 we briefly review binary and gray-scale morphology, in section 3 we discuss different approaches towards fuzzy morphology, and in section 4 we give some conclusions and final remarks.

2 Binary and Gray-Scale Mathematical Morphology

We adopt the definitions of Serra (1982). Recall that the translation $T_y(B)$ of a binary object B by the vector $y \in \mathbf{R}^n$ is defined as $T_y(B) = \{x \in \mathbf{R}^n | x \Leftrightarrow y \in B\}$.

Definition 1 *Let A and B be binary objects. The binary dilation $D(A, B)$ and the binary erosion $E(A, B)$ are the binary objects given by:*

$$D(A, B) = \{y | T_y(B) \cap A \neq \emptyset\}, \quad (1)$$

$$E(A, B) = \{y | T_y(B) \subseteq A\}. \quad (2)$$

Binary morphology has been successfully extended to gray-scale images. There are two different approaches, namely the *threshold* approach (only allowing binary structuring elements) and the *umbra* approach (also allowing gray-scale structuring elements).

The following morphological operations are based on the threshold approach, and are referred to as t-operations. Recall that the domain of an arbitrary \mathbf{R}^n - \mathbf{R} function F is given by $d_F = \{x | x \in \mathbf{R}^n \text{ and } (\exists t \in \mathbf{R})(F(x) = t)\}$.

Definition 2 Let A be a gray-scale object, and let B be a binary object. The t -dilation $D_t(A, B)$ and the t -erosion $E_t(A, B)$ are the gray-scale objects given by:

$$D_t(A, B)(y) = \sup_{x \in T_y(B) \cap d_A} A(x), \quad \forall y \in D(d_A, B), \quad (3)$$

$$E_t(A, B)(y) = \inf_{x \in T_y(B)} A(x), \quad \forall y \in E(d_A, B). \quad (4)$$

The following morphological operations are based on the umbra approach, and are referred to as u-operations. Note that these operations do not necessarily produce \mathbf{R}^n - $[0, 1]$ functions. In practice we will have to transform the obtained result in order to get a \mathbf{R}^n - $[0, 1]$ function.

Definition 3 Let A and B be gray-scale objects. The u -dilation $D_u(A, B)$ and the u -erosion $E_u(A, B)$ are the gray-scale objects given by:

$$D_u(A, B)(y) = \sup_{x \in T_y(d_B) \cap d_A} A(x) + B(x \Leftrightarrow y), \quad \forall y \in D(d_A, d_B) \quad (5)$$

$$E_u(A, B)(y) = \inf_{x \in T_y(d_B)} A(x) \Leftrightarrow B(x \Leftrightarrow y), \quad \forall y \in E(d_A, d_B). \quad (6)$$

It is just a matter of verification to see that the t-operations are extensions of their binary counterparts. The relation between t-operations and u-operations is given in the following proposition.

Proposition 1 Let A be a gray-scale object and let B be a binary structuring element, then it holds:

- (i) $D_u(A, B) = D_t(A, B) + 1$,
- (ii) $E_u(A, B) = E_t(A, B) \Leftrightarrow 1$.

3 Fuzzy Mathematical Morphology

3.1 The general logical framework

From the definitions of dilation and erosion it is clear that intersection and inclusion play an important role. To obtain a fuzzification, the idea of De Baets (1997) was to fuzzify the underlying logical operations.

Definition 4 A unary operator \mathcal{N} on $[0, 1]$ is a negator if it is a decreasing mapping that coincides with the Boolean negation on $\{0, 1\}$, i.e. $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. \mathcal{N} is called involutive if $(\forall x \in [0, 1])(\mathcal{N}(\mathcal{N}(x)) = x)$.

The standard negator N_s , defined by $N_s(x) = 1 \Leftrightarrow x$, is an involutive negator.

Definition 5 A binary operator \mathcal{C} on $[0, 1]$ is a conjunctor if it is an increasing mapping (i.e. a mapping with increasing partial mappings) that coincides with the Boolean conjunction on $\{0, 1\}^2$, i.e. $\mathcal{C}(0, 0) = \mathcal{C}(0, 1) = \mathcal{C}(1, 0) = 0$ and $\mathcal{C}(1, 1) = 1$. It is a semi-norm if it also satisfies $(\forall x \in [0, 1])(\mathcal{C}(1, x) = \mathcal{C}(x, 1) = x)$, and a semi-norm is called a t -norm if it is commutative and associative.

The minimum M , the algebraic product P and the Lukasiewicz t -norm W are very popular t -norms. They are given by $M(x, y) = \min(x, y)$, $P(x, y) = x \cdot y$ and $W(x, y) = \max(0, x + y \Leftrightarrow 1)$.

Definition 6 A binary operator \mathcal{I} on $[0,1]$ is an *implicator* if it is a hybrid monotonic mapping (i.e. with decreasing first and increasing second partial mappings) that coincides with the Boolean implication on $\{0,1\}^2$, i.e. $\mathcal{I}(0,0) = \mathcal{I}(0,1) = \mathcal{I}(1,1) = 1$ and $\mathcal{I}(1,0) = 0$. It is a *border implicator* if it also satisfies $(\forall x \in [0,1])(\mathcal{I}(0,x) = 1 \text{ and } \mathcal{I}(1,x) = x)$, and a border implicator is called a *model implicator* if it is contrapositive w.r.t. its induced negator (i.e. $(\forall (x,y) \in [0,1]^2)(\mathcal{I}(x,y) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(y), \mathcal{N}_{\mathcal{I}}(x)))$) and if it fulfills the exchange principle (i.e. $(\forall (x,y,z) \in [0,1]^3)(\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z)))$). Recall that the induced negator is given by $\mathcal{N}_{\mathcal{I}}(x) = \mathcal{I}(x, 0)$, for all $x \in [0,1]$.

The Łukasiewicz implicator I_L , the Kleene-Dienes implicator I_{KD} and the Reichenbach implicator I_R are well-known model implicators. They are given by $I_L(x,y) = \min(1, 1 \Leftrightarrow x + y)$, $I_{KD}(x,y) = \max(1 \Leftrightarrow x, y)$ and $I_R(x,y) = 1 \Leftrightarrow x + x \cdot y$.

Given a conjunctor one can construct an implicator from it, and vice versa. Indeed, let \mathcal{N} , \mathcal{C} and \mathcal{I} be respectively a negator, conjunctor and implicator on $[0,1]$. Then the operators $\mathcal{C}_{\mathcal{I},\mathcal{N}}$ and $\mathcal{I}_{\mathcal{C},\mathcal{N}}$ defined by $\mathcal{C}_{\mathcal{I},\mathcal{N}}(x,y) = \mathcal{N}(\mathcal{I}(x, \mathcal{N}(y)))$ and $\mathcal{I}_{\mathcal{C},\mathcal{N}}(x,y) = \mathcal{N}(\mathcal{C}(x, \mathcal{N}(y)))$ are a conjunctor, respectively an implicator on $[0,1]$. As an example, one easily verifies that $\mathcal{I}_{M,N_s} = I_{KD}$, $\mathcal{I}_{P,N_s} = I_R$ and $\mathcal{I}_{W,N_s} = I_L$.

Definition 7 Let A and B be gray-scale objects, let \mathcal{C} be a conjunctor and let \mathcal{I} be an implicator on $[0,1]$. The fuzzy dilation $D_{\mathcal{C}}(A,B)$ and the fuzzy erosion $E_{\mathcal{I}}(A,B)$ are the gray-scale objects given by:

$$D_{\mathcal{C}}(A,B)(y) = \sup_{x \in T_y(d_B) \cap d_A} \mathcal{C}(B(x \Leftrightarrow y), A(x)), \quad \forall y \in D(d_A, d_B) \quad (7)$$

$$E_{\mathcal{I}}(A,B)(y) = \inf_{x \in T_y(d_B)} \mathcal{I}(B(x \Leftrightarrow y), A(x)), \quad \forall y \in E(d_A, d_B). \quad (8)$$

Recall that two fuzzy morphological operations P and Q are called \mathcal{N} -dual if for any two fuzzy sets $A, B \in \mathcal{F}(\mathbf{R}^n)$ it holds that $P(A,B) = co_{\mathcal{N}} Q(co_{\mathcal{N}} A, B)$, with $(co_{\mathcal{N}} A)(x) = \mathcal{N}(A(x))$.

Proposition 2 Let \mathcal{N} be an involutive negator, let \mathcal{C} be a conjunctor and let \mathcal{I} be an implicator on $[0,1]$. Then the fuzzy dilation $D_{\mathcal{C}}$ and the fuzzy erosion $E_{\mathcal{I}}$ are \mathcal{N} -dual operations if and only if $\mathcal{I} = \mathcal{I}_{\mathcal{C},\mathcal{N}}$, or equivalently $\mathcal{C} = \mathcal{C}_{\mathcal{I},\mathcal{N}}$.

One easily verifies that the fuzzy dilation $D_{\mathcal{C}}$ and the fuzzy erosion $E_{\mathcal{I}}$ are extensions of their binary counterparts. It also turns out that the t-dilation and the t-erosion can be considered as special instances of fuzzy dilations and fuzzy erosions.

Proposition 3 Let A be a gray-scale object, and let B be a binary object. If \mathcal{C} is a semi-norm and \mathcal{I} is a border implicator, then it holds:

- (i) $D_{\mathcal{C}}(A,B) = D_t(A,B)$,
- (ii) $E_{\mathcal{I}}(A,B) = E_t(A,B)$.

In general, the u-dilation $D_u(A,B)$ is a \mathbf{R}^n - $[0,2]$ function, while the u-erosion $E_u(A,B)$ is a \mathbf{R}^n - $[\Leftrightarrow 1,1]$ function. However, there are several ways to transform these functions into \mathbf{R}^n - $[0,1]$ functions. Regarding the u-dilation, we could subtract $D_u(A,B)(y)$ by 1 (obtaining a value in $[-1,1]$), and then take the maximum of this new value and zero (obtaining a value in $[0,1]$). If we denote this modified u-dilation as D_u^* , then we have:

$$D_u^*(A,B)(y) = \sup_{x \in T_y(d_B) \cap d_A} \max(0, A(x) + B(x \Leftrightarrow y) \Leftrightarrow 1), \quad \forall y \in D(d_A, d_B).$$

In an analogous way, we obtain the modified u-erosion E_u^* , given by $E_u^*(A,B)(y) = \inf_{x \in T_y(d_B)} \min(1, A(x) \Leftrightarrow B(x \Leftrightarrow y) + 1)$. We immediately have the following result.

Proposition 4 Let A and B be two gray-scale objects, then it holds:

- (i) $D_W(A,B) = D_u^*(A,B)$,
- (ii) $E_{I_L}(A,B) = E_u^*(A,B)$.

We note that the modified u-operations can show a significant loss of information w.r.t. the original u-operations. However, both types of operations are equivalent if the structuring element B is normalized, i.e. if $(\exists z \in \mathbf{R}^n)(B(z) = 1)$.

Proposition 5 *Let A and B be two gray-scale objects. If the structuring element B is normalized, then it holds:*

- (i) $D_W(A, B) = D_u(A, B) \Leftrightarrow 1$,
- (ii) $E_{I_L}(A, B) = E_u(A, B) + 1$.

3.2 The approach of Bloch and Maitre

Bloch and Maitre (1995) followed a similar approach to define the fuzzy dilation and the fuzzy erosion. There are however two important differences. First, they use a t-norm \mathcal{T} instead of a conjunctor to define the fuzzy dilation, and secondly they use the associated model implicator $\mathcal{I}_{\mathcal{T}, \mathcal{N}}$ (with \mathcal{N} an involutive negator) to define the fuzzy erosion. This implies that duality is forced upon the fuzzy dilation and erosion from the beginning, whereas this is not the case in the general framework. Furthermore, several other approaches towards fuzzy morphology do not fit in this framework (see section 3.4).

3.3 The approach of Minkowski addition

If A and B are two subsets of \mathbf{R}^n , then their Minkowski sum $A \oplus B$ is defined as $A \oplus B = \cup_{b \in B} T_b(A)$. One easily verifies that $A \oplus B = D(A, \Leftrightarrow B)$. De Baets, Kerre & Gupta (1995) noticed that this suggests another way to construct a fuzzy mathematical morphology. Indeed, let A and B be two fuzzy sets in \mathbf{R}^n , then the fuzzy Minkowski addition of A and B is the fuzzy set given by $(A \oplus B)(y) = \sup_{x \in \mathbf{R}^n} \min(B(x), A(y \Leftrightarrow x))$. The fuzzy dilation $D_{\oplus}(A, B)$ can then be defined as $D_{\oplus}(A, B)(y) = (A \oplus (\Leftrightarrow B))(y)$, resulting in:

$$D_{\oplus}(A, B)(y) = \sup_{x \in T_y(d_B) \cap d_A} \min(B(x \Leftrightarrow y), A(x)), \quad \forall y \in D(d_A, d_B).$$

The fuzzy erosion $E_{\oplus}(A, B)$ can then be defined as the N_s -dual of the fuzzy dilation $D_{\oplus}(A, B)$.

Proposition 6 *Let A and B be two gray-scale objects, then it follows that:*

- (i) $D_{\oplus}(A, B) = D_M(A, B)$,
- (ii) $E_{\oplus}(A, B) = E_{I_{KD}}(A, B)$.

3.4 The approach of subset inclusion

Given a fuzzified set inclusion Inc , we can use it to extend the binary erosion to an operation on fuzzy sets in \mathbf{R}^n by putting $E_{Inc}(A, B)(y) = Inc(T_y(B), A)$. Having this fuzzy erosion, the fuzzy dilation will be defined by duality w.r.t. the standard negator.

3.4.1 Fuzzy set inclusion of Zadeh

Zadeh (1965) proposed the following definition.

Definition 8 *Let A and B be two fuzzy sets in \mathbf{R}^n . The fuzzy set inclusion \subseteq_z of Zadeh is defined as:*

$$A \subseteq_z B \Leftrightarrow A(x) \leq B(x), \quad \forall x \in \mathbf{R}^n.$$

A major drawback of this definition is that it doesn't allow any degree of subsetness (a fuzzy set is a subset of another fuzzy set, or it isn't). This also implies that the corresponding "fuzzy" dilation and erosion will be crisp.

Definition 9 *The Zadeh-dilation $D_z(A, B)$ and the Zadeh-erosion $E_z(A, B)$ are the binary objects defined by:*

$$y \in D_z(A, B) \Leftrightarrow \sup_{x \in T_y(d_B) \cap d_A} A(x) + B(x \Leftrightarrow y) > 1, \quad (9)$$

$$y \in E_z(A, B) \Leftrightarrow \inf_{x \in T_y(d_B)} A(x) \Leftrightarrow B(x \Leftrightarrow y) \geq 0. \quad (10)$$

We immediately have the following connection with the u-dilation and the u-erosion.

Proposition 7 Let A and B be two gray-scale objects, then it holds:

- (i) $y \in D_z(A, B) \Leftrightarrow D_u(A, B)(y) > 1$,
- (ii) $y \in E_z(A, B) \Leftrightarrow E_u(A, B)(y) \geq 0$.

The Zadeh-operations can also be linked to the general framework. In order to do so, we define the conjunctor C_z and the implicator I_z as follows:

$$C_z(x, y) = \begin{cases} 1 & \text{if } x + y > 1 \\ 0 & \text{else} \end{cases} \quad I_z(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{else} \end{cases}, \quad \forall (x, y) \in [0, 1]^2.$$

Proposition 8 Let A and B be two gray-scale objects, then it holds:

- (i) $D_z(A, B) = D_{C_z}(A, B)$,
- (ii) $E_z(A, B) = E_{I_z}(A, B)$.

3.4.2 Fuzzy Set Inclusion of Sinha and Dougherty

Sinha & Dougherty (1993) introduced a general indicator Inc_{SD} for fuzzified set inclusion, based on nine intuitive acceptable axioms. They propose the following indicator:

$$Inc_\lambda(A, B) = \inf_{x \in \mathbf{R}^n} \min(1, \lambda(A(x)) + \lambda(1 \Leftrightarrow B(x))), \quad (11)$$

where λ is a $[0, 1] \Leftrightarrow [0, 1]$ function satisfying a set of properties (e.g. λ must be decreasing, $\lambda(0) = 1$ and $\lambda(1) = 0$). Some examples of possible λ -functions are given by $\lambda_n(x) = 1 \Leftrightarrow x^n$ (with $n \geq 1$), and $\lambda_n(x) = \frac{1-x}{1+nx}$ (with $n \in]\Leftrightarrow 1, 0[$). Using the indicator Inc_λ , we obtain the following fuzzification of the binary dilation and erosion, see also Sinha & Dougherty (1995).

Definition 10 The λ -dilation $D_\lambda(A, B)$ and the λ -erosion $E_\lambda(A, B)$ are the gray-scale objects given by:

$$D_\lambda(A, B)(y) = \sup_{x \in T_y(d_B) \cap d_A} \max(0, 1 \Leftrightarrow \lambda(B(x \Leftrightarrow y)) + \lambda(A(x))), \quad \forall y \in D(d_A, d_B), \quad (12)$$

$$E_\lambda(A, B)(y) = \inf_{x \in T_y(d_B)} \min(1, \lambda(B(x \Leftrightarrow y)) + \lambda(1 \Leftrightarrow A(x))), \quad \forall y \in E(d_A, d_B). \quad (13)$$

Now, define the conjunctor C_λ and the implicator I_λ as follows:

$$\begin{aligned} C_\lambda(x, y) &= \max(0, 1 \Leftrightarrow \lambda(x) \Leftrightarrow \lambda(y)), \\ I_\lambda(x, y) &= \min(1, \lambda(x) + \lambda(1 \Leftrightarrow y)), \quad \forall (x, y) \in [0, 1]^2. \end{aligned}$$

We then have the following result, which shows that the λ -operations coincide with a specific choice of the conjunctor and the implicator in the general framework presented in section ??.

Proposition 9 Let A and B be two gray-scale objects, then it holds:

- (i) $D_\lambda(A, B) = D_{C_\lambda}(A, B)$,
- (ii) $E_\lambda(A, B) = E_{I_\lambda}(A, B)$.

One can also verify that C_λ is a semi-norm and that I_λ is a border implicator if and only if $\lambda(x) = 1 \Leftrightarrow x$. In that case the conjunctor coincides with the Lukasiewicz t-norm W , while the implicator is given by the Lukasiewicz implicator I_L .

3.4.3 Fuzzy set inclusion of Kitainik

Another approach towards fuzzified set inclusion is given by Kitainik (1993). Just as Sinha & Dougherty, Kitainik starts with a list of axioms that any fuzzified set inclusion should satisfy. Kitainik was also able to characterize all fuzzified set inclusions that satisfy the postulated axioms. In order to do so, he introduced the class Phi as the class of functions φ that satisfy the following conditions:

- (a) φ is a $T \Leftrightarrow [0, 1]$ function, where T is the triangle defined by $T = \{(x, y) | (x, y) \in [0, 1]^2 \text{ and } x \geq y\}$,
- (b) φ has decreasing partial mappings,
- (c) $\varphi(0, 0) = \varphi(1, 0) = 1$ and $\varphi(1, 1) = 0$.

Proposition 10 Let Inc be a $\mathcal{F}(\mathbf{R}^n) \times \mathcal{F}(\mathbf{R}^n) \Leftrightarrow [0, 1]$ mapping, then Inc satisfies the axioms of Kitainik if and only if there exists a function $\varphi \in Phi$ such that:

$$Inc(A, B) = Inc_\varphi(A, B) = \inf_{x \in \mathbf{R}^n} \varphi(\max(A(x), 1 \Leftrightarrow B(x)), \min(A(x), 1 \Leftrightarrow B(x))). \quad (14)$$

In the following, an indicator in Kitainik's sense will be denoted by Inc_φ .

Definition 11 The φ -dilation $D_\varphi(A, B)$ and the φ -erosion $E_\varphi(A, B)$ are the gray-scale objects given by:

$$D_\varphi(A, B)(y) = \sup_{x \in T_y(d_B) \cap d_A} (1 \Leftrightarrow \varphi[\max(B(x \Leftrightarrow y), A(x)), \min(B(x \Leftrightarrow y), A(x))]), \quad (15)$$

$$E_\varphi(A, B)(y) = \inf_{x \in T_y(d_B)} \varphi(\max(B(x \Leftrightarrow y), 1 \Leftrightarrow A(x)), \min(B(x \Leftrightarrow y), 1 \Leftrightarrow A(x))), \quad (16)$$

for respectively $y \in D(d_A, d_B)$ and $y \in E(d_A, d_B)$.

Also in this case the φ -operations coincide with a specific choice of the conjunctor \mathcal{C} and the implicator \mathcal{I} . Indeed, define the conjunctor \mathcal{C}_φ and the implicator \mathcal{I}_φ as follows:

$$\begin{aligned} \mathcal{C}_\varphi(x, y) &= 1 \Leftrightarrow \varphi(\max(x, y), \min(x, y)), \\ \mathcal{I}_\varphi(x, y) &= \varphi(\max(x, 1 \Leftrightarrow y), \min(x, 1 \Leftrightarrow y)), \quad \forall (x, y) \in [0, 1]^2. \end{aligned}$$

Proposition 11 Let A and B be two gray-scale objects and let $\varphi \in Phi$, then it holds:

- (i) $D_\varphi(A, B) = D_{\mathcal{C}_\varphi}(A, B)$,
- (ii) $E_\varphi(A, B) = E_{\mathcal{I}_\varphi}(A, B)$.

The relationship with the general logical framework is much stronger. To show this, we first define the class Op as the class of ordered pairs $(\mathcal{C}, \mathcal{I})$ that satisfy the following conditions:

- (a) \mathcal{C} is a commutative conjunctor,
- (b) \mathcal{I} is an implicator that is contrapositive w.r.t. the standard negator,
- (c) \mathcal{C} and \mathcal{I} are connected by the relationship $\mathcal{C} = \mathcal{C}_{\mathcal{I}, N_s}$ or, equivalently, $\mathcal{I} = \mathcal{I}_{\mathcal{C}, N_s}$.

Next, we introduce the functions $\varphi_{\mathcal{C}}$ and $\varphi_{\mathcal{I}}$, given by:

$$\begin{aligned} \varphi_{\mathcal{C}}(x, y) &= 1 \Leftrightarrow \mathcal{C}(x, y), \\ \varphi_{\mathcal{I}}(x, y) &= \mathcal{I}(x, 1 \Leftrightarrow y), \end{aligned}$$

for all $(x, y) \in T$.

Now, one can prove the following: if $\varphi \in Phi$, then $(\mathcal{C}_\varphi, \mathcal{I}_\varphi) \in Op$; conversely, if $(\mathcal{C}, \mathcal{I}) \in Op$, then $\varphi_{\mathcal{C}} = \varphi_{\mathcal{I}} \in Phi$. Furthermore, the correspondence between Phi and Op is a one-to-one correspondence. This is expressed in the following proposition.

Proposition 12 The mapping M , defined by:

$$\begin{aligned} M : Phi &\rightarrow Op \\ \varphi &\mapsto (\mathcal{C}_\varphi, \mathcal{I}_\varphi), \quad \forall \varphi \in Phi \end{aligned}$$

is a bijection.

As a consequence, there is a one-to-one correspondence between fuzzy mathematical morphologies derived from the fuzzified set inclusion in Kitainik's sense and fuzzy mathematical morphologies based on conjunctors \mathcal{C} and implicators \mathcal{I} from the class Op .

3.4.4 Fuzzy set inclusion of Bandler and Kohout

The approach of Bandler & Kohout (1980) towards a fuzzified set inclusion is a logical one, i.e. they fuzzify the binary expression $A \subseteq B \Leftrightarrow (\forall x \in \mathbf{R}^n)(x \in A \Rightarrow x \in B)$, using the notion of an implicator.

Definition 12 Let A and B be two fuzzy sets in \mathbf{R}^n , and let \mathcal{I} be an implicator on $[0, 1]$. The indicator $Inc_{\mathcal{I}}$ for fuzzified set inclusion is the $\mathcal{F}(\mathbf{R}^n) \times \mathcal{F}(\mathbf{R}^n) \Leftrightarrow [0, 1]$ mapping given by:

$$Inc_{\mathcal{I}}(A, B) = \inf_{x \in \mathbf{R}^n} \mathcal{I}(A(x), B(x)).$$

It immediately follows that the corresponding fuzzy erosion, defined by $E(A, B)_{Inc_{\mathcal{I}}} = Inc_{\mathcal{I}}(T_y(B), A)$ equals $E_{\mathcal{I}}(A, B)$. Since the corresponding fuzzy dilation is defined as the N_s -dual of the erosion, this operator is given by $D_{\mathcal{C}_{\mathcal{I}, N_s}}(A, B)$.

4 Conclusions

It was shown that binary morphology and gray-scale morphology based on the threshold approach can be considered as special instances of a class of fuzzy morphologies. Gray-scale morphology based on the umbra approach only coincides with a specific fuzzy morphology under a supplementary condition on the structuring element.

We have also discussed several alternative ways to extend binary morphology to fuzzy morphology. All these different approaches fit into the general logical framework introduced by De Baets. In particular, an equivalence between fuzzy morphologies based on Kitainik's fuzzified set inclusion and fuzzy morphologies based on a certain class of connected conjunctors and implicators has been shown.

These results also suggest that further theoretical research on fuzzy mathematical morphology should concentrate on the general logical framework, since that proves to be the most universal approach.

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