

# Fuzzy mathematical morphology based on the fuzzy inclusion in Kitainik's sense

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## Abstract

Fuzzy mathematical morphology can be seen as an extension of the binary morphological operations to gray-scale images based on the theory of fuzzy sets. This paper introduces the basic concepts for fuzzy mathematical morphology, based on Kitainik's fuzzy inclusion. More precisely, the fuzzy erosion, dilation, opening and closing are introduced. Their basic properties such as monotonicity and intersection are discussed in detail.

**Keywords** : fuzzy inclusion, fuzzy mathematical morphology, dilation, erosion, closing, opening.

## 1 Introduction

Mathematical morphology, i.e. the analysis of the shape of  $n$  - dimensional objects has applications varying from medicine to robotics. The use of morphological operators simplifies the image but it preserves certain characteristics of the image. Morphological operators based on set-theory were originally introduced for black-and-white images. A binary morphological operation transforms an object  $A$  by means of another object  $B$ , called a structuring element, into a new object.  $A$  is the object we wish to analyse while  $B$  is the "small" set we use to analyse  $A$ . The sets  $A$  and  $B$  are subsets of  $\mathbf{R}^n$ .

In crisp set theory the set  $-B$  is defined as

$$-B = \{-b \mid b \in B\}.$$

We define the fuzzy set  $-B$  for  $B \in \mathcal{F}(\mathbf{R}^n)$  as:

$$(\forall x \in \mathbf{R}^n)((-B)(x) = B(-x)).$$

Thus we have the following property for any family  $(A_i \mid i \in I)$  in  $\mathcal{F}(X)$ :

$$\begin{aligned} -\bigcup_{i \in I} A_i &= \bigcup_{i \in I} -A_i, \\ -\bigcap_{i \in I} A_i &= \bigcap_{i \in I} -A_i. \end{aligned}$$

Consider a subset  $B$  of  $\mathbf{R}^n$  and  $y \in \mathbf{R}^n$  then the translation  $T_y(B)$  of  $B$  by  $y$  is defined as:

$$T_y(B) = \{x \mid x - y \in B\}.$$

The translation of a fuzzy set  $B$  in  $x$  by  $y \in \mathbf{R}^n$  is defined by  $T_y(B)(x) = B(x - y)$ . A binary image is completely determined by its set of black pixels (with value 1), or equivalently by its set of white pixels (with value 0). By the object in the image we mean the set of pixels describing the foreground region. The complement of this set then describes the background. The primitive morphological operations are erosion and dilation. Erosion is the set of all  $y$  for which the translation  $T_y(B)$  is contained in  $A$ . These definitions are formalised in [?]

$$\mathcal{D}(A, B) = \{y \mid T_y(B) \cap A \neq \emptyset\}$$

$$\mathcal{E}(A, B) = \{y \mid T_y(B) \subseteq A\}.$$

Dilation and erosion are not inverse operations. Therefore, their “compositions” lead to new operations. The (binary) closing  $\mathcal{C}$  and (binary) opening  $\mathcal{O}$  are defined in [?]:

$$\mathcal{C}(A, B) = \mathcal{E}(\mathcal{D}(A, B), -B)$$

$$\mathcal{O}(A, B) = \mathcal{D}(\mathcal{E}(A, B), -B).$$

The closing of an object  $A$  with respect to the structuring element  $B$  is the set of all  $y$  for which any translation of  $B$  that contains  $y$  has a non-empty intersection with  $A$ . The opening of  $A$  with respect to  $B$  is the set of all  $y$  for which there exists a translation of  $B$  that contains  $y$  and that is contained in  $A$ . This can be formulated as:

$$\mathcal{C}(A, B) = \{y \mid (\forall z \in \mathbf{R}^n)(y \in T_z(B) \Rightarrow T_z(y) \cap A \neq \emptyset)\}$$

$$\mathcal{O}(A, B) = \{y \mid (\exists z \in \mathbf{R}^n)(y \in T_z(B) \text{ and } T_z(B) \subseteq A)\}.$$

Binary morphology has been extended to gray-scale images, in gray-scale morphology by Haralick [?]. He proved that all the relationships and properties valid for binary morphology still remained true. In this paper we wish to discuss an alternative gray-scale morphology based on fuzzy set theory.

Starting from the fuzzy inclusion as introduced by Kitainik [?], we define the basic morphological operators erosion and dilation in terms of this inclusion. We discuss basic properties of these fuzzy operators, as monotonicity and intersection.

## 2 Definition of the fuzzy inclusion in Kitainik’s sense

Kitainik [?] defined fuzzy inclusion as follows: Consider the following subset of  $[0, 1]^2$ :

$$T = \{(\alpha, \beta) \mid (\alpha, \beta) \in [0, 1]^2 \text{ and } \beta \leq \alpha\}.$$

Define the following set of mappings:

$$\Phi = \{\phi \mid \phi : T \rightarrow [0, 1] \text{ and } \phi \text{ is decreasing in both arguments, } \phi(0, 0) = \phi(1, 0) = 1, \phi(1, 1) = 0\}.$$

The fuzzy inclusion is defined as follows:

$$\text{Inc}_\phi(A, B) = \inf_{x \in \mathbf{R}^n} \phi(\max(A(x), 1 - B(x)), \min(A(x), 1 - B(x))),$$

for  $A \in \mathcal{F}(\mathbf{R}^n), B \in \mathcal{F}(\mathbf{R}^n)$ .

In the following we will omit the  $\phi$  in  $\text{Inc}_\phi$ .

## 3 The fuzzy erosion based on Kitainik’s fuzzy inclusion

### 3.1 Definition

We define the fuzzy erosion as follows

**Definition 1** Consider fuzzy sets  $A$  and  $B$  in  $\mathbf{R}^n$ . The fuzzy erosion based on the fuzzy inclusion by Kitainik is then defined as follows:

$$(\forall x \in \mathbf{R}^n)(\text{E}(A, B)(x) = \text{Inc}(T_x(B), A)),$$

or, equivalently;

$$\text{E}(A, B)(x) = \inf_{y \in \mathbf{R}^n} \phi(\max(T_x(B)(y), 1 - A(y)), \min(T_x(B)(y), 1 - A(y))).$$

### 3.2 Properties of the fuzzy erosion

**Proposition 1** 1. *Monotonicity (increasing in the first and decreasing in the second argument); for  $(A, B, C) \in \mathcal{F}(\mathbf{R}^n)^3$ :*

$$A \subseteq C \Rightarrow E(A, B) \subseteq E(C, B)$$

$$B \subseteq C \Rightarrow E(A, B) \supseteq E(A, C);$$

2. *Interaction with Zadeh's intersection; for any family  $(A_i | i \in I)$  in  $\mathcal{F}(\mathbf{R}^n)$  and  $B \in \mathcal{F}(\mathbf{R}^n)$ :*

$$E\left(\bigcap_{i \in I} A_i, B\right) \subseteq \bigcap_{i \in I} E(A_i, B) \subseteq \bigcup_{i \in I} E(A_i, B)$$

$$E\left(B, \bigcap_{i \in I} A_i\right) \supseteq \bigcup_{i \in I} E(B, A_i) \supseteq \bigcap_{i \in I} E(B, A_i);$$

3. *Interaction with Zadeh's union; for any family  $(A_i | i \in I)$  in  $\mathcal{F}(\mathbf{R}^n)$  and  $B \in \mathcal{F}(\mathbf{R}^n)$ :*

$$E\left(\bigcup_{i \in I} A_i, B\right) \supseteq \bigcup_{i \in I} E(A_i, B) \supseteq \bigcap_{i \in I} E(A_i, B)$$

$$E\left(B, \bigcup_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} E(B, A_i) \subseteq \bigcup_{i \in I} E(B, A_i).$$

#### Proof

1. The fuzzy erosion is increasing in the first argument, since the fuzzy inclusion Inc is increasing in the second. The fuzzy erosion E is decreasing in the first argument if we can conclude from  $B \subseteq C$  that  $(\forall x \in \mathbf{R}^n)(T_x(B) \subseteq T_x(C))$ . This is straightforward. We conclude that E is decreasing in the second argument since Inc is decreasing in the first argument.
2. As an example we prove the first inclusion. Let  $x \in \mathbf{R}^n$ .

$$\begin{aligned} E(\bigcap_{i \in I} A_i, B)(x) &= \inf_{y \in \mathbf{R}^n} \phi(\max(T_x(B)(y), 1 - \inf_{i \in I} A_i(y)), \min(T_x(B)(y), 1 - \inf_{i \in I} A_i(y))) \\ &= \inf_{y \in \mathbf{R}^n} \phi(\max(T_x(B)(y), \sup_{i \in I} (1 - A_i(y))), \min(T_x(B)(y), \sup_{i \in I} (1 - A_i(y)))) \\ &= \inf_{y \in \mathbf{R}^n} \phi(\sup_{i \in I} \max(T_x(B)(y), 1 - A_i(y)), \sup_{i \in I} \min(T_x(B)(y), 1 - A_i(y))) \\ &\leq \inf_{y \in \mathbf{R}^n} \inf_{i \in I} \phi(\max(T_x(B)(y), 1 - A_i(y)), \min(T_x(B)(y), 1 - A_i(y))) \\ &= \bigcap_{i \in I} E(A_i, B)(x) \\ &\leq \bigcup_{i \in I} E(A_i, B)(x). \end{aligned}$$

□

## 4 The fuzzy dilation based on Kitainik's fuzzy inclusion

### 4.1 Definition

We define the dilation as follows:

**Definition 2** Consider fuzzy sets  $A$  and  $B$  in  $\mathbf{R}^n$ . The fuzzy dilation based on the fuzzy inclusion by Kitainik is defined as follows:

$$(\forall x \in \mathbf{R}^n)(D(A, B)(x) = 1 - \text{Inc}(T_x(-B), \text{co } A)),$$

or, equivalently;

$$1 - D(A, B)(x) = \inf_{y \in \mathbf{R}^n} \phi(\max(T_x(-B)(y), A(y)), \min(T_x(-B)(y), A(y))).$$

## 4.2 Properties of the fuzzy dilation

**Proposition 2** 1. *Monotonicity (increasing in both arguments), for  $(A, B, C) \in \mathcal{F}(\mathbf{R}^n)^3$ :*

$$A \subseteq C \Rightarrow D(A, B) \subseteq D(C, B)$$

$$B \subseteq C \Rightarrow D(A, B) \subseteq D(A, C);$$

2. *Interaction with Zadeh's intersection; for any family  $(A_i | i \in I)$  in  $\mathcal{F}(\mathbf{R}^n)$  and  $B \in \mathcal{F}(\mathbf{R}^n)$ :*

$$D\left(\bigcap_{i \in I} A_i, B\right) \subseteq \bigcap_{i \in I} D(A_i, B) \subseteq \bigcup_{i \in I} D(A_i, B)$$

$$D\left(B, \bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} D(B, A_i) \subseteq \bigcup_{i \in I} D(B, A_i);$$

3. *Interaction with Zadeh's union, for any family  $(A_i | i \in I)$  in  $\mathcal{F}(\mathbf{R}^n)$  and  $B \in \mathcal{F}(\mathbf{R}^n)$ :*

$$D\left(\bigcup_{i \in I} A_i, B\right) \supseteq \bigcup_{i \in I} D(A_i, B) \supseteq \bigcap_{i \in I} D(A_i, B)$$

$$D\left(B, \bigcup_{i \in I} A_i\right) \supseteq \bigcup_{i \in I} D(B, A_i) \supseteq \bigcap_{i \in I} D(B, A_i).$$

### Proof

1. We prove that the dilation is increasing in the first argument. The second part can be proven similarly. Consider  $A \subseteq C$ . It is straightforward that

$$\max(T_x(-B)(y), A(y)) \leq \max(T_x(-B)(y), C(y)),$$

$\forall x, y \in \mathbf{R}^n$  and this holds for min as well. From the decreasingness of  $\phi$  in both arguments we can conclude the following:

$$\phi(\max(T_x(-B)(y), A(y)), \min(T_x(-B)(y), A(y))) \geq \phi(\max(T_x(-B)(y), C(y)), \min(T_x(-B)(y), C(y))).$$

Thus we have :  $D(A, B)(x) \leq D(C, B)(x)$ .

2. As an example we prove the first inclusion. Let  $x \in \mathbf{R}^n$ .

$$\begin{aligned} D(\bigcap_{i \in I} A_i, B)(x) &= 1 - \inf_{y \in \mathbf{R}^n} \phi(\max(\inf_{i \in I} A_i(y), T_x(-B)(y)), \min(\inf_{i \in I} A_i(y), T_x(-B)(y))) \\ &= 1 - \inf_{y \in \mathbf{R}^n} \phi(\inf_{i \in I} \max(A_i(y), T_x(-B)(y)), \inf_{i \in I} \min(A_i(y), T_x(-B)(y))) \\ &\leq 1 - \inf_{y \in \mathbf{R}^n} \sup_{i \in I} \phi(\max(A_i(y), T_x(-B)(y)), \min(A_i(y), T_x(-B)(y))) \\ &\leq 1 - \sup_{i \in I} \inf_{y \in \mathbf{R}^n} \phi(\max(A_i(y), T_x(-B)(y)), \min(A_i(y), T_x(-B)(y))) \\ &= \inf_{i \in I} (1 - \inf_{y \in \mathbf{R}^n} \phi(\max(A_i(y), T_x(-B)(y)), \max(A_i(y), T_x(-B)(y)))) \\ &= \bigcap_{i \in I} D(A_i, B)(x) \\ &\leq \bigcup_{i \in I} D(A_i, B)(x). \end{aligned}$$

□

## 5 The fuzzy closing based on Kitainik's fuzzy inclusion

We make compositions with dilation and erosion, starting with closing.

## 5.1 Definition

**Definition 3** Consider fuzzy sets  $A$  and  $B$  in  $\mathbf{R}^n$ . The fuzzy closing based on the fuzzy inclusion by Kitainik is defined as follows:

$$C(A, B) = E(D(A, B), -B),$$

or, equivalently, for  $x \in \mathbf{R}^n$ ,

$$\begin{aligned} C(A, B)(x) &= \text{Inc}(T_x(-B), D(A, B)) \\ &= \inf_{y \in \mathbf{R}^n} \phi(\max(T_x(-B)(y), 1 - D(A, B)(y)), \min(T_x(-B)(y), 1 - D(A, B)(y))) \\ &= \inf_{y \in \mathbf{R}^n} \phi(\max(T_x(-B)(y), \inf_{z \in \mathbf{R}^n} \phi(\max(T_y(-B)(z), A(z)), \min(T_y(-B)(z), A(z))), \\ &\quad \min(T_x(-B)(y), \inf_{z \in \mathbf{R}^n} \phi(\max(T_y(-B)(z), A(z)), \min(T_y(-B)(z), A(z)))). \end{aligned}$$

## 5.2 Properties of the fuzzy closing

**Proposition 3** 1. Interaction with Zadeh's intersection; for any family  $(A_i \mid i \in I)$  in  $\mathcal{F}(\mathbf{R}^n)$  and  $B \in \mathcal{F}(\mathbf{R}^n)$ :

$$C\left(\bigcap_{i \in I} A_i, B\right) \subseteq \bigcap_{i \in I} C(A_i, B);$$

2. Interaction with Zadeh's union; for any family  $(A_i \mid i \in I)$  in  $\mathcal{F}(\mathbf{R}^n)$  and  $B \in \mathcal{F}(\mathbf{R}^n)$ :

$$C\left(\bigcup_{i \in I} A_i, B\right) \supseteq \bigcup_{i \in I} C(A_i, B).$$

## Proof

1.

$$\begin{aligned} C(\cap_{i \in I} A_i, B)(x) &= E(D(\cap_{i \in I} A_i, B), -B) \\ &\subseteq E(\cap_{i \in I} D(A_i, B), -B) \\ &\subseteq \cap_{i \in I} E(D(A_i, B), -B) \\ &= \cap_{i \in I} C(A_i, B). \end{aligned}$$

2.

$$\begin{aligned} C(\cup_{i \in I} A_i, B) &= E(D(\cup_{i \in I} A_i, B), -B) \\ &\supseteq E(\cup_{i \in I} D(A_i, B), -B) \\ &\supseteq \cup_{i \in I} E(D(A_i, B), -B) \\ &= \cup_{i \in I} C(A_i, B). \end{aligned}$$

□

*Remark* Parts of these results may be obtained through a different approach [?] in which the starting point is a general formulation for the fuzzy erosion and fuzzy dilation in terms of a conjunctive and implicative.

## 6 The fuzzy opening based on Kitainik's fuzzy inclusion

### 6.1 Definition

**Definition 4** Consider fuzzy sets  $A$  and  $B$  in  $\mathbf{R}^n$ . The fuzzy opening based on the fuzzy inclusion by Kitainik is defined as follows:

$$O(A, B) = D(E(A, B), -B),$$

or, equivalently, for  $x \in \mathbf{R}^n$ ;

$$\begin{aligned} O(A, B)(x) &= 1 - \inf_{y \in \mathbf{R}^n} \phi(\max(T_x(B)(y), E(A, B)(y)), \min(T_x(B)(y), E(A, B)(y))) \\ &= 1 - \inf_{y \in \mathbf{R}^n} \phi(\max(T_x(B)(y), \inf_{z \in \mathbf{R}^n} \phi(\max(T_y(B)(z), 1 - A(z)), \min(T_y(B)(z), 1 - A(z))), \\ &\quad \min((T_x(B)(y), \inf_{z \in \mathbf{R}^n} \phi(\max(T_y(B)(z), 1 - A(z)), \min(T_y(B)(z), 1 - A(z)))). \end{aligned}$$

## 6.2 Properties of the fuzzy opening

**Proposition 4** 1. Interaction with Zadeh's intersection, for any family  $(A_i | i \in I)$  in  $\mathcal{F}(\mathbf{R}^n)$  and  $B \in \mathcal{F}(\mathbf{R}^n)$ :

$$O\left(\bigcap_{i \in I} A_i, B\right) \subseteq \bigcap_{i \in I} O(A_i, B);$$

2. Interaction with Zadeh's union, for any family  $(A_i | i \in I)$  in  $\mathcal{F}(\mathbf{R}^n)$  and  $B \in \mathcal{F}(\mathbf{R}^n)$ :

$$O\left(\bigcup_{i \in I} A_i, B\right) \supseteq \bigcup_{i \in I} O(A_i, B).$$

**Proof**

1.

$$\begin{aligned} O(\cap_{i \in I} A_i, B) &= D(E(\cap_{i \in I} A_i, B), -B) \\ &\subseteq D(\cap_{i \in I} E(A_i, B), -B) \\ &\subseteq \cap_{i \in I} D(E(A_i, B), -B) \\ &= \cap_{i \in I} O(A_i, B). \end{aligned}$$

2.

$$\begin{aligned} O(\cup_{i \in I} A_i, B) &= D(E(\cup_{i \in I} A_i, B), -B) \\ &\supseteq D(\cup_{i \in I} E(A_i, B), -B) \\ &\supseteq \cup_{i \in I} D(E(A_i, B), -B) \\ &= \cup_{i \in I} O(A_i, B). \end{aligned}$$

□

## 7 Duality properties

**Proposition 5** Let  $A$  and  $B$  be fuzzy sets in  $\mathbf{R}^n$  then the following properties hold:

1. The dilation-erosion duality

$$\begin{aligned} D(A, -B) &= \text{co } E(\text{co } A, B) \\ E(A, -B) &= \text{co } D(\text{co } A, B); \end{aligned}$$

2. The opening-closing duality

$$\begin{aligned} C(A, -B) &= \text{co } C(\text{co } A, B) \\ O(A, -B) &= \text{co } O(\text{co } A, B). \end{aligned}$$

**Proof**

1. Straightforward.

2. As an example we prove the first equality.

$$\begin{aligned} O(A, -B) &= D(E(A, -B), B) \\ &= D(\text{co } D(\text{co } A, B), B) \\ &= \text{co } E(D(\text{co } A, B), -B) \\ &= \text{co } C(\text{co } A, B). \end{aligned}$$

□

## 8 Conclusion

In this paper we have introduced a fuzzy mathematical morphology based on the fuzzy inclusion by Kitainik. We have investigated basic properties for the fuzzy erosion, the fuzzy dilation, the fuzzy opening and closing such as monotonicity, intersection and union.

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