TESTING FUZZY HYPOTHESES WITH VAGUE DATA

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1 Introduction

Testing hypotheses is one of the primary purposes of statistical inference. In classical statistics all model parameters, i.e. data, hypotheses and test requirements should be precise. However in real life we meet very often vague data, like "about ten", "more or less between five and seven", "rather greater than 100", etc. Moreover, sometimes we are quite satisfied in verifying fuzzy hypothesis like "the mean μ is about 40" instead of the crisp one μ@. We may also test a hypothesis on a significance level "not greater than α" instead of the crisp value of α.. Such statistical test under fuzzy constraints were considered by Arnold [1]. The problem of testing hypotheses with vague data was considered by Casals et al. [3], [4] and Grzegorzewski [8], [9]. Testing fuzzy hypotheses was discussed by Delgado et al. [5], Saade and Schwarzlander [17], [18], Watanabe and Imaizumi [19] and Arnold [2].

The present paper is devoted to testing fuzzy hypotheses in the presence of vague data. This problem was slightly touched in the excellent book by Kruse and Meier [13]. Unfortunately, their solution reveals many disadvantages and gaps (see [10]). Below, we propose other approach utilizing the Dubois-Prade necessity index of strict dominance, so popular in the possibility theory.

2 Basic concepts and notation

Assume that the investigated phenomenon is described by a probability distribution P_θ which belongs to a family of distributions $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$. We consider the null hypothesis $H : \theta \in \Theta_H$ concerning the parameter θ, with the alternative hypothesis $K : \theta \in \Theta_K$, where Θ_H and Θ_K are subsets of Θ such that $\Theta_H \cap \Theta_K = \emptyset$. We assume that if θ were known one would also know whether or not the hypothesis is true.

In the hypothesis testing problem we observe a random sample $V_1, \ldots, V_n$, and this observation can lead to one of two possible decisions: either to reject H (and to accept K), or to not reject H (usually identified with accepting H). Traditionally, rejection of H is denoted by zero and acceptance of H by one. Hence, a decision rule, called a statistical test, can be defined as a function $\varphi : \mathcal{R}^n \to \{0, 1\}$. Each statistical test divides the sample space $\mathcal{R}^n$ into two exclusive subsets: $\{(v_1, \ldots, v_n) \in \mathcal{R}^n : \varphi(v_1, \ldots, v_n) = 0\}$ - the set of the acceptance of H, and $\mathcal{K} = \{(v_1, \ldots, v_n) \in \mathcal{R}^n : \varphi(v_1, \ldots, v_n) = 1\}$ - the set of the rejection of H which is also called a *critical region.* In practice, we compute a certain test statistic $T(V_1, \ldots, V_n)$ (i.e. a function of the observations), then we find a critical region $\mathcal{K}$, and finally we reject the considered hypothesis if $T(V_1, \ldots, V_n) \in \mathcal{K}$ or accept it otherwise. A typical statistical test has a following form:

$$\varphi(V_1, \ldots, V_n) = \begin{cases} 1 & \text{if} \quad T(V_1, \ldots, V_n) \in \mathcal{K}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The critical region depends on a preselected upper bound of the probability of the type I error (i.e. rejecting H when it is actually true), called a *significance level* δ . Thus we have

$$P\{\varphi(V_1, \ldots, V_n) = 1 \mid H\} \leq \delta. \tag{2}$$

For more details concerning the traditional theory of testing statistical hypotheses we refer the reader to Lehmann [15].

Now let $X_1, \ldots, X_n$ denote a fuzzy sample which is a fuzzy perception of the usual random sample $V_1, \ldots, V_n$, from the population with the distribution P_θ. Assume that each X_i is a fuzzy number, i.e. X_i is a normal, fuzzy convex and bounded fuzzy subset of the real line $\mathcal{R}$ with an upper semicontinuous membership function μ_{X_i} (see, e.g., Dubois and Prade [6]). A space of all fuzzy numbers will be denoted by $\mathcal{FN}(\mathcal{R})$. Of course, $\mathcal{FN}(\mathcal{R}) \subset \mathcal{F}(\mathcal{R})$, where $\mathcal{F}(\mathcal{R})$ denotes a space of all fuzzy sets on the real line. An useful tool for dealing with fuzzy numbers are their $\alpha-$cuts. The $\alpha-$cut of a fuzzy number X with its membership function μ_X is a crisp set defined as

$$X_\alpha = \{t \in \mathcal{R} : \mu_X(t) \geq \alpha\}. \tag{3}$$

Every $\alpha-$cut of a fuzzy number is a closed interval. A following notation will be useful below: $X_\alpha = [X_\alpha^L, X_\alpha^U]$, where

$$\begin{aligned} X_\alpha^L &= \inf\{t \in \mathcal{R} : \mu_X(t) \geq \alpha\}, \\ X_\alpha^U &= \sup\{t \in \mathcal{R} : \mu_X(t) \geq \alpha\}. \end{aligned}$$

The precise definition of a fuzzy random variable and fuzzy sample can be found in Kwakernaak [14], Kruse and Meyer [13] and Puri and Ralescu [16].

In the presence of fuzzy data we cannot observe the parameter θ directly but only its vague image. Moreover, we cannot estimate θ better than to its fuzzy perception $\Lambda(\theta)$ defined as

$$\mu_{\Lambda(\theta)}(t) = \sup\left\{\nu(V_1, \ldots, V_n) : (V_1, \ldots, V_n) \in \chi^n, \theta(V_1) = t\right\}, \quad t \in \mathcal{R}, \tag{4}$$

where χ^n is a set of all possible originals of that fuzzy random sample with membership function

$$\nu(V_1, \ldots, V_n) = \min_{i=1,\ldots,n} \inf\{\mu_{X_i(\omega)}(V_i(\omega)) : \omega \in \Omega\}. \tag{5}$$

One can easily obtain $\alpha-$cuts of $\Lambda(\theta)$:

$$\begin{array}{l} (\Lambda(\theta))_\alpha = \{t \in \mathcal{R} : \exists (V_1, \ldots, V_n) \in \chi^n, \theta(V_1) = t, \text{ such that} \\ V_i(\omega) \in (X_i(\omega))_\alpha \text{ for } \omega \in \Omega \text{ and for } i = 1, \ldots, n\}. \end{array} \tag{6}$$

For more information we refer the reader to Kruse, Meyer [13].

Kruse and Meier concluded that since in estimation with vague data the best they could do was to find fuzzy perception $\Lambda(\theta)$ of the parameter θ under study, they also restricted hypotheses testing to hypotheses about $\Lambda(\theta)$. Thus actually they considered the problem of testing fuzzy hypotheses with fuzzy data. They suggested a following definition of a test:

Definition 1 *Let $X_1, \ldots, X_n$ denote a fuzzy sample, then a function $\phi : [\mathcal{FN}(\mathcal{R})]^n \to \{0, 1\}$ such that*

$$P\{\omega \in \Omega : \phi(X_1(\omega), \ldots, X_n(\omega)) = 1 \mid \Lambda(\theta) = \Lambda_0\} \leq \delta \tag{7}$$

is called a test for testing hypothesis $H : \Lambda(\theta) = \Lambda_0$, where $\Lambda_0 \in \mathcal{FN}(\mathcal{R})$, on the significance level $\delta \in (0, 1)$.

It is easily seen that the definition given above is a natural generalization of definition of the classical test and (7) reduces to (2) if all the data are crisp, i.e. $X_i = V_i$ and we consider crisp hypothesis $\theta = \theta_0$.

Kruse and Meier also proposed how to construct such a test for verifying hypothesis $H : \Lambda(\theta) = \Lambda_0$ against two-sided and one-sided alternatives. Unfortunately their method has many drawbacks. For example, the statement "$\Lambda(\theta)$ is greater than Λ_0" they use for one-sided alternative hypothesis, has no sense in the case of fuzzy numbers where there is no unique linear order. Any time we say that one fuzzy number is "greater" than the second one we have to explain what does it mean, i.e. we have to mention how we order fuzzy numbers. This and other problems connected with Kruse and Meier's approach to hypotheses testing were described in [10].

3 Hypotheses using NSD

Below we propose a new method of testing fuzzy hypotheses with vague data which can be used both for one-sided and two-sided alternatives and which satisfies the Kruse-Meier definition (7). To express this alternatives we use the necessity index of strict dominance (NSD) due to Dubois and Prade [7]. Let us recall that for any fuzzy numbers A and B with membership functions μ_A and μ_B, respectively, we can evaluate the degree of necessity to which the relation $A > B$ is fulfilled

$$Ness(A > B) = 1 - \sup_{x,y:x\leq y} \min\{\mu_A(x), \mu_B(y)\}. \tag{8}$$

Dubois and Prade proposed also the possibility of strict dominance index and other indices. However, we decided to use NSD index because of its natural interpretation and effectiveness in solving real-life problems (see, e.g. Hryniewicz [11], [12]).

Let us begin with the problem of testing the null hypothesis $H : \Lambda(\theta) = \Lambda_0$ against one-sided alternative $K : Ness(\Lambda(\theta) > \Lambda_0) \geq \xi$, where ξ is a fixed number from the interval $[0, 1]$. Although we have both fuzzy data and fuzzy hypotheses the order based on NSD leads to a very simple statistical test. Before we show the construction of that test a following lemma, used in the test construction, should be stated.

Lemma 2 *Let $X, Y \in \mathcal{FN}(\mathcal{R})$. The following conditions are equivalent:*

(i) $Ness(X > Y) \geq \xi$,

(ii) $X^L_{1-\xi} \geq Y^U_{1-\xi}$,

(iii) $X^L_\alpha \geq Y^U_\alpha \quad \forall\alpha \in [0, 1],\ \alpha \geq 1 - \xi$.

Therefore, according to the lemma, it is enough to consider only one $\alpha-$level in order to check whether the relation $Ness(\Lambda(\theta) > \Lambda_0) \geq \xi$ holds. This conclusion makes the starting point for our test construction. Moreover, we take advantage from the well known fact that there is an equivalence between the totality of parameters for which the null hypothesis is accepted and the structure of confidence intervals. More precisely, there is one-to-one correspondence between the acceptance region of the test for the hypothesis $H : \theta = \theta_0$ against $K : \theta > \theta_0$ on the significance level δ and one-sided confidence interval $[\pi_1, +\infty)$ for the parameter θ on the confidence level $1 - \delta$, where $\pi_1 = \pi_1(V_1, \ldots , V_n; \delta)$.

A following proposition holds

Proposition 3 *Let $X_1, \ldots , X_n$, denote a fuzzy random sample, where $X_i \in \mathcal{FN}(\mathcal{R})$ for $i = 1, \ldots , n$, from the distribution with unknown real parameter θ and let $\xi \in [0, 1]$. Let $\Lambda(\theta) \in \mathcal{FN}(\mathcal{R})$ denote a fuzzy perception of θ and let $[\pi_1, +\infty)$ be upper one-sided confidence interval for the parameter θ on the confidence level $1 - \delta$. Then a function $\phi : (\mathcal{FN}(\mathcal{R}))^n \to \{0, 1\}$ such that*

$$\phi(X_1, \ldots , X_n) = \begin{cases} 1 & \text{if} \quad (\Lambda_0)^U_{1-\xi} < \Pi^L_{1-\xi} \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

where

$$\begin{aligned} \Pi^L_{1-\xi} = \Pi^L_{1-\xi}(X_1, \ldots , X_n; \delta) = \inf \{t \in \mathcal{R} : \forall i \in \{1, \ldots , n\} \\ \exists x_i \in (X_i)_{1-\xi} \quad \text{such that} \quad \pi_1(x_1, \ldots , x_n) \leq t\}, \end{aligned} \tag{10}$$

is a test for hypothesis $H : \Lambda(\theta) = \Lambda_0$ against one-sided alternative $K : Ness(\Lambda(\theta) > \Lambda_0) \geq \xi$, on the significance level δ, i.e.

$$P\{\omega \in \Omega : \phi(X_1(\omega), \ldots , X_n(\omega)) = 1 \mid \Lambda(\theta) = \Lambda_0 \} \leq \delta.$$

Similarly, using one-to-one correspondence between the acceptance region of the test for the hypothesis $H : \theta = \theta_0$ against $K : \theta < \theta_0$ on the significance level δ and one-sided confidence interval $(-\infty, \pi_2]$ for the parameter θ on the confidence level $1 - \delta$, where $\pi_2 = \pi_2(V_1, \ldots , V_n; \delta)$, we get a test for the opposite one-sided alternative fuzzy hypothesis.

Proposition 4 *Let $X_1, \ldots, X_n$, denote a fuzzy random sample, where $X_i \in \mathcal{FN}(\mathcal{R})$ for $i = 1, \ldots, n$, from the distribution with unknown real parameter θ and let $\xi \in [0,1]$. Let $\Lambda(\theta) \in \mathcal{FN}(\mathcal{R})$ denote a fuzzy perception of θ and let $(-\infty, \pi_2]$ be lower one-sided confidence interval for the parameter θ on the confidence level $1-\delta$. Then a function $\phi : (\mathcal{FN}(\mathcal{R}))^n \to \{0,1\}$ such that*

$$\phi(X_1, \ldots, X_n) = \begin{cases} 1 & \textit{if} \quad (\Lambda_0)^L_{1-\xi} > \Pi^U_{1-\xi} \\ 0 & \textit{otherwise,} \end{cases} \tag{11}$$

where

$$\begin{aligned} \Pi^U_{1-\xi} = \Pi^U_{1-\xi}(X_1, \ldots, X_n; \delta) = \sup \{ & t \in \mathcal{R} : \forall i \in \{1, \ldots, n\} \\ & \exists x_i \in (X_i)_{1-\xi} \quad \textit{such that} \quad \pi_2(x_1, \ldots, x_n) \geq t \}, \end{aligned} \tag{12}$$

is a test for hypothesis $H : \Lambda(\theta) = \Lambda_0$ against one-sided alternative $K : Ness(\Lambda_0 > \Lambda(\theta)) \geq \xi$, on the significance level δ.

We can also use NSD index for testing our null hypothesis against two-sided alternative. Firstly we will define a following relation

Definition 5 *Let $X, Y \in \mathcal{FN}(\mathcal{R})$ and let $\xi \in [0,1]$. Then*

$$Ness(X \neq Y) \geq \xi \Leftrightarrow (Ness(X > Y) \geq \xi \quad or \quad Ness(Y > X) \geq \xi). \tag{13}$$

Now, keeping in mind that there is one-to-one correspondence between the acceptance region of the test for the hypothesis $H : \theta = \theta_0$ against $K : \theta \neq \theta_0$ on the significance level δ and two-sided confidence interval $[\pi_1, \pi_2]$ for the parameter θ on the confidence level $1 - \delta$, where $\pi_1 = \pi_1(V_1, \ldots, V_n; \frac{\delta}{2})$ $\pi_2 = \pi_2(V_1, \ldots, V_n; \frac{\delta}{2})$, we can state a following proposition

Proposition 6 *Let $X_1, \ldots, X_n$, denote a fuzzy random sample, where $X_i \in \mathcal{FN}(\mathcal{R})$ for $i = 1, \ldots, n$, from the distribution with unknown real parameter θ and let $\xi \in [0,1]$. Let $\Lambda(\theta) \in \mathcal{FN}(\mathcal{R})$ denote a fuzzy perception of θ and let $[\pi_1, \pi_2]$ be two-sided confidence interval for the parameter θ on the confidence level $1-\delta$. Then a function $\phi : (\mathcal{FN}(\mathcal{R}))^n \to \{0,1\}$ such that*

$$\phi(X_1, \ldots, X_n) = \begin{cases} 1 & \textit{if} \quad (\Lambda_0)^U_{1-\xi} < \Pi^L_{1-\xi} \quad \textit{or} \quad (\Lambda_0)^L_{1-\xi} > \Pi^U_{1-\xi} \\ 0 & \textit{otherwise,} \end{cases} \tag{14}$$

where

$$\begin{aligned} \Pi^L_{1-\xi} = \Pi^L_{1-\xi}(X_1, \ldots, X_n; \tfrac{\delta}{2}) = \inf \{ & t \in \mathcal{R} : \forall i \in \{1, \ldots, n\} \\ & \exists x_i \in (X_i)_{1-\xi} \quad \textit{such that} \quad \pi_1(x_1, \ldots, x_n) \leq t \}, \end{aligned} \tag{15}$$

$$\begin{aligned} \Pi^U_{1-\xi} = \Pi^U_{1-\xi}(X_1, \ldots, X_n; \tfrac{\delta}{2}) = \sup \{ & t \in \mathcal{R} : \forall i \in \{1, \ldots, n\} \\ & \exists x_i \in (X_i)_{1-\xi} \quad \textit{such that} \quad \pi_2(x_1, \ldots, x_n) \geq t \}, \end{aligned} \tag{16}$$

is a test for hypothesis $H : \Lambda(\theta) = \Lambda_0$ against one-sided alternative $K : Ness(\Lambda_0 \neq \Lambda(\theta)) \geq \xi$, on the significance level δ.

4 Conclusions

The propositions given above show how to construct statistical tests for verifying fuzzy hypotheses with fuzzy data. In defining fuzzy alternatives we have used the Dubois-Prade necessity index of strict dominance. Of course, one may construct similar tests for other indices, like the possibility index of strict dominance, etc.

Our tests are well defined, because if we use crisp data instead of fuzzy observations and if we replace fuzzy hypotheses by crisp ones our tests reduce to the classical tests of significance. These tests are also very simple in use. Although we consider fuzziness both in data and in hypotheses the output of these test is crisp, i.e. our tests lead to precisely described decision: to rejection or to acceptance of the hypothesis under study. Thus they do not require any defuzzification method, which is also their advantage.

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