Logical representation and generalization of discrete Choquet integral

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Abstract

In this paper is given a new representation of Choquet integral - the conjunctive logical representation of discrete fuzzy Choquet integral. The new representation is based on a newly defined conjunctive logical fuzzy measure and the following three properties: (a) the Choquet integral is linear by the measure; (b) a discrete Choquet integral for a conjunctive fuzzy measure is equivalent to a conjunctive logical expression; and (c) any fuzzy measure (vector) can be represented by conjunctive fuzzy logic measures (vectors). The new logical representation enables, among other things, a direct (non-iterative) determination of fuzzy measure and/or Möbius coefficients from the data. In this paper are defined new fuzzy discrete integrals, based on the new conjunctive logical representation. The generalization has been done by replacing a min function by another AND function or a family of AND functions. The new generalized families of discrete fuzzy integrals are very promising for practical applications: data mining, classification, pattern recognition, reliability analysis, diagnostics, picture analysis, reliability analysis etc.

Keywords: Fuzzy measure, Non-monotonic measure, Choquet integral, AND operator (t-norms), fuzzy integral

1 Introduction

The discrete fuzzy Choquet integral as a tool for aggregating attribute values in the presence of interactions among them, offers great potentials for use in multi-attribute decision making, [4], [6], data mining, classification [5], similarity determination, game theory - modelling the importance of coalitions [8], nonstandard reasoning - case-based reasoning, multivariable statistical analysis, etc.

The interpretation and understanding of the discrete fuzzy Choquet integral based on the logical representation are given in [17]. Discrete fuzzy Choquet integral can be represented by logical expressions of attributes. In equivalent logical expressions there exist AND and OR logical operators, defined as min and max functions, respectively. This logical representation is based on the following properties of discrete fuzzy Choquet integral: (1) it is linear by the measure [14],[3], (2) for logical (0-1) fuzzy measure it is logical functions on variables (attributes), [13], [17]; (3) any fuzzy measure can be expressed as a linear convex combination of logical (0-1) fuzzy measure, [17]. The equivalent logical representation is of great importance for practical applications, especially for a consistent explication of a preferential structure in the presence of interaction between attributes. In [17], it is defined a fuzzy measure vector and a fuzzy measure space, which is also used for new representation of Choquet integral.

In this paper are (a) introduced conjunctive fuzzy logical measures; (b) defined the most natural orthonormal basis of fuzzy space, as a function of a conjunctive fuzzy logic measure; (c) shown that any fuzzy measure can be expressed as a linear combination of the most natural orthonormal basis vectors; finally, as a consequence of the linearity of Choquet integral by the measure and previous property (d) given -a conjunctive logical representation of discrete fuzzy Choquet integral. The conjunctive logical representation of discrete fuzzy Choquet integral is even simpler when Möbius transform of fuzzy measure is used instead of the fuzzy measure itself.

In this paper are also given new fuzzy integrals, based on the generalization of discrete fuzzy Choquet integral . The generalization is based on the new conjunctive logical representation of discrete fuzzy Choquet integral. The new logical representation is based only on the AND logical operator, defined as a min function. The generalization of discrete Choquet integral is based on replacing the min function by families of AND functions (other t-norms), [10]. What is common to all new fuzzy discrete integrals is a logical interpretation as in the case of original discrete fuzzy Choquet integral, but with different functions for the logical AND operator.

2 A logical representation of discrete fuzzy Choquet integral

We consider discrete spaces only. In the whole paper, we will work on a finite universe Ω of n elements, $\Omega = \{a_1, ..., a_n\}$. $\mathcal{P}(\Omega)$ is the power set of Ω, while $|\Omega|$ denotes the cardinal of a set Ω, and $A \setminus B$ denotes the set difference.

In this section is given the logical representation of discrete fuzzy Choquet integral [17]. This logical representation enables expressing what a given fuzzy measure means in terms of behavior in decision making and/or enables a consistent explication of a decision maker's preference structure in the presence of interaction among attributes, [17]. In [17], is defined a fuzzy measure vector and fuzzy measure space, which are used for new representation of Choquet integral - the conjunctive logical representations of discrete fuzzy Choquet integral.

2.1 Fuzzy measure vector and fuzzy measure space

A fuzzy measure vector and a fuzzy measure space are introduced in [17]. Any fuzzy measure of a finite crisp set of attributes, Ω, can be represented as a vector, $\overrightarrow{\mu}$, with 2^n components, where $n = |\Omega|$.

Definition 1 *Fuzzy measure vector components, of a finite crisp set of attributes, Ω, are measures of the elements of a power set, $\mathcal{P}(\Omega)$.*

The first and the last fuzzy measure vector components are fixed values, $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$, respectively.

Definition 2 *A logical fuzzy measure vector, $\overrightarrow{\mu^L}$, is a fuzzy measure vector, whose components take the values from $\{0, 1\}$.*

Definition 3 *A space defined by all possible fuzzy measure vectors is a fuzzy measure space.*

2.2 Any fuzzy measure and logical fuzzy measures

Proposition 1 *Any fuzzy measure vector, $\overrightarrow{\mu}$, can be represented as a linear convex combination of logical fuzzy measure vectors $\overrightarrow{\mu_q^L}$, $q = 1, ..., Q$.*

$$\overrightarrow{\mu} = \sum_{q=1}^{Q} \lambda_q \overrightarrow{\mu_q^L}$$

where $\sum_{q=1}^{Q} \lambda_q = 1$ and $\lambda_q \geq 0$, $q = 1, ..., Q$.

Proof is given in [17].

2.3 The logical representation of Choquet integral for any discrete fuzzy measure

From propositions: (a.) about any fuzzy measure and logical fuzzy measures and (b.) the linearity of discrete fuzzy Choquet integral by measures, follows the following proposition, [17]:

Proposition 2 *The fuzzy Choquet integral for any fuzzy measure can be represented as a convex combination of logical fuzzy Choquet integrals with clear logical interpretation.*

$$\begin{aligned} C_\mu(a_1, ..., a_n) &= \sum_{q=1}^{Q} \lambda_q C_{\mu_q^L}(a_1, ..., a_n) \\ &= \sum_{q=1}^{Q} \lambda_q \bigvee_{A:\mu_q^L(A)=1} \left(\bigwedge_{a_k \in A} a_k \right) \end{aligned}$$

The new representation of Choquet integral-the logical representation of discrete conjunctive fuzzy Choquet integral is based on a conjunctive fuzzy logical measure.

3 A conjunctive and disjunctive logical representation of discrete fuzzy Choquet integral

In this section are given the new representations of Choquet integral - a conjunctive and a disjunctive logical representation of discrete fuzzy Choquet integral. The new logical representations enable: data mining, easy calculation of fuzzy measure from the data, multivariable statistical analysis in the presence of interaction among variables (classification, discriminant analysis, canonical correlation analysis, regression analysis, etc.), similarity of object determination. The conjunctive and the disjunctive logical representation of discrete fuzzy Choquet integral is suitable as a basis for defining new classes of integrals with the same property as a Choquet integral in the sense of linearity by measure.

3.1 A conjunctive fuzzy logical measure

Definition 4 *Conjunctive fuzzy logical measure on* Ω *is fuzzy logical (0 -1) measure, described by the following set function:*

$$\mu_{(A_i)}^{CFL}(A) = \begin{cases} 1 & A \supseteq A_i \\ 0 & A \not\supseteq A_i \end{cases} ; \quad A,\ A_i \in \mathcal{P}(\Omega).$$

The most relevant characteristic of conjunctive fuzzy logical measure, for new representation, is given by the following proposition.

Proposition 3 *A discrete Choquet integral of* a_i, $i = 1, ..., n$; *for a conjunctive fuzzy logical measure* $\mu_{(A_i)}^{CFL}$ *is equal to a conjuctive (AND) relation on the elements of* A_i *or to a minimal value element of subset* A_i :

$$\begin{aligned} C_{\mu_{(A_i)}^{CFL}}(a_1, ..., a_n) &= \min_{a_p \in A_i} a_p \\ &= \bigwedge_{a_p \in A_i} a_p. \end{aligned}$$

All proofs are given in [18].

3.2 Orthonormal basis of fuzzy measure space

The most natural orthonormal basis of fuzzy measure space for the new representation is given by the following definition.

Definition 5 *The most natural orthonormal basis of (fuzzy) measure space is a set of vectors* $\left\{ \overrightarrow{b}_{(A_i)} \mid A_i \in \mathcal{P}(\Omega) \right\}$, *whose components are defined by set functions:*

$$b_{(A_i)}(A) = \begin{cases} 1 & A = A_i \\ 0 & A \neq A_i \end{cases} ; \quad A,\ A_i \in \mathcal{P}(\Omega).$$

The most natural orthonormal basis of fuzzy measure space can be represented as a function of conjunctive logical fuzzy measures as it is given by the following proposition.

Proposition 4 *The most natural orthonormal basis, of attribute set measure space, is given, as a function of conjunctive fuzzy logical measure, by the following set functions*

$$b_{(A_i)}(A) = \sum_{B_k \subset \Omega \setminus A_i \cup \emptyset} (-1)^{|B_k|} \mu_{(A_i \cup B_k)}^{CFL}(A); \quad A,\ A_i \in \mathcal{P}(\Omega).$$

or in vector form:

$$\overrightarrow{b}_{(A_i)} = \sum_{B_k \subset \Omega \setminus A_i \cup \emptyset} (-1)^{|B_k|} \overrightarrow{\mu}_{(A_i \cup B_k)}^{CFL}; \quad A_i \in \mathcal{P}(\Omega).$$

Measure $\mu(A_i)$ of $A_i \in \mathcal{P}(\Omega)$, is a geometrical projection of fuzzy measure vector $\overrightarrow{\mu}$ from the measure space on the most natural basis vector $\overrightarrow{b}_{(A_i)}$. As a consequence, any fuzzy measure μ can be described as a set function, given in the following proposition.

Proposition 5 *Any measure μ in the measure space can be represented using the basis of measure space by the following set function:*

$$\mu(A) = \sum_{A_i \subset \Omega} \mu(A_i)\, b_{(A_i)}(A); \quad A \in \mathcal{P}(\Omega).$$

or in vector form:

$$\overrightarrow{\mu} = \sum_{A_i \subset \Omega} \mu(A_i)\, \overrightarrow{b}_{(A_i)}.$$

3.3 Conjunctive fuzzy logical measure and Choquet integral

Since (a) Choquet integral is linear by the measure and (b) any fuzzy measure (vector) can be represented by the most natural orthonormal basis of measure space, which is described by conjunctive fuzzy logic measures, and Choquet integral for a conjunctive fuzzy measure is a conjunctive logical expression on attributes, it follows that Choquet integral for any discrete fuzzy measure can be represented by - a conjunctive logical representation of discrete fuzzy Choquet integral.

Proposition 6 *A discrete Choquet integral of a_i, $i = 1, ..., n$; for a fuzzy measure μ, can be represented as:*

$$\begin{aligned} C_\mu(a_1, ..., a_n) &= \sum_{A_i \subset \Omega} \mu(A_i) \sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \bigwedge_{a_p \in A_i} a_p \bigwedge_{a_q \in B_k} a_q \\ &= \sum_{A_i \subset \Omega} \mu(A_i) \sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \bigwedge_{a_p \in A_i \cup B_k} a_p \end{aligned}$$

Another representation of this result is based on Möbius transformation of fuzzy measure μ.

Proposition 7 *Discrete Choquet integral of a_i, $i = 1, ..., n$; for any fuzzy measure μ, can be represented as:*

$$\begin{aligned} C_m(a_1, ..., a_n) &= C_\mu(a_1, ..., a_n) \\ &= \sum_{A_i \subset \Omega} m(A_i) \bigwedge_{a_p \in A_i} a_p, \end{aligned}$$

where m is Möbius transformation of μ defined as:

$$m(A) := \sum_{B \subset A} (-1)^{|A \setminus B|} \mu(B), \quad A \in \mathcal{P}(\Omega).$$

Möbius coefficients can be interpreted as weighted coefficients of conjunctive logical representation of Choquet integral or as conjunctive regression coefficients.

The conjunctive logical representation of discrete fuzzy Choquet integral, in both forms, fuzzy measure and Möbius transform of fuzzy measure, is a very useful representation for: fuzzy measure determination, data mining, multi-variable statistical analysis in the presence of interaction among attributes, pattern recognition, etc. New logical representation possibility of discrete fuzzy Choquet integral will be illustrated on practical examples in a forthcoming paper. This new representation is a basis for generating new discrete fuzzy integrals described in the following Section.

4 New discrete fuzzy integrals

The new representations - the conjunctive logical representation and disjunctive logical representation of discrete fuzzy Choquet integral enable the definition of new fuzzy integrals.

The AND function immanent to a classical discrete fuzzy Choquet integral is a min function. The generalization of discrete Choquet integral, here, is based on other known types of AND operators (t-norms) and on the following two representations of discrete Choquet integral :(a) as a function of fuzzy measure $C_\mu(a_1, ..., a_n) = \sum_{A_i \subset \Omega} \mu(A_i) \sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \bigwedge_{a_p \in A_i \cup B_k} a_p$, and (b) as a function of Möbius transformation of fuzzy measure $C_m(a_1, ..., a_n) = \sum_{A_i \subset \Omega} m(A_i) \bigwedge_{a_p \in A_i} a_p$.

Example 1 *Choquet-Yager discrete fuzzy integral. The Yager AND function [10]:*

$$a_1 \wedge a_2 = 1 - \min\left\{1, \left[(1-a_1)^\lambda + (1-a_2)^\lambda\right]^{1/\lambda}\right\}; \quad \lambda > 0.$$

A generalized Yager AND function is:

$$\bigwedge_{a_p \in \Omega} a_p = 1 - \min\left\{1, \left[\sum_{a_p \in \Omega} (1-a_p)^\lambda\right]^{1/\lambda}\right\}; \quad \lambda > 0.$$

The Choquet - Yager discrete fuzzy integral:

$$CY_\mu\left(a_1, ..., a_n, \lambda\right) =$$
$$\sum_{A_i \subset \Omega} \mu\left(A_i\right) \sum_{B_k \subset \Omega \backslash A_i} (-1)^{|B_k|} \left(1 - \min\left\{1, \left[\sum_{a_p \in A_i \cup B_k} (1-a_p)^\lambda\right]^{1/\lambda}\right\}\right); \quad \lambda > 0.$$

The Choquet - Yager discrete fuzzy integral represented as a function of Möbius coefficients:

$$CY_m\left(a_1, ..., a_n, \lambda\right) = \sum_{A_i \subset \Omega} m\left(A_i\right) \left(1 - \min\left\{1, \left[\sum_{a_p \in A_i} (1-a_p)^\lambda\right]^{1/\lambda}\right\}\right); \quad \lambda > 0.$$

5 Conclusion

In this paper is given a new logical representation of Choquet integral - a conjunctive logical representation of discrete fuzzy Choquet integral. The new representation is based on a newly defined conjunctive logical fuzzy measure and the following three properties: (a) Choquet integral is linear by the measure; (b) a discrete Choquet integral for a conjunctive fuzzy measure is equivalent to a conjunctive logical expression; and (c) any fuzzy measure (vector) can be represented by conjunctive fuzzy logic measures (vectors); the simpler conjunctive logical representation of discrete Choquet integral as a function of Möbius coefficients instead of fuzzy measure is also given. New logical representation enables, among other things, direct (non-iterative) determination of fuzzy measure and/or Möbius coefficients from the data. The conjunctive logical representation of discrete fuzzy Choquet integral is very useful representation for: data mining, multi-variable statistical analysis (classification, discriminant analysis, canonical correlation analysis etc.) in the presence of interaction among attributes [19], pattern recognition, etc. New logical representation of discrete fuzzy Choquet integral (possibility), will be illustrated on practical examples in the next paper.

In this paper are defined new fuzzy discrete integrals, based on the new logical representation of discrete fuzzy Choquet integral. The conjunctive logical representation consists of logical expressions in which only AND operator figures, defined as a min function. The generalization has been done by replacing the min function by another AND function or a family of AND functions,[10]. In this way, whole families of new discrete "ala Choquet" integral have been obtained. As a special case, a multiplicative Choquet integral, when the AND operator is a multiplicative function, is very promising for reliability analysis of conceptually complex systems, with an unclear structure (system operator, ecological system, economical system etc.) [11].

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