Logical reperesentation of order weight aggregation (OWA) operator

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Abstract

In this paper new Logical representations and generalization of OWA operator are given. Generalization is based on substitution of AND function, which is min faultiness in logical representation of classical definition, by any other AND function or class of AND functions.

Keywords: OWA operator, logical representation, generalization, AND operators, symmetric fuzzy measure, symmetric Möbius coefficients

1 Introduction

Order weight aggregation (OWA), [3], is very popular in many fields of applications, [4]. The reason for this is, among others, the facts that (a) the complexity of the aggregation problem by OWA operator is same as complexity of aggregation by weighted coefficients and (b) by OWA operator can be modeled the interaction among attributes which is impossible by weighted coefficients approach.

In this paper logical representation and generalization of OWA operator is given. Logical representation and generalization of order weight aggregation operator is based on the fact that classical OWA operator is particular case of discrete fuzzy Choquet integral and on logical representation and generalization of discrete fuzzy Choquet integral [1].

In the first section the logical representation of OWA operator is given, and symmetric and logical symmetric fuzzy measure and Möbius coefficients are defined. Logical representation of OWA operator in function of symmetric fuzzy measure and symmetric Möbius coefficients are given too. Relationship between symmetric fuzzy measure and/or symmetric Möbius coefficients with weighted coefficients in OWA operator are derived.

In the second section generalization of OWA operator is given. Generalization is based on substitution of AND operator, which is in logical representation of classical OWA operator defined as a min function, by some other AND function or class of AND functions.

2 Order weight aggregation - OWA

We consider discrete spaces only. In the whole paper, we will work on a finite universe Ω of n elements, $\Omega = \{a_1, ..., a_n\}$. $\mathcal{P}(\Omega)$ is the power set of Ω, while $|\Omega|$ denotes the cardinal of a set Ω, and $A \setminus B$ denotes the set difference.

2.1 OWA classical definition, [3]:

$$OWA_w(a_1, ..., a_n) = \sum_{i=1}^{n} w_i a_{(i)}$$

$$\begin{aligned} \forall w_j &\geq 0, \qquad \sum_{i=1}^{n} w_i = 1 \\ a_{(1)} &\geq a_{(1)} \geq ... \geq a_{(n)} \end{aligned}$$

2.2 OWA - new logical representation as a function of w

Proposition 1 *OWA operator can be represented as*

$$OWA_w(a_1, ..., a_n) = \sum_{i=1}^{n} w_i \bigvee_{\substack{A \subset \Omega; \\ |A|=i}} \bigwedge_{a_p \in A} a_p, \quad \forall w_j \geq 0, \qquad \sum_{i=1}^{n} w_i = 1.$$

Proofs can be found in [1].

Example 1 *For the case of three attributes* $\Omega = \{a_1, a_2, a_3\}$*:*

$$OWA_w(a_1, a_2, a_3) = w_1(a_1 \vee a_2 \vee a) + w_2(a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3) + w_3(a_1 \wedge a_2 \wedge a_3).$$

Proposition 2 *OWA in function of w and AND operator only can be represented also as:*

$$\begin{aligned} OWA_w(a_1, ..., a_n) &= \sum_{i=1}^{n} w_i \sum_{\substack{A \subset \Omega; \\ |A| \geq i}} (-1)^{|A|-i} \bigwedge_{a_p \in A} a_p \\ \forall w_j &\geq 0, \qquad \sum_{i=1}^{n} w_i = 1 \end{aligned}$$

Example 2 *For the case of three attributes* $\Omega = \{a_1, a_2, a_3\}$ *:*

$$\begin{aligned} OWA_w(a_1, a_2, a_3) = \; & w_1(a_1 + a_2 + a_3 - a_1 \wedge a_2 - a_1 \wedge a_3 - a_2 \wedge a_3 + a_1 \wedge a_2 \wedge a_3) + \\ & + w_2(a_1 \wedge a_2 + a_1 \wedge a_3 + a_2 \wedge a_3 - a_1 \wedge a_2 \wedge a_3) + \\ & + w_3(a_1 \wedge a_2 \wedge a_3). \end{aligned}$$

2.3 OWA - new logical representation as a function of $\mu_{|\bullet|}$−symmetric fuzzy measure

Definition 1 *Symmetric fuzzy measure is fuzzy measure which is a function of only cardinal number:*

$$\mu_{|j|}^{S}(A) = \mu(|A|); \qquad |A| = j; \qquad A \subset \Omega.$$

Definition 2 *Logical symmetric fuzzy measure is symmetric measure* μ^{LS} *if it takes the values from* $\{0, 1\}$.

Example 3 *Logical symmetric fuzzy measure for the universal set* $\Omega = \{a_1, a_2, a_3\}$ *:*

	μ_1	μ_2	μ_2	$\mu_{1,2}$	$\mu_{1,3}$	$\mu_{2,3}$	$\mu_{1,2,3}$
$\mu_{a_1 \wedge a_2 \wedge a_3}^{LS}$	*0*	*0*	*0*	*0*	*0*	*0*	*1*
$\mu_{(a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3)}$	*0*	*0*	*0*	*1*	*1*	*1*	*1*
$\mu_{a_1 \vee a_2 \vee a_3}^{LS}$	*1*	*1*	*1*	*1*	*1*	*1*	*1*

.

2.3.1 Conjunctive form

Proposition 3 *OWA operator as a function of symmetric fuzzy measure can be represented by next conjunctive logical expression, [1]:*

$$OWA_{\mu}\left(a_1,...,a_n\right)=\sum_{i=1}^{n}\mu_{|i|}\left(A\right)\sum_{\substack{A\subset\Omega;\\|A|=i}}\sum_{B\subset\Omega\backslash A}\left(-1\right)^{|B|}\bigwedge_{a_p\in A\cup B}a_p.$$

Example 4 *New conjunctive logical representation of OWA operator for three attribute case:*

$$\begin{aligned}OWA_{\mu}\left(a_1,a_2,a_3\right)=\mu_{|1|}\left(a_1+a_2+a_3-2a_1\wedge a_2-2a_1\wedge a_3-2a_2\wedge a_3+3a_1\wedge a_2\wedge a_3\right)+\\+\mu_{|2|}\left(a_1\wedge a_2+a_1\wedge a_3+a_2\wedge a_3-3a_1\wedge a_2\wedge a_3\right)+\\+\mu_{|3|}a_1\wedge a_2\wedge a_3.\\=m_{|1|}\left(a_1+a_2+a_3\right)+m_{|2|}\left(a_1\wedge a_2+a_1\wedge a_3+a_2\wedge a_3\right)+m_{|3|}\left(a_1\wedge a_2\wedge a_3\right)\end{aligned}$$

where $m_{|i|}$ are symmetric Möbius coefficients:

$$\begin{aligned}m_{|1|}&=\mu_{|1|}\\m_{|2|}&=\mu_{|2|}-2\mu_{|1|}\\m_{|3|}&=3\mu_{|1|}-3\mu_{|2|}+\mu_{|3|}.\end{aligned}$$

2.4 OWA - new logical representation as a function of symmetric Möbius coefficients

Definition 3 *Symmetric Möbius coefficient is Möbius coefficient which is the function of only cardinal number:*

$$m_{|i|}=m\left(|A|\right);\qquad |A|=i;\qquad A\subset\Omega.$$

Proposition 4 *OWA_m operator as a function of symmetric Möbius coefficients:*

$$OWA_m\left(a_1,...,a_n\right)=\sum_{i=1}^{n}m_{|i|}\sum_{\substack{A\subset\Omega;\\|A|=i}}\bigwedge_{a_p\in A}a_p.$$

Proposition 5 *Connection between symmetric fuzzy measure $\mu_{|i|}$, $i=1,...,n$, and symmetric Möbius coefficients $m_{|i|}$ $i=1,...,n$, for the case of n attributes, is given by the next expression:*

$$\mu_{|i|}=i\sum_{j=1}^{i-1}m_{|j|}+m_{|i|}\qquad i=1,...,n.$$

2.5 Connection between symmetric Möbius coefficients and OWA weights

There is connection between symmetric Möbius coefficients $m_{|i|}$, $i=1,...,n$, and weights in OWA operator w_j $j=1,...,n$, given by next proposition:

Proposition 6 *Connection between symmetric Möbius coefficients $m_{|i|}$, $i=1,...,n$, and weights in OWA operator w_j $j=1,...,n$ is:*

$$w_i=m_{|i-1|}+m_{|i|};\qquad i=1,...,n;\quad m_{|0|}:=0.$$

and inverse

$$m_{|i|}=\sum_{j=1}^{i}\left(-1\right)^{i-j}w_j,\qquad i=1,...,n.$$

2.6 Connection between symmetric fuzzy measure $\mu_{|\bullet|}$ and weights w in OWA

Proposition 7 *Connection between symmetric fuzzy measure $\mu_{|i|}$, $i = 1, ..., n$, and weights w_j, $j = 1, ..., n$ in OWA for the case of n attributes (variables) is given by the next relation:*

$$w_j = \sum_{k=0}^{j-1} (-1)^k \binom{j-1}{k} \mu_{|j-k|}; \qquad j = 1, ..., n;$$

3 Generalization of OWA operator

Generalization of OWA operator is based on replacing AND operator defined as min function, in logical OWA representation with some other function or family of functions, used as AND operator.

3.1 Multiplicative OWA operator

In equivalent logical representation of multiplicative OWA fulgurate as AND operator the multiplicative function

$$\bigwedge_{a_p \in A} a_p = \prod_{a_p \in A} a_p.$$

instead of *min* function

$$\bigwedge_{a_p \in A} a_p = \min_{a_p \in A} \{a_p\}$$

as it is the case in classical OWA operator.

3.1.1 Multiplicative OWA operator in function of OWA weights

$$\begin{aligned} OWA_w(a_1, ..., a_n) &= \sum_{i=1}^{n} w_i \sum_{\substack{A \subset \Omega; \\ |A| \geq i}} (-1)^{|A|-i} \prod_{a_p \in A} a_p \\ \forall w_j &\geq 0, \qquad \sum_{i=1}^{n} w_i = 1 \end{aligned}$$

Example 5 *Multiplicative OWA operator in function of w for the three attributes:*

$$\begin{aligned} OWA_w(a_1, a_2, a_3) &= w_1 (a_1 + a_2 + a_3 - a_1 a_2 - a_1 a_3 - a_2 a_3 + a_1 a_2 a_3) + \\ &+ w_2 (a_1 a_2 + a_1 a_3 + a_2 a_3 - a_1 a_2 a_3) + \\ &+ w_3 a_1 a_2 a_3 \end{aligned}$$

3.1.2 Multiplicative OWA operator in function of symmetric fuzzy measure

$$OWA_\mu(a_1, ..., a_n) = \sum_{i=1}^{n} \mu_{|i|} \sum_{\substack{A \subset \Omega; \\ |A| = i}} \sum_{B \subset \Omega \backslash A} (-1)^{|B|} \prod_{a_p \in A \cup B} a_p.$$

Example 6 *Multiplicative OWA operator in function of μ for the three attributes:*

$$\begin{aligned} OWA_\mu(a_1, a_2, a_3) &= \mu_{|1|} (a_1 + a_2 + a_3 - 2a_1 a_2 - 2a_1 a_3 - 2a_2 a_3 + 3a_1 a_2 a_3) + \\ &+ \mu_{|2|} (a_1 a_2 + a_1 a_3 + a_2 a_3 - 3a_1 a_2 a_3) + \\ &+ \mu_{|3|} (a_1 a_2 a_3). \end{aligned}$$

3.1.3 Multiplicative OWA operator in function of symmetric Möbius coefficients

$$OWA_m\left(a_1,...,a_n\right)=\sum_{i=1}^{n}m_{|i|}\sum_{\substack{A\subset\Omega;\\|A|=i}}\prod_{a_p\in A\cup B}a_p.$$

Example 7 *Multiplicative OWA operator in function of m for the three attributes:*

$$OWA_m\left(a_1,a_2,a_3\right)=m_{|1|}\left(a_1+a_2+a_3\right)+m_{|2|}\left(a_1a_2+a_1a_3+a_2a_3\right)+m_{|3|}a_1a_2a_3$$

3.2 Yager Yager OWA operator

If for AND operator in equivalent logical representation of OWA operator is used Yager generalized AND parametric operator:

$$\bigwedge_{a_p\in\Omega}a_p=1-\min\left\{1,\left[\sum_{a_p\in\Omega}\left(1-a_p\right)^{\lambda}\right]^{\frac{1}{\lambda}}\right\};\quad\lambda>0,$$

it was got the Yager-Yager operator

3.2.1 Yager-Yager OWA operator in function of OWA weights

$$\begin{aligned}OWA_{w,\lambda}\left(a_1,...,a_n\right)&=\sum_{i=1}^{n}w_i\sum_{\substack{A\subset\Omega;\\|A|\geq i}}(-1)^{|A|-i}\left(1-\min\left\{1,\left[\sum_{a_p\in A\cup B}\left(1-a_p\right)^{\lambda}\right]^{\frac{1}{\lambda}}\right\}\right);\quad\lambda>0.\\ \forall w_j&\geq 0,\qquad\sum_{i=1}^{n}w_i=1\end{aligned}$$

3.2.2 Yager-Yager OWA operator in function of symmetric fuzzy measure

$$OWA_{\mu,\lambda}\left(a_1,...,a_n\right)=\sum_{i=1}^{n}\mu_{|i|}\sum_{\substack{A\subset\Omega;\\|A|=i}}\sum_{B\subset\Omega\backslash A}(-1)^{|B|}\left(1-\min\left\{1,\left[\sum_{a_p\in A\cup B}\left(1-a_p\right)^{\lambda}\right]^{\frac{1}{\lambda}}\right\}\right);\quad\lambda>0.$$

3.2.3 Yager-Yager OWA operator in function of symmetric Möbius coefficients

$$OWA_{m,\lambda}\left(a_1,...,a_n\right)=\sum_{i=1}^{n}m_{|i|}\sum_{\substack{A\subset\Omega;\\|A|=i}}\left[1-\min\left\{1,\left[\sum_{a_p\in A\cup B}\left(1-a_p\right)^{\lambda}\right]^{\frac{1}{\lambda}}\right\}\right];\quad\lambda>0.$$

4 Conclusion

In this paper is given logical representation of OWA operator, based on logical representation of discrete fuzzy Choquet integral [1]. Symmetric fuzzy measure and symmetric Möbius coefficients are defined. Relationship between symmetric fuzzy measure and/or symmetric Möbius coefficients with weighted coefficients in OWA operator are derived. Equivalent representations of OWA operator in function of symmetric fuzzy measure and symmetric Möbius coefficients are given.

Generalization of OWA operator is based on substitution of AND operator, defined as a min function, in logical representation, by some other continual AND operator - for example: multiplicative function.

Possibilities of applying the logical representation of OWA operator and generalized OWA operators in different fields (such as multi-variable statistical analysis, data mining, reliability analysis, pattern recognition, fuzzy-neural systems, etc.) will be the subject of our forthcoming paper.

References

[1] D. Radojevic, Logical representation and generalization of discrete Choquet integral, EUFIT'99.

[2] D. Radojevic, Logical representation and generalization of order weight aggregation operator, Submitted for publication in: International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems.

[3] R. Yager, On ordered weighted averaging aggregators in multi-criteria decision making, IEEE Trans. Sys.Man & Cybernetics, 18. (1988) pp.183-190.

[4] R. Yager, Families of OWA operator, Fuzzy Sets and Systems, 59 (1993) pp. 125-148.