System reliability analysis and multiplicative Choquet integral

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Abstract

Multiplicative fuzzy Choquet integral is proposed as a mathematical tool for system reliability calculation. In this context, a logical fuzzy measure is structural function defined in reliability theory. Fuzzy measure is generalization of structural function and can be use for reliability analysis of the system with "unclear" structure (biological, system operator etc.)

Keywords: Fuzzy measure, logical fuzzy measure, structural function, multiplicative Choquet integral, reliability analysis

1 Introduction

There are many papers which deal with application of fuzzy logic in reliability engineering, [2]. The main object in almost all of these papers is management of uncertainty immanent to component reliability (or probability of failure of components) and their influence on system reliability.

Uncertainty of system structure is very important and interesting problem in reliability engineering (non-standard system: biological systems, system operator, etc.).

In this paper is given mathematical tool for system reliability analysis in the presence of system structure uncertainty. New approach is based on fuzzy measure and multiplicative Choquet integral as a special case of generalized fuzzy discrete Choquet integral, [4]. It was shown that logical fuzzy measure is determined by structural function from system reliability theory and fuzzy measure can be viewed as a generalization of structural function for a "unclear" structure. The value of multiplicative fuzzy Choquet integral on values of system elements reliability for logical measure, which corresponds to analyze structure is equal to the system reliability. In general case, when structure is "unclear", i.e. it can be described as something between some clear structures, a fuzzy measure should be used instead of a logical fuzzy measure.

In the first section are given the known definitions of fuzzy measure, discrete Choquet integral and defined multiplicative discrete fuzzy Choquet integral.

In the second section among the known definitions from system reliability theory are given: (a) the connections between structural function and fuzzy logical measure and (b) system reliability calculation from the system elements reliability by multiplicative fuzzy Choquet integral.

Possibilities of applying the fuzzy measure and multiplicative fuzzy integral to reliability analysis of system with "unclear" structure (biological systems, system operator etc.) will be the subject of our forthcoming paper.

2 Multiplicative Choquet integral

In this section are given the known definitions of fuzzy measure and Choquet integral and is introduced logical representation of multiplicative Choquet integral. We consider discrete spaces only. In the whole paper, we will work on a finite universe Ω of n elements, $\Omega = \{R_1, ..., R_n\}$. $\mathcal{P}(\Omega)$ is the power set of Ω, while $|\Omega|$ denotes the cardinal of a set Ω, and $A \setminus B$ denotes the set difference.

2.1 Fuzzy measure and Choquet integral [3]

Definition 1 *[5] A fuzzy measure μ is a mapping $\mu : F(\Omega) \rightarrow [0,1]$ such that, for every A and B in $F(\Omega)$:*

1. $\mu(\emptyset) = 0, \qquad \mu(\Omega) = 1$ *(boundary condition)*
2. *if $B \subseteq A$, then $\mu(B) \leq \mu(A)$,* *(monotonicity)*

where: Ω is any set of elements, $F(\Omega)$ is the set of fuzzy subsets of Ω, and A, B are fuzzy sets.

Definition 2 *A logical fuzzy measure μ^L is a fuzzy measure, which is mapping $\mu^L : F(\Omega) \rightarrow \{0,1\}$*

Definition 3 *[1] Let μ be a discrete fuzzy measure on Ω, whose elements are denoted $R_1, ..., R_n$ here. The discrete Choquet integral with respect to μ is defined by*

$$C_\mu(R_1, ..., R_n) := \sum_{i=1}^{n} \left(R_{(i)} - R_{(i-1)}\right) \mu\left(A_{(i)}\right)$$

where $\cdot_{(i)}$indicates that the indices have been permuted so that $0 \leq R_{(1)} \leq ... \leq R_{(n)}$, and $A_{(i)} := \left\{R_{(i)}, ..., R_{(n)}\right\}$, and $R_{(0)} = 0$.

An equivalent definition of discrete fuzzy Choquet integral is:

Definition 4 *Let μ be a discrete fuzzy measure on Ω, whose elements are denoted $R_1, ..., R_n$ here. The discrete Choquet integral with respect to μ is defined by*

$$C_\mu(R_1, ..., R_n) := \sum_{i=1}^{n} R_{(i)}(\mu\left(A_{(i)}\right) - \mu\left(A_{(i+1)}\right))$$

where $\cdot_{(i)}$indicates that the indices have been permuted so that $0 \leq R_{(1)} \leq ... \leq R_{(n)}$, and $A_{(i)} := \left\{R_{(i)}, ..., R_{(n)}\right\}$, and $\mu\left(A_{(n+1)}\right) = 0$.

A discrete fuzzy Choquet integral enables modeling of positive interaction and redundancy between attributes. For reliability analysis multiplicative Choquet integral is very suitable:

2.2 Logical representation and multiplicative Choquet integral

Logical representations of discrete Choquet integral from [4] are

$$CM_\mu(R_1, ..., R_n) = \sum_{A_i \subset \Omega} \mu(A_i) \sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \bigwedge_{R_p \in A_i} R_p \bigwedge_{R_q \in B_k} R_q,$$

and

$$CM_m(R_1, ..., R_n) = \sum_{A_i \subset \Omega} m(A_i) \bigwedge_{R_p \in A_i} R_p,$$

where: m is Möbius transform of fuzzy measure μ, defined as:

$$m(A) := \sum_{B \subset A} (-1)^{|A \setminus B|} \mu(B), \quad A \in \mathcal{P}(\Omega).$$

Multiplicative AND function (t-norm) is:

$$R_1 \wedge R_2 = R_1 R_2,$$

and in the general case (more than two attributes):

$$\bigwedge_{R_p \in A} R_p = \prod_{R_p \in A} R_p.$$

Multiplicative discrete Choquet integral CM_μ is above logical representations of discrete Choquet integral with multiplicative AND function:

$$CM_\mu (R_1, ..., R_n) = \sum_{A_i \subset \Omega} \mu (A_i) \sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \prod R_p \prod R_q$$

or

$$CM_m (R_1, ..., R_n) = \sum_{A_i \subset \Omega} m (A_i) \prod R_p.$$

Proposition 1 *A multiplicative discrete fuzzy Choquet integral is:*

$$CM_\mu (R_1, ..., R_n) = \sum_{A_i \subset \Omega} \mu (A_i) \prod_{R_p \in A_i} R_p \prod_{R_q \in \Omega \setminus A_i} \overline{R}_q$$

where: $\overline{R}_q = 1 - R_q$.
Proof. From

$$CM_\mu (R_1, ..., R_n) = \sum_{A_i \subset \Omega} \mu (A_i) \sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \prod_{R_p \in A_i} R_p \prod_{R_q \in B_k} R_q$$

and

$$\sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \prod_{R_p \in A_i \cup B_k} R_p = \prod_{R_p \in A_i} R_p \sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \prod_{R_q \in B_k} R_q$$

and

$$\begin{aligned} \sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \prod_{R_q \in B_k} R_q &= \prod_{R_q \in \Omega \setminus A_i} (1 - R_q) \\ &= \prod_{R_q \in \Omega \setminus A_i} \overline{R}_q \end{aligned}$$

it follows:

$$\sum_{B_k \subset \Omega \setminus A_i} (-1)^{|B_k|} \prod_{R_p \in A_i \cup B_k} R_p = \prod_{R_p \in A_i} R_p \prod_{R_q \in \Omega \setminus A_i} \overline{R}_q. \blacksquare$$

A multiplicative discrete fuzzy Choquet integral is a very promising tool for reliability analysis of systems with a non-clear structure (for example, reliability of an operator).

3 System reliability analysis based on multiplicative Choquet integral

Here is given revue of the basic elements for system reliability analysis and their connection with basic elements defined in the field of fuzzy measure and fuzzy integrals.

3.1 Elements of classical system reliability analysis

3.1.1 Notation

P_i = probability of failure of component i.

R_i = reliability of component i, is defined as:

$$R_i = 1 - P_i$$

P_{sys} =system probability of failure

R_{sys} = system reliability, defined as:

$$R_{sys} = 1 - P_{sys}$$

C_i, $i = 1, ..., I$ Components of system

y_i indicator of the i-th element state

$$y_i = \begin{cases} 1 & i \text{ element is corect} \\ 0 & i \text{ element is not corect} \end{cases}$$

$\overline{y}$ vector of performance indicators of system elements $\overline{y} = (y_1, ..., y_n)$

$h(\overline{y})$ system structural function (SSF) gives information about performance of system based on system components performance

3.1.2 Some characteristics of system structural function

- SSF is binary (Bool) function
- Boundary condition of SSF:

$$h(\overline{0}) = 0; \qquad \overline{0} = (0, ..., 0)$$
$$h(\overline{1}) = 1; \qquad \overline{1} = (1, ..., 1)$$

- SSF is monotone function

$$h(\overline{y}) \geq h(\overline{z}), \qquad y_i \geq z_i, \ i = 1, ..., I$$

System reliability for characteristic structures:

3.1.3 Parallel Systems

System probability of failure in function of components probability of failures for parallel system is:

$$P_{sys} = P_1 \times P_2 \times ... \times P_n.$$

System reliability in function of components reliability for parallel system is:

$$R_{sys} = 1 - \left[(1 - R_1)(1 - R_2) ... (1 - R_n)\right].$$

3.1.4 Series Systems

System reliability in function of components reliability for series system is

$$R_{sys} = R_1 \times R_2 \times ... \times R_n$$

System probability of failure in function of components probability of failures for series system is:

$$P_{sys} = 1 - \left[(1 - P_1)(1 - P_2) ... (1 - P_n)\right]$$

3.2 Structural function and fuzzy logical measure

System structural function from system reliability theory corresponds to logical fuzzy measure (logical function, boundary conditions and monotonicity) in the field of fuzzy measure and fuzzy integrals:

$$\mu^L(A) = h(\overline{y}) \qquad y_i = \begin{cases} 1 & i \in A \\ 0 & i \notin A \end{cases}$$

and as a consequence to Möbius transform of logical fuzzy measures.

Example 1 *Structural function and logical measure as well as Möbius transform of logical fuzzy measures for series and parallel systems with two components, in tabular form are given in next two tables:*

	Series System	*Parallel System*
$\mu^L(\{R_1\}) = h(1,0)$	0	1
$\mu^L(\{R_2\}) = h(0,1)$	0	1
$\mu^L(\{R_1,R_2\}) = h(1,1)$	1	1

	Series System	*Parallel System*
$m(\{R_1\}) = h(1,0)$	0	1
$m(\{R_2\}) = h(0,1)$	0	1
$m(\{R_1,R_2\}) = h(1,1) - h(1,0) - h(0,1)$	1	-1

3.3 Function of system reliability and multiplicative fuzzy Choquet integral

System reliability can be calculated by application of multiplicative fuzzy Choquet integral on values of components reliability for logical fuzzy measure which corresponds to analyzed structure

$$R_{sys} = CM_{\mu^L}(R_1, ..., R_n),$$

or to Möbius transform of relative fuzzy measure:

$$R_{sys} = CM_m(R_1, ..., R_n).$$

Example 2 *Expression for reliability of the two elements system in the function of fuzzy measure (structural function) in the general case is*

$$R_{sys}(R_1,R_2) = CM_{\mu^L}(R_1,R_2) = \mu^L(\{R_1\})R_1(1-R_2) + \mu^L(\{R_2\})R_2(1-R_1) + \mu^L(\{R_1,R_2\})R_1R_2.$$

From the example 1:

$R_{sys}(R_1,R_2) = R_1R_2$	*Series System*
$R_{sys}(R_1,R_2) = R_1(1-R_2) + R_2(1-R_1) + R_1R_2 = R_1 + R_2 - R_1R_2$	*Parallel System*

Expression for reliability of the two elements system in the function of Möbius coefficients in the general case is

$$R_{sys}(R_1,R_2) = CM_m(R_1,R_2) = m(\{R_1\})R_1 + m(\{R_2\})R_2 + m(\{R_1,R_2\})R_1R_2$$

From the example 1:

$R_{sys}(R_1,R_2) = CM_m(R_1,R_2) = R_1R_2$	*Series System*
$R_{sys}(R_1,R_2) = CM_m(R_1,R_2) = R_1 + R_2 - R_1R_2$	*Parallel System*

4 Conclusion

In this paper is shown that multiplicative discrete fuzzy Choquet integral can be used for system reliability analysis and correspondences between basic elements from reliability theory and fuzzy measure and fuzzy integral are established. Multiplicative Choquet integral is a special case of generalized fuzzy discrete Choquet integral [4]. Generalization is based on logical representation of fuzzy integral [4]. In logical representation of fuzzy discrete Choquet integral AND operator is Min - function. AND operator in the case of multiplicative fuzzy Choquet integral is multiplicative function.

Logical fuzzy measure of system structure corresponds to system structural function, defined in reliability engineering. Fuzzy measure is generalization of structural function and can be used for reliability analysis of the system with "unclear" - fuzzy structure, structure which can be described by "interpolation" between some classical - crisp structures. Value of multiplicative fuzzy discrete Choquet integral for components reliability by fuzzy measure which corresponds to system structure is equal to system reliability.

Possibilities of applying the fuzzy measure and multiplicative fuzzy integral to reliability analysis of system with "unclear" structure (biological systems, system operator etc.) will be the subject of our forthcoming paper.

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