Stability Analysis of Dynamic Fuzzy Systems

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ABSTRACT: Qualitative modeling is often the only way towards an automation if quantitative process models are not available. In this contribution, a concept of qualitative modeling of dynamic processes using Dynamic Fuzzy Systems is presented and a new approach for stability analysis of these systems is developed. In contrast to common approaches of fuzzy modeling [Babuš ka (1996)], the dynamic system is completely described in the fuzzy domain: The fuzzy information about the previous state is directly applied to compute the system's current state, i.e. the delayed fuzzy output is fed back to the input without defuzzification. Since the current state of a Dynamic Fuzzy System is represented by fuzzy numbers, a new inference method, the so called Inference with Interpolating Rules is developed and a new stability definition is derived. For stability analysis, an analytic procedure for first order Dynamic Fuzzy Systems is expounded and a numerical approach for the analysis of higher order Dynamic Fuzzy Systems is presented.

KEYWORDS: Stability Analysis, Dynamic Fuzzy Systems, Inference with Interpolating Rules, Qualitative Modeling

1 INTRODUCTION

The analysis and control of complex plants often requires the introduction of qualitative process models since quantitative process models are not available. However, human experts as operators are usually able to accomplish control tasks, taking into consideration only imprecise knowledge about the process. Vague and imprecise knowledge may be described linguistically by a set of rules like

If valve is ”open wide” **Then** liquid level is ”rising fast”

what naturally leads to the concept of fuzzy modeling.
In this contribution, modeling is achieved using a particular class of Dynamic Fuzzy Systems where the nonlinear static characteristics of the process and – in contrast to common approaches [Babuš ka (1996)] - as well its dynamics are represented in the fuzzy domain.
To be more specific, Figure 1 shows an autonomous first order Dynamic Fuzzy System.

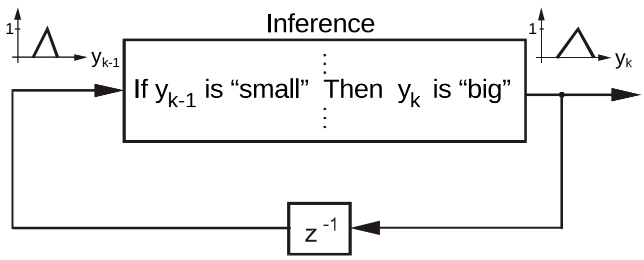

Figure 1: Autonomous first order Dynamic Fuzzy System

The rule base may consist of rules like

If y_{k-1} is ”small” **Then** y_k is ”big”.

Linguistic terms like ”small” are modeled by fuzzy sets. The knowledge propagation is carried out by a fuzzy inference method.

Since the fuzzy output is fed back without a prior defuzzification, the linguistic information about the system is completely modeled in the fuzzy domain. As a consequence, a new inference scheme has to be derived for the following reason:

An inference method has to evaluate a set of fuzzy rules corresponding to the human way of approximate reasoning. Human beings are able to process only such fuzzy sets that might be properly adjoined to linguistic values. Therefore, only this kind of interpretable fuzzy sets are appropriate inputs of Fuzzy Systems. Since the fuzzy output of a Dynamic Fuzzy System has to be processed by the inference in subsequent steps, it has to be guaranteed that the inference maps interpretable fuzzy inputs onto an interpretable fuzzy output.

In the sequel, fuzzy numbers with triangular shaped membership functions which are often used to characterize linguistic values like ”small” or ”big” will be used as interpretable fuzzy sets.

Conventional reasoning methods like the ”max-min-inference” [Driankov (1993)] do not generate an interpretable fuzzy output. Therefore, a new fuzzy inference method called the ”Inference with Interpolating Rules” was developed which is outlined in the second section. This method is the central element of a new systems theory covering processes represented by a set of fuzzy rules.

Within the scope of this systems theory, stability analysis is of particular importance as stability is an essential property for any dynamic process. Thus, in the third section a new stability definition for Dynamic Fuzzy Systems is expounded. The stability of first order Dynamic Fuzzy Systems which can be analyzed analytically is investigated in the fourth section. For the analysis of second and higher order systems, numerical approaches are presented in the fifth section.

2 INFERENCE WITH INTERPOLATING RULES

Considering a simple tank system, the underlying interpretation of the fuzzy rules becomes obvious:

A small influx q leads to a slowly rising height h in the tank, whereas a large influx results in a fast rising height. Thus, the two fundamental rules

If q is ”small” **Then** h is ”rising slowly”
If q is ”large” **Then** h is ”rising fast”

may be sufficient as a linguistic description of the process. The linguistic values ”small”, ”large”, ”rising slowly”, and ”rising fast” are modeled with the fuzzy sets *A, B, C,* and *D*[1], respectively. Figure 2 shows the membership functions of these fuzzy sets.

Such membership functions with gaps in between define a so called sparse rule base. Obviously, the rules cannot be evaluated with conventional reasoning methods for arbitrary influx.

For example, if q is ”medium”, the height is ”rising moderately”. This rule is hidden in the two fundamental rules given above since we know that an increasing influx leads to a faster rising height. A max-min-inference would not result in a suitable output since there is no rule in the rule set for a ”medium” fuzzy input.

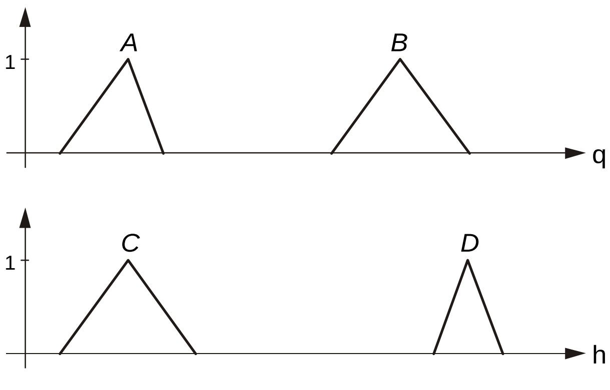

Figure 2: Membership functions defined for the rule set.

The new inference mechanism automatically generates an interpolating rule from the fundamental rules depending on the current input. This interpolating rule is evaluated in the next step to map the fuzzy input onto the fuzzy output.

The interpolating rule

If q is *IP* **Then** h is *IC*

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[1] Italic capital letters are used to label fuzzy sets and fuzzy mappings

consists of an interpolating premise *IP* and an interpolating conclusion *IC* (figure 3).

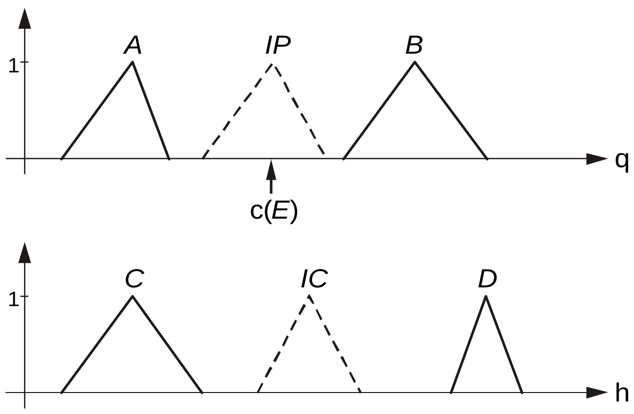

Figure 3: Interpolating premise and interpolating conclusion.

Both, *IP* and *IC* result from fuzzy mappings [Krebs (1998), Schäfers (1999)]. A fuzzy mapping is an operator mapping fuzzy or crisp inputs onto a fuzzy output depending on fuzzy or crisp parameters.
The fuzzy set *IP* results from a fuzzy mapping G_{IP}:

$$IP = G_{IP}\left(c(E); A, B\right).$$

The fuzzy parameters of the mapping G_{IP} are the fuzzy sets *A* and *B* defined for the premises of the fundamental rules. The input of G_{IP} is the center c(*E*) of the fuzzy rule input.
The mapping G_{IC} generates the interpolating conclusion *IC*

$$IC = G_{IC}\left(c(E); c(A), c(B), C, D\right)$$

where *c*(*E*) is the crisp input of G_{IC}, while the centers *c*(*A*) and *c*(*B*) are the crisp parameters of the mapping and the sets *C* and *D* represent the fuzzy parameters.
Both G_{IP} and G_{IC} are subject to certain constraints. E.g., the mappings must be steady in all parameters and inputs. *IP* and *IC* have to be interpretable fuzzy sets. Furthermore, the implicit knowledge represented by the interpolating rule must not be in conflict with the explicit knowledge given with the fundamental rules:
If the center of the fuzzy input is equivalent to the center of a fuzzy set defined for the premise of the fundamental rules, the interpolating rule must be identical to the corresponding fundamental rule, i.e.

$$c(E) = c(A) \;\Rightarrow\; (IP = A) \wedge (IC = C)$$
$$c(E) = c(B) \;\Rightarrow\; (IP = B) \wedge (IC = D).$$

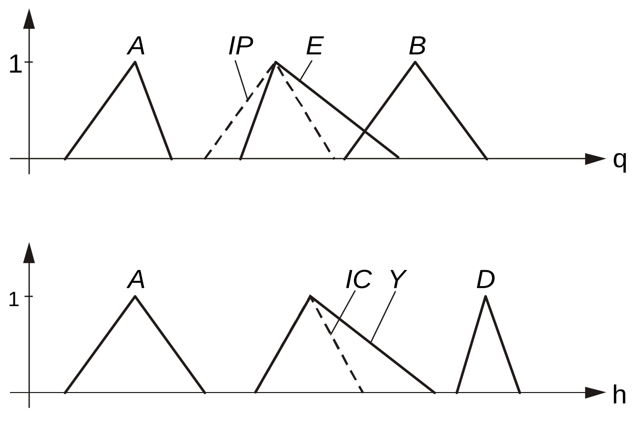

Figure 4: Evaluating the interpolating rule.

The interpolating rule is evaluated in the next step. If the fuzzy input *E* is a subset of the interpolating premise (*E* Í *IP)*, the fuzzy output *Y* equals the interpolating conclusion (*Y=IC)*:

$$E \subseteq IP \;\Rightarrow\; Y = IC\,.$$

Otherwise, the fuzziness of the output is increased applying the operator *F* (figure 4):

$$Y = F(E; A, B, C, D, IC, IP)\,.$$

In addition to the fuzzy sets defined for the fundamental rules, the interpolating premise and the interpolating conclusion are fuzzy parameters of the mapping *F*. Like the fuzzy mappings G_{IP} and G_{IC} the mapping *F* has to satisfy some particular conditions [Krebs (1998), Schäfers (1999)].
The basic ideas of the Inference with Interpolating Rules have been presented so far. For systems with multiple inputs the inference is slightly modified [Schäfers (1999)].
To conclude this section, the Inference with Interpolating Rules represents the essential framework for process modeling with Dynamic Fuzzy Systems. The following sections outline the fundamentals of stability analysis for Dynamic Fuzzy Systems.

3 A NEW STABILITY DEFINITION FOR DYNAMIC FUZZY SYSTEMS

To show some typical behavior of Dynamic Fuzzy Systems and to obtain an appropriate stability definition, a simple autonomous Fuzzy System represented by the following two rules is considered:

If y_{k-1} is ”negative” **Then** y_k is ”positive”
If y_{k-1} is ”positive” **Then** y_k is ”negative”.

The membership functions defined on the input domain are shown in figure 5.

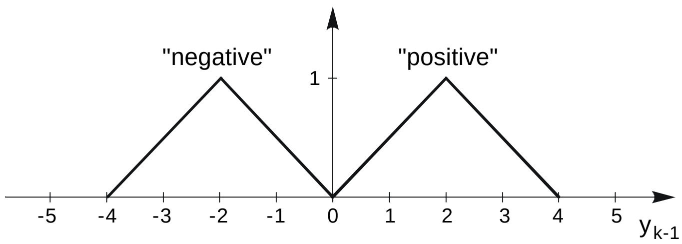

Figure 5: Membership functions defined for y_{k-1}.

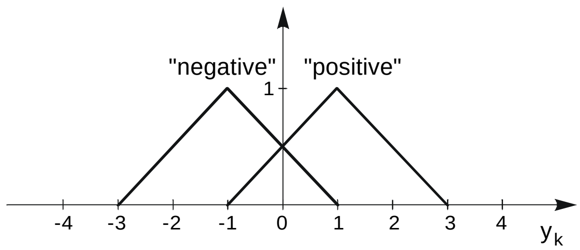

Figure 6: Output membership functions of system 1.

Depending on the output membership functions, the system exhibits different dynamic behavior. Given the output membership functions of figure 6, we obtain system 1 which is stable since the output converges to the fuzzy number Y_∞ with the center 0, the left foot -2 and the right foot +2. Figure 7 depicts the fuzzy output resulting from a crisp initial state y_0=2.

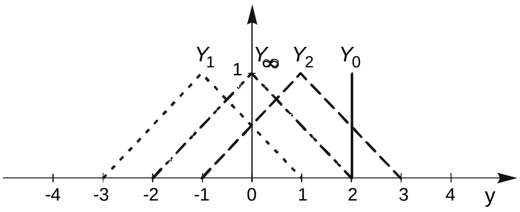

Figure 7: Dynamic behavior of system 1.

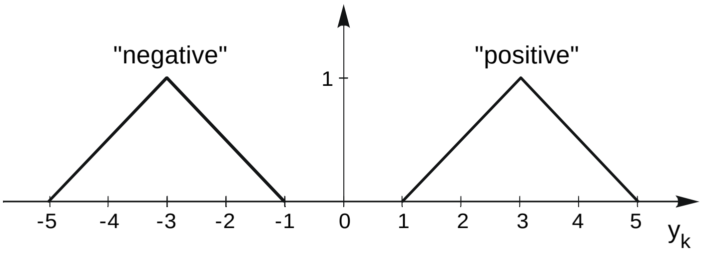

Figure 8: Output membership functions of system 2.

In contrast to system 1, the centers of the output membership functions of system 2 (figure 8) are further away from the origin than the centers of the input membership functions (figure 5). As figure 9 shows, an unstable system behavior results since the fuzzy output moves away from the origin. Thus, the origin is an unstable equilibrium point for the center of the output of system 2, whereas it is an asymptotically stable equilibrium point for the center of the output of system 1.

But, a Dynamic Fuzzy System may show an unsatisfactory behavior even if the origin is a stable equilibrium point for the center of the output. Figure 10 shows the output membership functions of system 3: The centers of the output membership functions are closer to the origin than the centers of the input membership functions, but the output membership functions are fuzzier than the ones defined on the input domain. Therefore, the center of the output converges to 0 for any initial state, but its left and right foot move to infinity (figure 11). Since the output becomes fuzzier with every step, the specificity of the output vanishes for $k \to \infty$. Finally, no information about the actual output of the process modeled by the Dynamic Fuzzy System is left. Consequently, such an equilibrium point is called unstable.

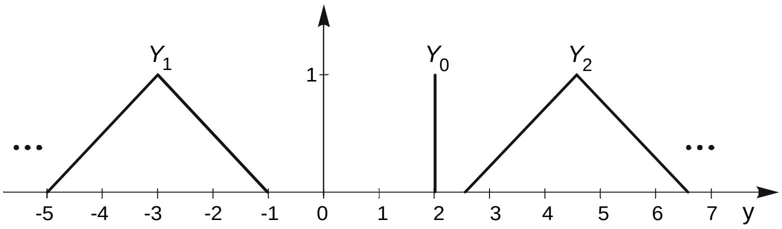

Figure 9: Dynamic behavior of system 2.

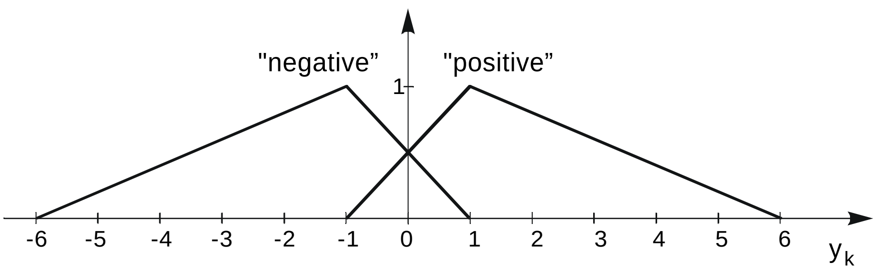

Figure 10: Output membership functions of system 3.

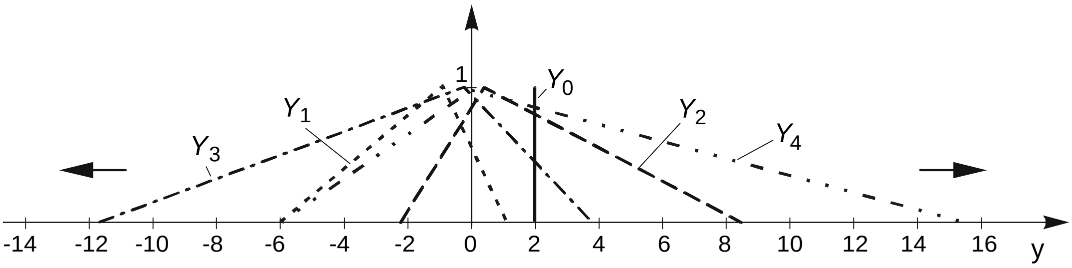

Figure 11: Dynamic behavior of system 3.

These simple examples suggest the following stability definition for Dynamic Fuzzy Systems:

Definition: Stability of Dynamic Fuzzy Systems

An equilibrium point of a Dynamic Fuzzy System marked by a crisp value R_0 is stable iff

- R_0 is an asymptotically stable equilibrium point for the center of the output $c(Y_k)$
- the feet of the fuzzy output stay in a bounded environment of R_0.

In the examples above, $R_0=0$ marks the equilibrium point. Only system 1 has a stable equilibrium point, whereas the equilibrium points of system 2 and system 3 are unstable.
For stability analysis, we make use of a particular property of the Inference with Interpolating Rules [Krebs (1998), Schäfers (1999)]: The center of the fuzzy output $c(Y)$ of the inference exclusively depends on the centers of the fuzzy inputs $c(E_1)$, $c(E_2)$,... $c(E_p)$[2]. This relationship is expressed by the center equation

$$c(Y)=f_c(c(E_1), c(E_2),... c(E_p)).$$

Therefore, to analyze whether R_0 is an asymptotically stable equilibrium point for the center of the output, it is sufficient to consider the center equation.
The analysis for first order Dynamic Fuzzy Systems can be realized analytically whereas numerical methods have to be applied for second and higher order systems. The following section focuses on the analysis of first order systems.

4 STABILITY ANALYSIS OF FIRST ORDER DYNAMIC FUZZY SYSTEMS

The dynamics of the center of a first order Dynamic Fuzzy System (figure 1) is expressed by the piecewise linear center equation

$$c(Y_k) = f\left(c(Y_{k-1})\right)$$

and can be analyzed independently from the dynamics of the feet of the output. Without loss of generality, the equilibrium point is assumed to be zero. To simplify the notation, y_k and y_{k-1} instead of $c(Y_k)$ and $c(Y_{k-1})$ will be used.
The dynamics of the center of the fuzzy output may be illustrated in the y_k-y_{k-1}-plane. Figure 12 shows how to determine the sequence y_1, y_2,... step by step starting with an initial value y_0: The center of the fuzzy output in subsequent steps follows from the center equation $y_k=f(y_{k-1})$. As illustrated with the dotted arrows in figure 12, the center is fed back to the input using the bisector $y_k=y_{k-1}$. The center equation is evaluated again in the following step.
In accordance with [LaSalle (1986)], it is possible to conclude from the stability of the linearized system to the stability of the equilibrium point. Thus, if the linearized system

$$Dy_k = f'(0) \cdot Dy_{k-1}$$

is stable, i.e. $|f'(0)| < 1$, the equilibrium point is asymptotically stable. Hence, in the neighborhood of the origin, the center equation has to be in between the two bisectors $y_k=y_{k-1}$ and $y_k=-y_{k-1}$.

[2] This property even holds for any inference scheme which is suitable for Dynamic Fuzzy Systems of the considered class [Krebs (1998), Schäfers (1999)].

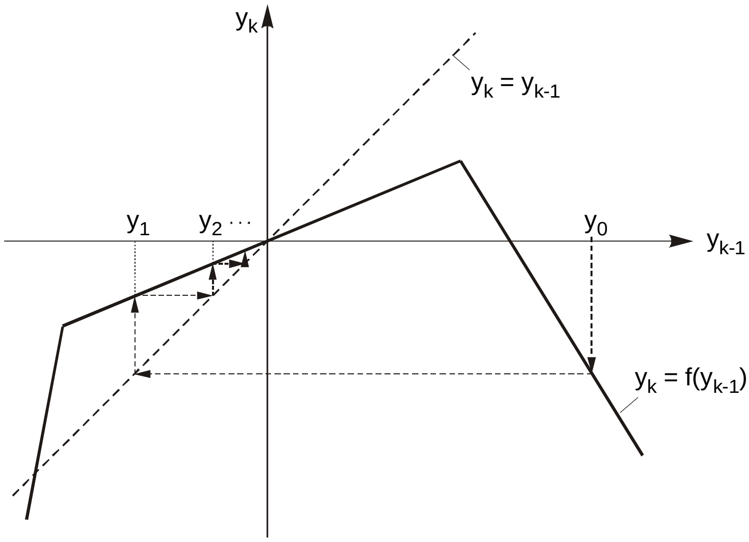

Figure 12: Determining the sequence of the center of the output for a first order Dynamic Fuzzy System

The domain of attraction of a stable equilibrium point is determined following the procedure illustrated in figure 13: Apparently, the equilibrium point is stable. In the first region E_1, the characteristics is in between the sector bounded by the bisector y_k=y_{k-1} and the y_{k-1}-axis. Every initial value in E_1 results in a sequence converging to zero. Therefore, E_1 is part of the domain of attraction of the equilibrium point. Region $\tilde{E}_1$ comprises the same range of values as E_1 and is defined on y_k.

All sequences starting from the second region E_2 have their first value inside $\tilde{E}_1$. Region E_2 allows to determine region E_3. All initial values inside E_3 result in sequences with their first value inside $\tilde{E}_2$ and their second value inside $\tilde{E}_1$. Thus, all sequences following from initial values inside E_2 and E_3 converge to zero. Hence, both E_2 and E_3 are part of the domain of attraction.

Accordingly, further parts of the domain of attraction may be determined: E_4 follows from E_3, E_5 from E_4 and so on. If it is not possible to determine another region from E_q, the domain of attraction E is entirely known:

$$E = E_1 \cup E_2 \cup \ldots E_q .$$

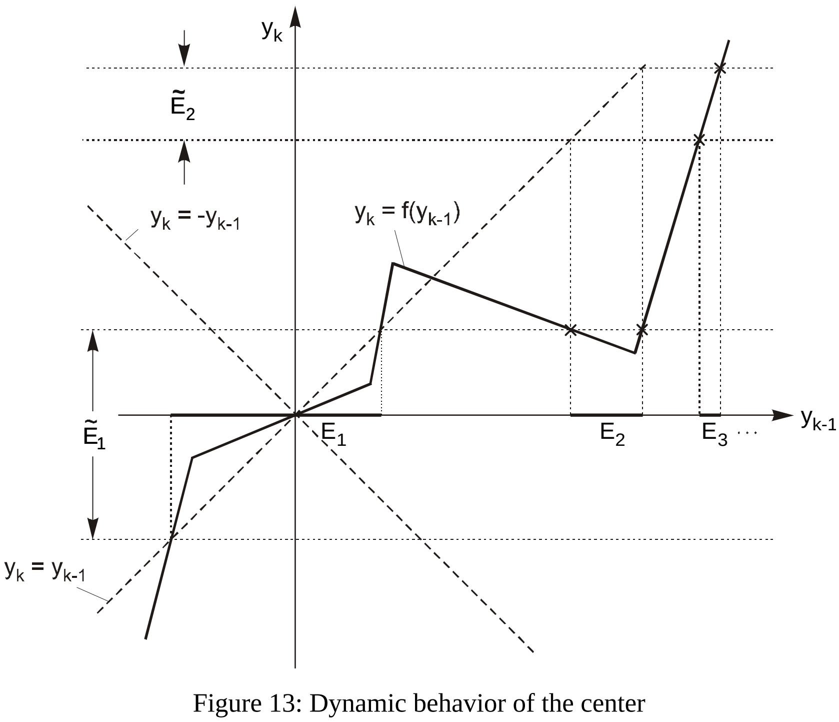

Figure 13: Dynamic behavior of the center

The example illustrated in figure 14 shows a domain of attraction consisting of two parts (q=2).

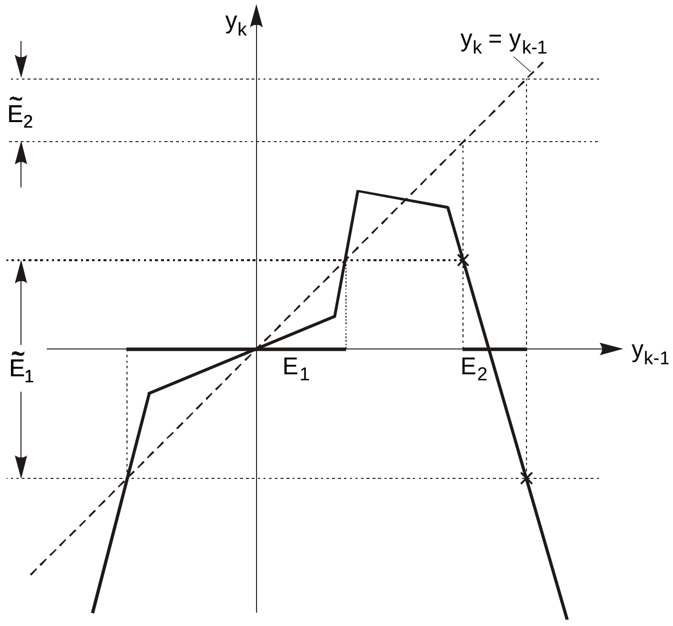

Figure 14: The domain of attraction consists of two regions

In the examples depicted in figure 13 and figure 14, the characteristics is inside the first and third quadrant in the environment of the origin. Thus, region E_1 follows from the intersections of the characteristics with the bisector $y_k=y_{k-1}$ or the y_{k-1}-axis.
If the characteristic is inside the second and fourth quadrant in an environment of the origin, the sign of the output changes with every step. Considering only every second value of the sequence, we come to the mapping

$$y_{k+1} = f\left(y_k\right) = f\left(f\left(y_{k-1}\right)\right) = \tilde{f}\,(y_{k-1})$$

with a characteristic $\tilde{f}\left(y_{k-1}\right)$ inside the first and third quadrant. Thus, the above procedure may be applied to determine the domain of attraction of the system $y_{k+1} = \tilde{f}\,(y_{k-1})$ which is equivalent to the domain of attraction of the original system [Karweina (1989)].
If the characteristics is identical with the y_{k-1}-axis in the environment of the origin, E_1 is equivalent to the region with $f(y_{k-1})=0$. From E_1, the output moves to the origin in a single step. Further parts of the domain of attraction are determined following the procedure above.
Assuming a characteristics with a continuos derivative in the neighborhood of the origin, all cases have been discussed so far. For characteristics with discontinuous derivatives, a similar procedure may be developed.
If the equilibrium point R_0 of the center of the fuzzy state is stable, the dynamics of the feet of the fuzzy output has to be analyzed to judge the stability of the fuzzy equilibrium marked by R_0. With a constant center $c\left(Y\right)$, the dynamics of the left and right foot *l(Y)* and *r(Y)* is described by the equation

$$\begin{pmatrix} r\left(Y_k\right) \\ l\left(Y_k\right) \end{pmatrix} = \underline{A}_{c(Y)} \cdot \begin{pmatrix} r\left(Y_{k-1}\right) \\ l\left(Y_{k-1}\right) \end{pmatrix} + \underline{b}_{c(Y)} \qquad (1)$$

with a constant matrix $\underline{A}_{c(Y)}$ and a constant vector $\underline{b}_{c(Y)}$ [Schäfers (1999)]. Thus, the dynamics of the feet can be analyzed considering the eigenvalues of $\underline{A}_{c(Y)}$ when the center is in its equilibrium, e.i. the eigenvalues of $\underline{A}_{R_0}$ have to be investigated. Due to the constant matrix in relationship (1), the domain of attraction of a fuzzy equilibrium point is equivalent to the domain of attraction determined for the center of the fuzzy state.

5 STABILITY ANALYSIS OF SECOND AND HIGHER ORDER DYNAMIC FUZZY SYSTEMS

For stability analysis of higher order Dynamic Fuzzy Systems, it is assumed that all interpolating premises defined on $y_{k-1}, \ldots y_{k-n}$ are fuzzier than the interpolating conclusion with the same center defined on y_k. Therefore, the fed back fuzzy output is always a subset of the interpolating premise and the fuzziness operator F is without effect on the fuzzy system output. Hence, the stability of an equilibrium point directly follows from the stability of the center equation [5].

The stability of an equilibrium point y_0 can be proved by considering the first derivative of the center equation [LaSalle (1986)]. To determine parts of the domain of attraction of a stable equilibrium point, the "Convex Decomposition" [Kiendl (1986), Karweina (1989), Rumpf (1997)] as an efficient numerical stability analysis method and an approach based on "Integral Ljapunov Functions" [Kiendl (1991), Scheel (1996)] have been successfully applied to Dynamic Fuzzy Systems.
The following example shows which parts of the domain of attraction of an equilibrium point can be determined using both the Convex Decomposition and Integral Ljapunov functions:
The fuzzy output Y_k of the considered Dynamic Fuzzy System depends on its delays Y_{k-1} and Y_{k-2}. The rule base consists of 9 rules like

If $Y_{k-1} = A$ **And** $Y_{k-2} = D$ **Then** $Y_k = AD$.

The corresponding membership functions are depicted in figure 15. Since all membership functions defined for Y_{k-1} and Y_{k-2} are fuzzier than the membership functions defined for the output Y_k, the fed back fuzzy output is always a subset of the interpolating premises. Consequently, for stability analysis only the center equation has to be examined.

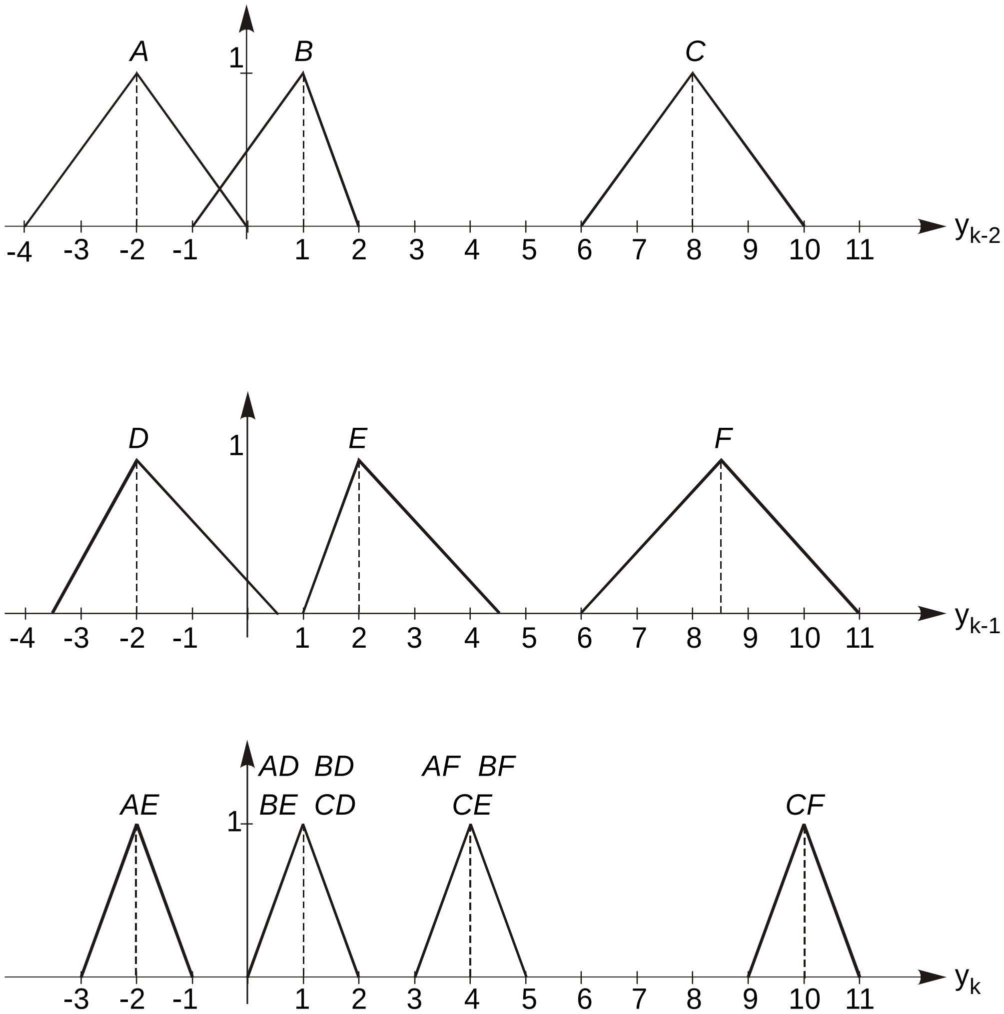

Figure 15: Membership functions

The stable equilibrium point of the considered system is marked by the crisp value R_0=1. Therefore, the domain of attraction of the equilibrium point y_R=1 has to be found analyzing the center equation

$$y_k = f_C\left(y_{k-1}, y_{k-2}\right) \qquad (2)$$

where y_k, y_{k-1}, y_{k-2} are the centers of the fuzzy output and the once and twice delayed fuzzy output.
As shown in [Schäfers (1999)], the center equation of a Dynamic Fuzzy Systems may be considered as a connection of piecewise multilinear characteristics.
To apply the Convex Decomposition, center equation (2) has to be approximated first by a piecewise linear characteristics since it is only applicable for systems with nonlinearities of this type. With the approximated center equation, the Convex Decomposition allows to determine whether a certain region belongs to the domain of attraction of the equilibrium point under investigation. G_{con} is the biggest rectangular region that could be found:

$$G_{conv} = \left\{y_{k-1}, y_{k-2} \middle| -4 \le y_{k-1} \le 4; -4 \le y_{k-2} \le 7\right\}.$$

Since the method based on Integral Ljapunov Functions is made to analyze time-discrete systems with nonlinearities consisting of connections of piecewise multilinear characteristics, center equation (2) could be analyzed without a prior approximation. The resulting region G_{int} is depicted in figure 16 together with region G_{con}.

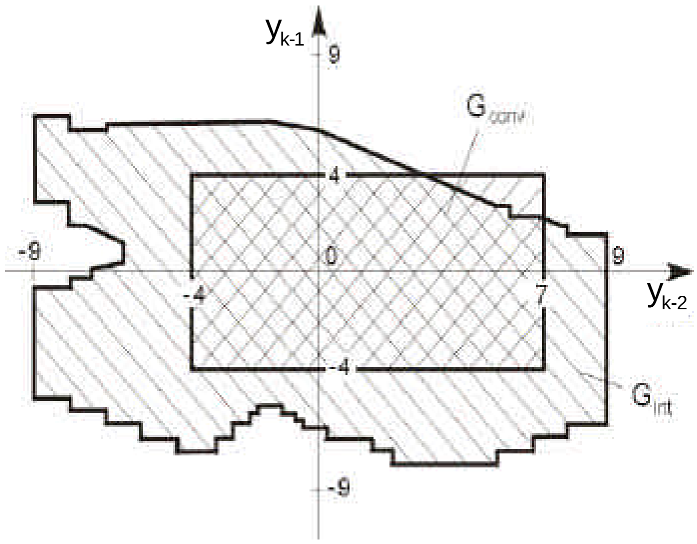

Figure 16: Stability regions

For analyzing Dynamic Fuzzy Systems, the method based on Integral Ljapunov Functions has two advantages in comparison with the Convex Decomposition: As mentioned above, no preceding approximation of the center equation is necessary. Furthermore, a stability region is determined automatically, i.e. the user does not have to propose different possible stability regions for investigation.

6 CONCLUSIONS

This contribution presented an approach for qualitative modeling of dynamic processes using a new class of Dynamic Fuzzy Systems. The dynamics of the process behavior is modeled by appropriate time delay and feed back of the fuzzy output to the system's input without previous defuzzification. Hence, an important feature of the new modeling approach is the particular procedure for rule propagation which was developed for this class of systems and is called Inference with Interpolating Rules.
Furthermore, the essentials of stability analysis for Dynamic Fuzzy Systems were outlined:
A closer look at the dynamic behavior of simple Dynamic Fuzzy Systems inspired an appropriate stability definition.
For first order Dynamic Fuzzy Systems, the complete domain of attraction of a stable equilibrium point can be determined analytically. An iterative approach was proposed based on the graphical representation of the center equation.

Stability of second and higher order Dynamic Fuzzy Systems can be analyzed numerically using both the ”Convex Decomposition” and a method based on ”Integral Ljapunov Functions”. The analysis of a second order Dynamic Fuzzy System was used as an representative example to discuss the two numerical methods.
Concluding, the new inference scheme allows qualitative modeling of complex plants as well as the analysis of these systems. Since the qualitative approach is often the only way to obtain an appropriate process representation, the new concept of modeling offers a considerable potential towards the automation of complex processes.

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