Statistical decisions in reliability testing - a fuzzy approach

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ABSTRACT: The paper presents a fuzzy approach to the statistical decision problem characteristic for the reliability or life-time testing. Three cases are considered. First, in a classical setting a decision is taken after the evaluation of its necessity. Second, the problem is generalised for the case of an imprecise (fuzzy) reliability requirement. Third, the problem is further generalised by allowing test data to be imprecisely described.

KEYWORDS: Statistical decisions, composite hypotheses, reliability tests, fuzzy requirement, fuzzy data

INTRODUCTION

In statistical testing decisions are made upon values of parameters describing the probability distribution of a random variable of interest. Suppose that T is a random variable distributed according to a density function $f(t,\theta)$, where θ is an unknown parameter. In many statistical textbooks only tests for testing a null hypothesis H: $\theta=\theta_0$ against an alternative hypothesis K: $\theta=\theta_1$ are usually considered. However, in many applications a test for a composite null hypothesis H: $\theta\geq\theta_0$ against an alternative K: $\theta<\theta_0$ is needed. This is also the case in reliability testing where T describes the life-time of tested items, θ is a parameter describing the mean life-time, and the fulfilment of the requirement $\theta>\theta_0$ (or $\theta\geq\theta_0$) is needed. Statistical textbooks suggest in such a case (we call this type of approach - the approach A, and the corresponding statistical test we call - the test A) to verify the hypothesis H_A: $\theta=\theta_0$ against the alternative K_A: $\theta<\theta_0$. A random sample $(T_1,T_2,...T_n)$ is observed, and the value of a test statistics $\hat{\theta}\,(T_1,T_2,...T_n)$ is calculated. When its observed value θ^* belongs to a certain critical region which (depends upon the value of a significance level α of this test) the null hypothesis H_A: $\theta=\theta_0$ is rejected in favour of the alternative K_A: $\theta<\theta_0$, and this means that the requirement $\theta\geq\theta_0$ is not fulfilled. For values of α suggested in all textbooks this procedure rejects the null hypothesis only in cases when there is a very strong evidence that the requirement $\theta\geq\theta_0$ is not fulfilled. However, in many practical cases we need to be convinced that the requirement $\theta\geq\theta_0$ is fulfilled. In such a case (we call this type of approach - the approach B, and the corresponding statistical test we call - the test B) we could verify the hypothesis H_B: $\theta\leq\theta_0$ against the alternative K_B: $\theta>\theta_0$. In this case the rejection of the null hypothesis in favour of the alternative may be regarded as a very strong evidence that the requirement $\theta\geq\theta_0$ is fulfilled. When we fix the value of the significance level α to a relatively low value (e.g. 0.05, as it is very often suggested in statistical textbooks) we may face a situation when we have not enough evidence neither to reject the hypothesis H_A: $\theta\geq\theta_0$ nor to reject the hypothesis H_B: $\theta\leq\theta_0$. This situation may confuse a decision maker when he needs to decide whether the reliability requirement is fulfilled or not. In the second section of this paper we propose a simple procedure which may add some additional information about a necessity measure of a decision.

In many practical cases reliability requirements are expressed in an imprecise way. We may say, for example, that the required mean life-time should be „much greater” than a specified value. In such a case we may use a fuzzy set approach to describe formally such imprecise requirements. Moreover, in the most general situation also test data may be expressed in an imprecise way. In such a case we have both imprecise test data and reliability requirements, and the problems with choosing an appropriate decision. Such situations arise very often when the test data comes from the field tests. Examples are given in Grzegorzewski and Hryniewicz (1999). Statistical decision procedures applied in the presence of fuzzy requirements and/or fuzzy data have been proposed by many authors. An overview of the most important papers on this topic can be found in Grzegorzewski and Hryniewicz (1997). Applications of statistical procedures in reliability are described e.g. in Hryniewicz (1995), and in Viertl and Gurker (1995). In the third section of this paper a new statistical decision procedure for the case of fuzzy reliability requirements is proposed. A similar procedure which is applicable for the general case of fuzzy requirements and fuzzy test data is proposed in the last section of this paper.

STATISTICAL DECISIONS BASED ON A NECESSITY MEASURE

Suppose that we are in the situation described in the previous section. We need to verify whether the mean life-time θ is larger than a certain required value θ_0. For a given significance level α (say, α=0.05) we may be confronted with the problem what to do if neither the hypothesis H_A: $\theta \geq \theta_0$ (when we adopt the approach A) nor the hypothesis H_B: $\theta \leq \theta_0$ (when we adopt the approach B) can be rejected. To cope with this problem it is advised to calculate the observed size of each statistical test, see Bickel and Doksum (1977), i.e. the so called *p-value* for this test. Let $C_A(\theta_0,\alpha)$ be the critical region for the test A , i.e. for testing the hypothesis H_A: $\theta \geq \theta_0$ on the significance level α. We reject the hypothesis H_A when the observed value of the test statistics θ^* belongs to this region, i.e. $\theta^* \in C(\theta_0, \alpha)$. The critical region is closely related to the confidence interval of the test statistics $\hat{\theta}(T_1, T_2, \ldots T_n)$. To illustrate this relation let us assume (and this assumption is frequently made in the statistical analysis of reliability data) that the life-time T is exponentially distributed with the density function

$$f(t,\theta) = \begin{cases} 0 & , \quad t \leq 0 \\ \frac{1}{\theta} exp\left(-\frac{1}{\theta}\right) & , \quad t > 0 \, . \end{cases} \tag{1}$$

Let $\overline{\theta}(1-\alpha)$ be the upper limit of the one-sided confidence interval of the parameter θ on the confidence level $1-\alpha$. In the case of the exponentially distributed life-times we have

$$\overline{\theta}(1-\alpha) = \frac{2Q}{\chi^2(2m, \alpha)} \quad , \tag{2}$$

where Q is the total-time-on-test statistics, m is the number of failures observed during the test, and $\chi^2(2m,\alpha)$ is the quantile of the α-th order of the chi-square distribution with $2m$ degrees of freedom. The hypothesis H_A: $\theta \geq \theta_0$ is rejected on the significance level α when the upper limit of the one-sided confidence interval of the parameter θ is smaller than the required value θ_0.

The observed test size α^* is the maximum value of α for which the null hypothesis cannot be rejected (or the minimum value of α for which the null hypothesis should be rejected) for the observed value of the test statistics. In the case of the exponential distribution the observed test size α^*_A (or the value p_A) of the test A is such that

$$\frac{2Q}{\chi^2(2m, \alpha^*_A)} = \theta_0 \, . \tag{3}$$

For the test B (i.e. for testing the hypothesis H_B: $\theta \leq \theta_0$ against the alternative K_B: $\theta > \theta_0$) the observed test size α^*_B (or the value p_B) of this test is such that

$$\frac{2Q}{\chi^2(2m, 1-\alpha^*_B)} = \theta_0 \, . \tag{4}$$

It is easy to notice that the following equation holds for both tests

$$\alpha^*_A = 1 - \alpha^\bullet_B \, . \tag{5}$$

From the theory of statistical tests we know that a large value (close to 1) of the observed test size means that there is a strong evidence in favour of the verified null hypothesis. Thus, the large value of α^*_A (and, correspondingly, the small value of α^*_B) supports the hypothesis that $\theta > \theta_0$. On the other hand, the large value of α^*_B (and, correspondingly, a small value of α^*_A) supports the hypothesis that $\theta < \theta_0$. We might say that if the inequality $\alpha^*_A > \alpha^*_B$ holds, we should accept the hypothesis $\theta \geq \theta_0$ rather than the hypothesis $\theta \leq \theta_0$. However, the question remains open about the necessary difference between the both values of the observed test sizes which is needed for making firm decisions. To answer this question we propose to apply the notion of the necessity measure from the fuzzy sets theory.

In the paper by Hryniewicz (1995) it has been noted that any number *a* such that $0 \leq a \leq 1$ can be represented by a fuzzy set $\tilde{x}$ with the membership function

$$\mu(x) = \begin{cases} \min\left[1, 2(1-a)\right] & , x = 0 \\ \min\left[1, 2a\right] & , x = 1 \end{cases}. \tag{6}$$

Thus, the observed test size α^*_A may be described by a fuzzy set $\tilde{A}$ with the membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} \min\left[1, 2(1-\alpha^*_A)\right] & , x = 0 \\ \min\left[1, 2\alpha^*_A\right] & , x = 1 \end{cases}. \tag{7}$$

and the observed test size α^*_B may be described by a fuzzy set $\tilde{B}$ with the membership function

$$\mu_{\tilde{B}}(y) = \begin{cases} \min\left[1, 2\alpha^*_A\right] & , y = 0 \\ \min\left[1, 2(1-\alpha^*_A)\right] & , y = 1 \end{cases}. \tag{8}$$

As we have said previously, when the relation $\alpha^*_A > \alpha^*_B$ is true we should accept the hypothesis $\theta \geq \theta_0$ rather than the hypothesis $\theta \leq \theta_0$. The same holds for the relation $\tilde{A} > \tilde{B}$. However, for this type of inequality we can adopt the approach of Dubois and Prade (1983), and to calculate the grade of its possibility or necessity. For this purpose we propose to use the Necessity of Strict Dominance index by Dubois and Prade (1983) defined as

$$NSD = Ness(\tilde{A} > \tilde{B}) = 1 - \sup_{x,y:\ x \leq y} \min\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\right\}. \tag{9}$$

In our case it is easy to show that

$$NSD = \begin{Bmatrix} 0 & , \text{when } \alpha^*_A \leq 0.5 \\ 1 - \min\left[1, 2(1-\alpha^*_A)\right] & , \text{when } \alpha^*_A > 0.5 \end{Bmatrix}. \tag{10}$$

The value of *NSD* shows us the grade of conviction that the results of the test support the hypothesis that $\theta \geq \theta_0$. In a similar way we can evaluate the grade of necessity that the results of the test support the hypothesis that $\theta \leq \theta_0$. The values of *NSD* give us an additional information for making the appropriate decision. For example, we could require that only values of *NSD* that are greater than 0.5 could confirm our hypothesis that $\theta \geq \theta_0$. Otherwise, an additional information should be acquired.

STATISTICAL DECISIONS IN CASE OF FUZZY REQUIREMENTS

In the previous section we assumed that the reliability requirement θ_0 is precisely defined. However, in certain circumstances such a requirement may not be precisely defined. There are subjective and objective reasons for such a situation. For example, a warranty time may be expressed exactly as a one year of exploitation of a car, but in terms of kilometres or miles such a reliability requirement may be hardly precise. Therefore, it is quite possible that real requirements for the parameter θ may be expressed as a fuzzy number $\tilde{\theta}_0$ with a certain membership function $\mu_0(\theta)$.

When the reliability requirement is expressed as a fuzzy number $\tilde{\theta}_0$ the test of the hypothesis $\theta > \tilde{\theta}_0$ becomes a genuine fuzzy test. The critical region $C_A(\theta_0, \alpha)$ becomes a fuzzy set $C_A(\tilde{\theta}_A, \alpha)$, and the observed test size α^* is now a fuzzy number $\tilde{\alpha}^*$. By applying the extension principle we could easily find the membership function of $\tilde{\alpha}^*$.

Let $[\theta_{0,min}(\alpha), \theta_{0,max}(\alpha)]$ be the α-cut of $\tilde{\theta}_0$. From the extension principle we find that the interval $[\alpha^*_{min}(\alpha), \alpha^*_{max}(\alpha)]$, where $\alpha^*_{min}(\alpha)$ is the observed test size corresponding to the value $\theta_{0,max}(\alpha)$, and $\alpha^*_{max}(\alpha)$ is the observed test size corresponding to the value $\theta_{0,min}(\alpha)$, is the α-cut of the fuzzy test size $\tilde{\alpha}^*$. In the case of the exponentially distributed test data we have for the test *A*:

$$\frac{2Q}{\chi^2(2m, \alpha^*_{A,min}(\alpha))} = \theta_{0,max}(\alpha), \tag{11}$$

and

$$\frac{2Q}{\chi^2(2m, \alpha^*_{A,max}(\alpha))} = \theta_{0,min}(\alpha). \tag{12}$$

For the test *B* we have

$$\frac{2Q}{\chi^2(2m, 1-\alpha^*_{B,max}(\alpha))} = \theta_{0,max}(\alpha), \tag{13}$$

and

$$\frac{2Q}{\chi^2(2m, 1-\alpha^*_{B,min}(\alpha))} = \theta_{0,min}(\alpha). \tag{14}$$

Let us notice now that $\alpha^*_{A,\max} = 1-\alpha^*_{B,\min}$, and $\alpha^*_{A,\min} = 1-\alpha^*_{B,\max}$. Moreover, we can notice that for each number $\alpha^*_A \in \left[\alpha^*_{A,\min}, \alpha^*_{A,\max}\right]$ there exists a corresponding number $\alpha^*_B = 1-\alpha^*_A$ with the same value of the membership function which is equal to the value of the membership function $\mu_0(\theta)$ for that value of θ which corresponds to α^*_A. Now, we can proceed exactly as in the previous section, and for each pair (α^*_A, α^*_B) we can find the value of the *NSD* index of the relation $\tilde{A} > \tilde{B}$. With this value of the *NSD* index there is associated a corresponding value of the membership function of α^*_A. Thus, for each $\alpha^*_A \in \left[\alpha^*_{A,min}, \alpha^*_{A,max}\right]$ there exists the value of $NSD(\tilde{A} > \tilde{B})$ with a given value of the membership function $\mu_{NSD}(x)$. That means that instead of a one index $NSD(\tilde{A} > \tilde{B})$ we have a fuzzy number $\widetilde{\widetilde{NSD}}(\tilde{A} > \tilde{B})$ which describes the grade of conviction that the results of the test support the hypothesis $\theta \geq \theta_0$.

In practice, we usually need only one number to show the grade of conviction that the results of the test support the hypothesis $\theta \geq \theta_0$. There are many methods which might be used to represent $\widetilde{\widetilde{NSD}}(\tilde{A} > \tilde{B})$ by a one number. We propose to use the well known F_1 index proposed by Yager given by the following expression :

$$F_1\left(\widetilde{\widetilde{NSD}}\right) = \frac{\int_0^1 x\mu_{NSD}(x)dx}{\int_0^1 \mu_{NSD}(x)dx}. \tag{15}$$

We may assume that only the large values of this index (say, larger than 0.5) strongly support the hypothesis $\theta \geq \theta_0$.

STATISTICAL DECISIONS IN CASE OF FUZZY REQUIREMENTS AND FUZZY TEST DATA

In a more general case we can additionally assume that the test data are also expressed imprecisely. This situation is described, for example, in Hryniewicz (1995) or in Grzegorzewski and Hryniewicz (1999). In the case of exponentially distributed life-time data this lack of precision can be expressed by a fuzzy form of the total-time-on-test statistics Q. In such a case instead of the one precise value Q we have a fuzzy number $\tilde{Q}$ described by a membership function $\eta(Q)$.

Let $supp(\tilde{Q})$ and $supp(\tilde{\theta}_0)$ be the supports of the fuzzy numbers $\tilde{Q}$ and $\tilde{\theta}_0$, respectively. Then, for each pair of numbers $Q \in supp(\tilde{Q})$ and $\theta_0 \in supp(\tilde{\theta}_0)$ we can calculate the corresponding observed test size $\alpha^*_A(Q, \theta_0, m)$. For this value of $\alpha^*_A(Q, \theta_0, m)$ we propose to calculate the value of the corresponding membership function from the following formula

$$\mu(\alpha^*) = min\left[\eta(Q), \mu_0(\theta_0)\right]. \quad (16)$$

Hence, we can find two fuzzy sets $\tilde{\alpha}^*_A$ and $\tilde{\alpha}^*_B$ that can be called the observed fuzzy test-sizes of test A and B, respectively. Having these fuzzy sets we can proceed exactly as in the previous section, arriving finally at the one value of the Necessity of Strict Dominance index *NSD* which describes the grade of our conviction that the reliability requirements have been fulfilled.

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