

Upper and lower bounds in the class of continuous Archimedean t -norms¹

R. Mesiar

Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic,
Pod vodárenskou věží 4, 18208 Prague, Czech Republic
and STU Bratislava, Radlinského 11, 81368 Bratislava, Slovakia

1 Introduction

Triangular norms, i.e., $[0, 1]^2 \rightarrow [0, 1]$ commutative, associative, non-decreasing mappings with neutral element 1 were introduced in [15], see also [9, 16]. The ordering in the class of continuous Archimedean t -norms was investigated by Schweizer and Sklar [16]. Let T_1 and T_2 be two continuous Archimedean t -norms with additive generators f_1 and f_2 , respectively. Then $T_1 \leq T_2$ if and only if the composite function $h = f_1 \circ f_2^{-1} : \text{Ran } f_2 \rightarrow [0, \infty]$ is subadditive, i.e., $h(x+y) \leq h(x) + h(y)$ whenever $x, y, x+y \in \text{Ran } f_2$. Similarly, the composite function $g = f_2 \circ f_1^{-1}$ (i.e. $g = h^{-1}$) should be superadditive, $g(x+y) \geq g(x) + g(y)$ whenever $x, y, x+y \in \text{Ran } f_1$.

For example, it is well known that $T_L \leq T_P$. Put $h = f_L \circ f_P^{-1}$ and $g = f_P \circ f_L^{-1}$, i.e. $h(x) = 1 - e^{-x}$, $x \in [0, \infty]$, and $g(x) = -\log(1-x)$, $x \in [0, 1]$. It is easy to see that h is subadditive on $[0, \infty]$ and g is superadditive on $[0, 1]$.

The ordering of general continuous t -norms was studied by Klement et al. in [5]. However, the problem of finding a lower (an upper) bound for two t -norms in the class of continuous Archimedean t -norms resisted for years. The first partial result was recently obtained by Marko and Mesiar [10] by finding an upper nilpotent bound for two given nilpotent t -norms. We want to discuss the existence of strict (nilpotent) upper and lower bounds for any two given strict (nilpotent) t -norms T_1 and T_2 . By the induction, any obtained result can be extended to any finite system of strict (nilpotent) t -norms, and more, for a given finite system of continuous Archimedean t -norms.

For the class of continuous t -norms, a genuine continuous upper bound of any system is T_M . However, the problem of finding a continuous lower bound remains open. Note only that for infinite systems (even of Archimedean continuous t -norms) this continuous lower bounds need not exist, in general. As an example take the family of Yager's nilpotent t -norms T_p^Y , $p \in]0, \infty[$, generated by additive generators $f_p(x) = (1-x)^p$, where the only lower bound is the weakest t -norm T_W which is not continuous.

2 Reformulation of the problem

Problem 2.1. Given two nilpotent t -norms T_1 and T_2 , find the nilpotent t -norms T_* and T^* so that $T_* \leq \min(T_1, T_2) \leq \max(T_1, T_2) \leq T^*$.

¹The work on this paper was supported by grants GA ČR 402/99/0032 and VEGA 1/4064/97.

Lemma 2.1. Let τ be the class of all increasing one-to-one mappings $g : [0, 1] \rightarrow [0, 1]$. Problem 2.1 is equivalent with the next one: given two mappings $u, v \in \tau$, find another two mappings $p, r \in \tau$ so that the composites $p \circ u$ and $p \circ v$ are subadditive on $[0, 1]$ and the composites $r \circ u$ and $r \circ v$ are superadditive on $[0, 1]$.

Problem 2.2. Given two strict t -norms T_1 and T_2 , find the strict t -norms T_* and T^* so that

$$T_* \leq \min(T_1, T_2) \leq \max(T_1, T_2) \leq T^*.$$

Lemma 2.2. Let \mathcal{F} be the class of all $[0, \infty] \rightarrow [0, \infty]$ increasing one-to-one mappings crossing the point $(1, 1)$. Problem 2.2 is equivalent with the next one: given two mappings $u, v \in \mathcal{F}$, find another two mappings $p, r \in \mathcal{F}$ so that the composites $p \circ u$ and $p \circ v$ are subadditive on $[0, \infty]$ and the composites $r \circ u$ and $r \circ v$ are superadditive on $[0, \infty]$.

It is evident that for any $u \in \mathcal{F}$, the restriction $u|_{[0, 1]} \in \tau$. Similarly, if $p, u \in \mathcal{F}$ and $p \circ u$ is subadditive on $[0, \infty]$, then $(p|_{[0, 1]}) \circ (u|_{[0, 1]}) = (p \circ u)|_{[0, 1]}$ is subadditive on $[0, 1]$, etc., leading to the next corollary.

Corollary 2.1. If there exists a solution of the problem stated in Lemma 2.2 then there exists also a solution of the problem stated in Lemma 2.1, i. e., the existence of lower and upper strict bounds of any couple of strict t -norms ensures a similar result in the class of nilpotent t -norms.

3 Main results

In this section, we give the essential theorems for the finding a lower (an upper) continuous generated bound of continuous generated t -norms. The corresponding complete proofs can be found in [11].

Theorem 3.1. Let $u, v \in \mathcal{F}$. Then there exists $w \in \mathcal{F}$ such that both composites $w \circ u$ and $w \circ v$ are superadditive.

Proof: For a finite sequence $0 \leq x_0 < x_1 < \dots < x_n \leq \infty$, put

$$P(x_0, \dots, x_n) = \prod_{i=1}^n \min \left(\frac{u^{-1}(x_{i-1})}{u^{-1}(x_i)}, \frac{v^{-1}(x_{i-1})}{v^{-1}(x_i)} \right).$$

Further, for $0 \leq x < y \leq \infty$, define

$$Q(x, y) = \inf (P(x_0, \dots, x_n); x = x_0 < \dots < x_n = y, n \in \mathbb{N}).$$

It is evident that $Q(x, y) = Q(x, z) Q(z, y)$ for any $z \in]x, y[$ and

$$\frac{u^{-1}(x) v^{-1}(x)}{u^{-1}(y) v^{-1}(y)} \leq Q(x, y) \leq \frac{u^{-1}(x)}{u^{-1}(y)}.$$

We can define an element $w \in \mathcal{F}$ by

$$w(x) = \begin{cases} Q(x, 1) & \text{if } x < 1 \\ 1 & x = 1 \\ 1/Q(1, x) & x > 1. \end{cases}$$

For the rest of the proof see [11].

Note that if the mappings u^{-1} , v^{-1} are smooth on an interval $]a, b[\subset [0, \infty]$ and for all $x \in]a, b[$ it is

$$\frac{(u^{-1})'(x)}{u^{-1}(x)} \geq \frac{(v^{-1})'(x)}{v^{-1}(x)},$$

then for all $x, y \in [a, b]$, $x \leq y$, it is

$$Q(x, y) = \frac{u^{-1}(x)}{u^{-1}(y)}.$$

This observation allows a fast computation of the function w in the case when both u^{-1} and v^{-1} are smooth.

Theorem 3.2. Let $u, v \in \mathcal{F}$. Then there exists $w \in \mathcal{F}$ such that both composites $w \circ u$ and $w \circ v$ are subadditive.

Proof: Denote $w \circ v = h$. Then $w \circ u = w \circ v \circ v^{-1} \circ u = h \circ g$, $h \in \mathcal{F}$, $g = v^{-1} \circ u$, $g \in \mathcal{F}$. Theorem 3.2 is equivalent with the claim: for any $g \in \mathcal{F}$ there exists a subadditive $h \in \mathcal{F}$ so that the composite $h \circ g$ is subadditive (then $w = h \circ v^{-1}$).

For given $g \in \mathcal{F}$, the function h will be defined first in points 2^k , $k \in \mathbb{Z}$, as follows:

$$h(1) = 1 \quad (\text{i. e., } k = 0);$$

for $k = 1, 2, 3, \dots$,

$$h(2^k) = \min \left(h(2^{k-1}) \cdot \frac{g^{-1}(2^k)}{g^{-1}(2^{k-1})}, h(2^{k-1}) + \frac{1}{3h(2^{k-1})} \right);$$

for $k = -1, -2, -3, \dots$,

$$(*) \quad h(2^k) = \max \left(h(2^{k+1}) \cdot \frac{g^{-1}(2^k)}{g^{-1}(2^{k+1})}, h(2^{k+1}) \cdot 0.8, \sup_{x \in [2^{k+2}, \infty[} (h(x) - h(x - 2^{k+1})) \right).$$

Note that first we define recursively $h(2^k)$ for $k = 0, 1, 2, \dots$ and for $x \in [2^k, 2^{k+1}]$, $k = 0, 1, \dots$, we define h so that the composite $h \circ g$ is a linear function on $[g^{-1}(2^k), g^{-1}(2^{k+1})]$, $k = 0, 1, \dots$. Then we define $h(2^{-1})$ by means of formula (*) and h on $[2^{-1}, 1]$ so that $h \circ g$ is linear on $[g^{-1}(2^{-1}), 1]$. Recursively we define $h(2^k)$, $k = -2, -3, \dots$ and h on $[2^k, 2^{k+1}]$, $k = -2, -3, \dots$, so that $h \circ g$ is linear on $[g^{-1}(2^k), g^{-1}(2^{k+1})]$. For the rest of the proof see [11].

4 Lower and upper bounds of continuous Archimedean t -norms and t -conorms

Due to Theorems 3.1 and 3.2, for any couple T_1, T_2 of continuous Archimedean t -norms we can find two continuous Archimedean t -norms T_* and T^* so that

$$T_* \leq \min(T_1, T_2) \leq \max(T_1, T_2) \leq T^*.$$

To see this fact, it is enough to observe that due to $T_L \leq T_P$ we can find to any nilpotent t -norm a stronger strict t -norm, and vice-versa, to any strict t -norm we can find a weaker nilpotent t -norm.

More, if both T_1, T_2 are nilpotent (strict), also T_* and T^* can be chosen to be nilpotent (strict). In general, T_* is nilpotent and T^* strict if T_1 is nilpotent and T_2 strict (or vice versa). The construction of the lower bound T_* and the upper bound T^* was described in the previous sections. Note that our construction need not give the best lower (upper) bound in the class of Archimedean continuous t -norms (which even need not exist!). Even if $T_1 \leq T_2$, we can get $T_* \neq T_1$ and $T^* \neq T_2$. Consequently, the looking for more efficient methods of finding the corresponding bounds is desirable. Concerning the triangular conorms, note that if $S : [0, 1]^2 \rightarrow [0, 1]$ is a t -conorm, i. e., an associative, commutative, non-decreasing mapping with neutral element 0 [8, 9, 16], then the dual mapping $T : [0, 1]^2 \rightarrow [0, 1]$ defined by $T(x, y) = 1 - S(1 - x, 1 - y)$ is a t -norm. More, for the continuous Archimedean t -conorm S ($S(x, x) > x$ for all $x \in]0, 1[$) the corresponding dual T is continuous Archimedean t -norm. If S_1, S_2 are two t -conorms and T_1, T_2 are their dual t -norms, respectively, then $S_1 \leq S_2$ if and only if $T_1 \geq T_2$. But then for any couple S_1, S_2 of continuous Archimedean t -conorms we can always find an upper bound S^* and a lower bound S_* in the class of continuous Archimedean t -conorms,

$$S_* \leq \min(S_1, S_2) \leq \max(S_1, S_2) \leq S^*.$$

Indeed, for the corresponding dual t -norms T_1, T_2 we find T_* and T^* as mentioned before,

$$T_* \leq \min(T_1, T_2) \leq \max(T_1, T_2) \leq T^*.$$

Now, it is enough to take S_* the dual to T^* and S^* the dual to T_* .

Example 4.1. It is well known that the product t -norm T_P and the Yager t -norm T_2^Y are incomparable [5, 17]. It is evident that

$$T_L \leq \min(T_P, T_2^Y).$$

However, applying the Theorem 3.2 (with necessary modifications for the nilpotent case), we get a nilpotent lower bound $T_* < \min(T_L, T_2^Y) = T_L \leq \min(T_P, T_2^Y)$. Concerning the upper bound, we have to look for a strict upper bound.

For the nilpotent t -norm T_2^Y with the normed additive generator $f_2^Y(x) = (1 - x)^2$, the isomorphism between T_2^Y and T_L is given by $\varphi = f_L^{-1} \circ f_2^Y$, i. e., $\varphi(x) = 1 - (1 - x)^2$. Take the strict t -norm $(T_P)_\varphi(x, y) = \varphi^{-1}(\varphi(x) \varphi(y))$. Its additive generator is $f = f_P \circ \varphi$ and $T_2^Y = (T_L)_\varphi \leq (T_P)_\varphi$.

To apply Theorem 3.1, put $u = \frac{f_P}{f_P(0.5)} \circ j^{-1}$ and $v = \frac{f}{f(0.5)} \circ j^{-1}$, where $j : [0, 1] \rightarrow [0, \infty]$, $j(x) = \frac{1-x}{x}$, i.e.,

$$u(x) = \frac{-\log\left(\frac{1}{1+x}\right)}{-\log 0.5} = \log_2(1+x) \quad \text{and}$$

$$v(x) = \frac{-\log\left(1 - \left(\frac{1}{1+x}\right)^2\right)}{-\log 0.75} = \log_{\frac{4}{3}}\left(\frac{1+2x+x^2}{1+2x}\right), \quad x \in [0, \infty].$$

Then $u, v \in \mathcal{F}$ and we can apply Theorem 3.1 to find a mapping $w \in \mathcal{F}$ such that both composites $w \circ u$ and $w \circ v$ are superadditive. Then the function $g = w \circ j$, $g(x) = w\left(\frac{1-x}{x}\right)$, $x \in [0, 1]$, is an additive generator of a strict t -norm T^* which is stronger than both T_P and $(T_P)_\varphi$. But then

$$\max(T_P, T_2^Y) \leq (T_P, (T_P)_\varphi) \leq T^*,$$

i. e., T^* is a strict upper bound of T_P and T_2^Y .

5 Conclusions

We have found a method of constructing the lower (upper) bound of finite systems of t -norms in the class of continuous Archimedean t -norms. Among several applications in different areas, recall, e.g., the domain of T -fuzzy equivalence relations [2, 3, 4], when discussing the refinement of T -fuzzy equivalence relations, the Cartesian product, etc.

Another possible field of applications is the calculus of fuzzy numbers based on the Zadeh extension principle with the conjunction modelled by some t -norm T . A special role (corresponding to zero in the crisp case) is played by the T -idempotents, i. e. fuzzy numbers $A \in \mathcal{F}(R)$ for which $A \oplus_T A = A$. For more details we recommend [7, 8, 12, 13, 14].

References

- [1] J. Bezdek, I. Harris, Fuzzy partitions and relations, an axiomatic basis for clustering. Fuzzy Sets and Systems 1 (1978), 111–127.
- [2] B. De Baets, R. Mesiar, Pseudo-metrics and T -equivalences and pseudo-metrics. Proc. IFSA'97, Prague, 1997, vol. I, 187–192.
- [3] B. De Baets, R. Mesiar, T -partitions. Fuzzy Sets and Systems 97 (1998), 211–224.
- [4] S. Gottwald, Mathematical Aspects of Fuzzy Sets and Fuzzy Logic. Vieweg Verlag, Braunschweig, 1993.
- [5] E. P. Klement, R. Mesiar, E. Pap, A characterization of the ordering of continuous t -norms. Fuzzy Sets and Systems 86 (1997), 189–195.
- [6] E. P. Klement, R. Mesiar, E. Pap, Triangular Norms. Monograph in preparation.

- [7] A. Kolesárová, Similarity preserving additions of fuzzy numbers. Proc. IFSA'97, Prague, 1997, vol. II, 165–168.
- [8] A. Kolesárová, Triangular norm–based addition preserving linearity of T –sums of linear fuzzy intervals. Mathware and Soft Computing 5 (1998), 91–98.
- [9] C. M. Ling, Representation of associative functions. Publ. Math. Debrecen 12 (1965), 189–212.
- [10] V. Marko, R. Mesiar, A note on a nilpotent lower bound of nilpotent triangular norms. Fuzzy Sets and Systems, in press.
- [11] V. Marko, R. Mesiar, Continuous Archimedean t-norms and their bounds, submitted.
- [12] A. Marková–Stupňanová, A note to the addition of fuzzy numbers based on a continuous Archimedean t –norm. Fuzzy Sets and Systems 91 (1997), 253–258.
- [13] A. Marková–Stupňanová, Pseudo–convolutions and their idempotents. Proc. IFSA'97, Prague, 1997, vol. I, 484–487.
- [14] R. Mesiar: Triangular norm–based addition of fuzzy intervals. Fuzzy Sets and Systems 91 (1997), 231–237.
- [15] B. Schweizer, A. Sklar, Statistical metric spaces. Pacific J. Math. 10 (1960), 313–334.
- [16] B. Schweizer, A. Sklar, Probabilistic Metric Spaces. North Holland, New York, 1983.
- [17] R. R. Yager, On a general class of fuzzy connectives. Fuzzy Sets and Systems 4 (1980), 235–242.
- [18] L. Zadeh, Similarity relations and fuzzy orderings. Inform. Sci. 3 (1971), 177–200.