

# FUZZY ROUGH SETS REVISITED

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**ABSTRACT:** The notion of a rough set was originally proposed by Pawlak (see Pawlak (1982)). Later on, in Dubois and Prade (1990), fuzzy rough sets were introduced as a fuzzy generalization of rough sets. In this paper, we propose a general approach to the fuzzification of rough sets. More specifically, we define a family of fuzzy rough sets, each one of which, called an  $(\mathcal{I}, \mathcal{T})$ -fuzzy rough set, is determined by a fuzzy implicator  $\mathcal{I}$  and a triangular norm  $\mathcal{T}$ . Several basic properties of fuzzy rough sets are investigated. In particular, basing on three classes of fuzzy implicators well-known in the literature, we define the corresponding three classes of fuzzy rough sets and essentially focus on investigating properties of these classes in the context of basic rough equalities.

**KEYWORDS:** Uncertainty Modelling, Rough Set Theory, Fuzzy Set Theory.

## 1 INTRODUCTION

The concept of a rough set was originally proposed by Pawlak (see Pawlak (1982)) as a formal tool for modelling and processing incomplete information in information systems. It soon invoked a natural question concerning possible connections between rough sets and fuzzy sets. Basically, both theories address the problem of information granulation: the theory of fuzzy sets is centered upon fuzzy information granulation, whereas rough set theory is focused on crisp information granulation. Originally, the basic notion in rough sets theory was indistinguishability (i.e. indistinguishability between objects in information systems induced by different values of attributes characterizing these objects), yet in recent extensions (see, for example, Orłowska (1998)) the focus moves to the notion of similarity, which is in fact the fuzzy concept. It is therefore apparent that these two theories have become much closer to each other.

In Dubois and Prade (1990) the problem of fuzzification of rough sets was investigated.<sup>1</sup> In the present paper, we propose a more general approach to this issue. Specifically, by applying the extension principle, we define fuzzy rough sets dependently on fuzzy conjunction and fuzzy implication operators. As a consequence, we propose a broad class of fuzzy rough sets, each one of which, called  $(\mathcal{I}, \mathcal{T})$ -fuzzy rough sets, is represented by a fuzzy implicator  $\mathcal{I}$  and a triangular norm  $\mathcal{T}$ . In perspective, this approach allows for considering a variety of reasoning methods from both incomplete and imprecise information.

The paper is organized as follows. In Section 2 we recall basic notions of rough set theory. Section 3 introduces the definition of fuzzy rough approximation and fuzzy rough set. Taking into account three classes of fuzzy implicators well-known in the literature, the corresponding three classes of fuzzy rough sets are defined.

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<sup>1</sup>Another approach was proposed in Nakamura (1988).

properties of the distinguished three classes of fuzzy rough sets in the context of basic rough equalities. The paper is completed by concluding remarks and some options for further work.

For the extended version of the present paper, we refer to Radzikowska and Kerre (1999).

## 2 ROUGH SETS

Let  $\mathfrak{X}$  be a nonempty universe and let  $R$  be an equivalence relation on  $\mathfrak{X}$ . The main question addressed by rough sets is: *how to represent subsets  $A \subseteq \mathfrak{X}$  by means of elements of the quotient set  $\mathfrak{X}/R$ .*

By an *approximation space* we mean a pair  $As = (\mathfrak{X}, R)$ , where  $\mathfrak{X}$  is a (nonempty) universe and  $R$  is an equivalence relation on  $\mathfrak{X}$ .

Given an approximation space  $As = (\mathfrak{X}, R)$ , a *rough approximation in  $As$*  is a mapping

$$\text{Apr}_{As} : \wp(\mathfrak{X}) \rightarrow \wp(\mathfrak{X}) \times \wp(\mathfrak{X})$$

defined by: for every  $A \in \wp(\mathfrak{X})$ ,  $\text{Apr}_{As}(A) = (\underline{As}(A), \overline{As}(A))$ , where

$$\underline{As}(A) = \{x \in \mathfrak{X} : [x]_R \subseteq A\} \quad (1)$$

$$\overline{As}(A) = \{x \in \mathfrak{X} : [x]_R \cap A \neq \emptyset\} \quad (2)$$

and  $[x]_R$  is the equivalence class of  $R$  with the representant  $x$ .

$\underline{As}(A)$  is called a *lower rough approximation of  $A$  in  $As$* , whereas  $\overline{As}(A)$  is called an *upper rough approximation of  $A$  in  $As$* .

Let  $As = (\mathfrak{X}, R)$  be an approximation space. A pair  $(L, U) \in \wp(\mathfrak{X}) \times \wp(\mathfrak{X})$  is called a *rough set in  $As$*  iff  $(L, U) = \text{Apr}_{As}(A)$  for some  $A \in \wp(\mathfrak{X})$ .

Given an approximation space  $As = (\mathfrak{X}, R)$ , a subset  $A \subseteq \mathfrak{X}$  is said to be *definable in  $As$*  iff  $\underline{As}(A) = \overline{As}(A)$ .

## 3 FUZZY ROUGH SETS

In this section we introduce definitions of fuzzy rough approximations and fuzzy rough sets. To begin with, let us recall some basic terminology.

### 3.1 BASIC NOTIONS

A triangular norm  $\mathcal{T}$ , or shortly t-norm, is an increasing, associative and commutative  $[0, 1]^2 - [0, 1]$  mapping satisfying the boundary condition:  $(\forall x \in [0, 1])(\mathcal{T}(x, 1) = x)$ . A triangular conorm  $\mathcal{S}$  (t-conorm, for short) is an increasing, associative and commutative  $[0, 1]^2 - [0, 1]$  mapping satisfying the boundary condition  $(\forall x \in [0, 1])(\mathcal{S}(x, 0) = x)$ . The most popular continuous triangular norms are: the standard *min* operation  $\mathcal{T}_M(x, y) = \min\{x, y\}$ , the algebraic product  $\mathcal{T}_P(x, y) = x \cdot y$  and the bold intersection  $\mathcal{T}_B(x, y) = \max\{0, x + y - 1\}$ ; three well-known continuous triangular conorms are: the standard *max* operation  $\mathcal{S}_M(x, y) = \max\{x, y\}$ , the probabilistic sum  $\mathcal{S}_P(x, y) = x + y - x \cdot y$  and the bounded sum  $\mathcal{S}_B(x, y) = \min\{1, x + y\}$ .

A fuzzy negator  $\mathcal{C}$  is a decreasing  $[0, 1] - [0, 1]$  mapping satisfying  $\mathcal{C}(0) = 1$  and  $\mathcal{C}(1) = 0$ . The standard fuzzy negator  $\mathcal{C}(x) = 1 - x$  will be denoted by  $\mathcal{C}_s$ . A negator  $\mathcal{C}$  is called involutive iff  $\mathcal{C}(\mathcal{C}(x)) = x$  for all  $x \in [0, 1]$ . Given an involutive fuzzy negator  $\mathcal{C}$ , a triangular norm  $\mathcal{T}$  and a triangular conorm  $\mathcal{S}$  are called dual with respect to  $\mathcal{C}$  iff  $\mathcal{T}(x, y) = \mathcal{C}(\mathcal{S}(\mathcal{C}(x), \mathcal{C}(y)))$  for all  $(x, y) \in [0, 1]^2$ . Given a fuzzy negator  $\mathcal{C}$ , we will write  $co_{\mathcal{C}}$  to denote fuzzy complementation operator determined by  $\mathcal{C}$ .

Let  $\mathcal{I}$  be a fuzzy impicator,<sup>2</sup> that is a  $[0, 1]^2 - [0, 1]$  mapping which satisfies the conditions  $\mathcal{I}(1, 0) = 0$  and  $\mathcal{I}(1, 1) = \mathcal{I}(0, 1) = \mathcal{I}(0, 0) = 1$ . A fuzzy impicator  $\mathcal{I}$  is called *l-monotonic* (resp. *r-monotonic*) iff for every  $x \in [0, 1]$ ,  $\mathcal{I}(\cdot, x)$  is decreasing (resp.  $\mathcal{I}(x, \cdot)$  is increasing). If  $\mathcal{I}$  is both *l-monotonic* and *r-monotonic*, then it is called *hybrid monotonic*.

Several classes of fuzzy impicators have been studied in the literature. Recall the definitions of three main classes of these operators.

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<sup>2</sup>For a detailed study of 18 widely used fuzzy impicators we refer to Ruan and Kerre (1993).

- an  $S$ -implicator based on  $\mathcal{S}$  and  $\mathcal{C}$  iff  $\mathcal{I}(x, y) = \mathcal{S}(\mathcal{C}(x), y)$  for all  $x, y \in [0, 1]$ ,
- an  $R$ -implicator based on  $\mathcal{T}$  iff  $\mathcal{I}(x, y) = \sup \{ \lambda \in [0, 1] : \mathcal{T}(x, \lambda) \leq y \}$  for all  $x, y \in [0, 1]$ , provided that  $\mathcal{T}$  is continuous.
- a  $QL$ -implicator based on  $\mathcal{S}$ ,  $\mathcal{T}$  and  $\mathcal{C}$  iff  $\mathcal{I}(x, y) = \mathcal{S}(\mathcal{C}(x), \mathcal{T}(x, y))$  for all  $x, y \in [0, 1]$ , provided that  $\mathcal{T}$  and  $\mathcal{S}$  are dual wrt  $\mathcal{C}$ .

Three most popular  $S$ -implicators are: the Łukasiewicz implicator  $\mathcal{I}_a(x, y) = \min\{1, 1-x+y\}$  (based on the bounded sum  $\mathcal{S}_B$  and the standard fuzzy negator  $\mathcal{C}_s$ ), the Kleene–Dienes implicator  $\mathcal{I}_b(x, y) = \max\{1-x, y\}$  (based on  $\mathcal{S}_M$  and  $\mathcal{C}_s$ ), and the Kleene–Dienes–Łukasiewicz implicator  $\mathcal{I}_*(x, y) = 1-x+x*y$  (based on  $\mathcal{S}_P$  and  $\mathcal{C}_s$ ). The Łukasiewicz implicator is also an  $R$ -implicator (based on the bold intersection  $\mathcal{T}_B$ ); the Standard Star implicator  $\mathcal{I}_g(x, y) = 1$  for  $x \leq y$  and  $\mathcal{I}_g(x, y) = y$  elsewhere, and the Gaines implicator  $\mathcal{I}_\Delta(x, y) = 1$  for  $x \leq y$  and  $\mathcal{I}_\Delta(x, y) = \frac{y}{x}$  elsewhere, are also  $R$ -implicators, based on  $\mathcal{T}_M$  and  $\mathcal{T}_P$ , respectively. Finally, the Early Zadeh implicator  $\mathcal{I}_m(x, y) = \max\{1-x, \min\{x, y\}\}$  is a  $QL$ -implicator based on  $\mathcal{T}_M$ ,  $\mathcal{S}_M$  and  $\mathcal{C}_s$ .

A fuzzy implicator  $\mathcal{I}$  is said to be a *border implicator*<sup>3</sup> ( $B$ -implicator, for short) iff for every  $x \in [0, 1]$ ,  $\mathcal{I}(1, x) = x$ .

Let  $\mathfrak{X}$  be a nonempty universe. A fuzzy binary relation  $R$  on  $\mathfrak{X}$  is called a *similarity relation* iff  $R$  is reflexive, symmetric and sup–min transitive (i.e. for all  $x, y \in \mathfrak{X}$ ,  $R(x, y) \geq \sup_{z \in \mathfrak{X}} \min\{R(x, z), R(z, y)\}$ ).

Recall that the similarity class  $[x]_R$  with the representant  $x$  is a fuzzy set on  $\mathfrak{X}$  defined by:

$$[x]_R : \mathfrak{X} \rightarrow [0, 1], \\ [x]_R(y) = R(x, y) \text{ for all } y \in \mathfrak{X}.$$

### 3.2 FUZZY ROUGH APPROXIMATIONS

The following notion is a fuzzy counterpart of an approximation space.

**Definition 1** Let  $\mathfrak{X}$  be a nonempty universe and let  $R$  be a similarity relation on  $\mathfrak{X}$ . A pair  $\text{FAS} = (\mathfrak{X}, R)$  is called a **fuzzy approximation space**.

It is easy to note that a *lower* (resp. an *upper*) *rough approximation* of  $A \subseteq \mathfrak{X}$  in  $\text{AS} = (\mathfrak{X}, R)$  may be equivalently defined by:

$$\text{for every } x \in \mathfrak{X}, \\ \chi_{\underline{\text{AS}}(A)}(x) = 1 \text{ iff } (\forall y \in \mathfrak{X}) (\chi_R(x, y) = 1 \Rightarrow \chi_A(y) = 1), \\ \chi_{\overline{\text{AS}}(A)}(x) = 1 \text{ iff } (\exists y \in \mathfrak{X}) (\chi_R(x, y) = 1 \wedge \chi_A(y) = 1),$$

where, as usual,  $\chi_A$  stands for the characteristic function of the set  $A$ .

While interpreting  $\Rightarrow$  and  $\wedge$  as a fuzzy implicator and a  $t$ -norm, respectively, we get by extension the following definition.

**Definition 2** Let  $\text{FAS} = (\mathfrak{X}, R)$  be a fuzzy approximation space. For any  $B$ -implicator  $\mathcal{I}$  and any  $t$ -norm  $\mathcal{T}$ , an  $(\mathcal{I}, \mathcal{T})$ -**fuzzy rough approximation** in  $\text{FAS}$  is a mapping  $\text{Apr}_{\text{FAS}}^{\mathcal{I}, \mathcal{T}} : \mathcal{F}(\mathfrak{X}) \rightarrow \mathcal{F}(\mathfrak{X}) \times \mathcal{F}(\mathfrak{X})$  defined by:

$$\text{for every } A \in \mathcal{F}(\mathfrak{X}), \\ \text{Apr}_{\text{FAS}}^{\mathcal{I}, \mathcal{T}}(A) = (\underline{\text{FAS}}_{\mathcal{I}}(A), \overline{\text{FAS}}_{\mathcal{T}}(A)),$$

where

$$\left( \forall x \in \mathfrak{X} \right) \left( \underline{\text{FAS}}_{\mathcal{I}}(A)(x) = \inf_{y \in \mathfrak{X}} \mathcal{I}(R(x, y), A(y)) \right) \\ \left( \forall x \in \mathfrak{X} \right) \left( \overline{\text{FAS}}_{\mathcal{T}}(A)(x) = \sup_{y \in \mathfrak{X}} \mathcal{T}(R(x, y), A(y)) \right).$$

The fuzzy set  $\underline{\text{FAS}}_{\mathcal{I}}(A)$  (resp.  $\overline{\text{FAS}}_{\mathcal{T}}(A)$ ) is called an  $\mathcal{I}$ -*lower* (resp.  $\mathcal{T}$ -*upper*) **fuzzy rough approximation of  $A$  in  $\text{FAS}$** . □

<sup>3</sup>In Nachtegaal and Kerre (1998) one can get an idea about the huge number of border implicators defined on a finite chain.

fuzzy rough approximation of  $A \in \mathcal{F}(\mathfrak{X})$  in FAS iff  $F = \underline{\text{FAS}}_{\mathcal{I}}(A)$  for some  $B$ -implicator  $\mathcal{I}$  (resp.  $F = \overline{\text{FAS}}^{\mathcal{T}}(A)$  for some  $t$ -norm  $\mathcal{T}$ ).

**Definition 3** Let  $\text{FAS} = (\mathfrak{X}, R)$  be a fuzzy approximation space and let  $\mathcal{I}$  and  $\mathcal{T}$  be a  $B$ -implicator and a  $t$ -norm, respectively. A pair  $(L, U) \in \mathcal{F}(\mathfrak{X}) \times \mathcal{F}(\mathfrak{X})$  is called an  $(\mathcal{I}, \mathcal{T})$ -fuzzy rough set iff  $(L, U) = \text{Apr}_{\text{FAS}}^{\mathcal{I}, \mathcal{T}}(A)$  for some  $A \in \mathcal{F}(\mathfrak{X})$ .  $\square$

Let us define three classes of fuzzy rough sets:

- ◇  $\mathbb{S}$ -fuzzy rough sets, determined by  $(\mathcal{I}_{\mathcal{S}}, \mathcal{T}_{\mathcal{S}})$ , where  $\mathcal{I}_{\mathcal{S}}$  is a continuous  $S$ -implicator based on a continuous  $t$ -conorm  $\mathcal{S}$  and a continuous and involutive negator  $\mathcal{C}$ , and  $\mathcal{T}_{\mathcal{S}}(x, y) = \mathcal{C}(S(\mathcal{C}(x), \mathcal{C}(y)))$  for all  $x, y \in [0, 1]$ ;
- ◇  $\mathbb{R}$ -fuzzy rough sets, determined by  $(\mathcal{I}_{\mathcal{T}}, \mathcal{T})$ , where  $\mathcal{I}_{\mathcal{T}}$  is a continuous  $R$ -implicator based on a continuous  $t$ -norm  $\mathcal{T}$ ;
- ◇  $\mathbb{Q}$ -fuzzy rough sets, determined by  $(\mathcal{I}, \mathcal{T})$ , where  $\mathcal{I}$  is a  $QL$ -implicator based on a continuous  $t$ -norm  $\mathcal{T}$  and a continuous and involutive negator  $\mathcal{C}$ .

**Remark 1** Let us recall the definition of fuzzy rough sets as proposed in Dubois and Prade (1990). Given a nonempty universe  $\mathfrak{X}$ , a similarity relation  $R$  on  $\mathfrak{X}$  and a fuzzy set  $F \in \mathcal{F}(\mathfrak{X})$ , a fuzzy rough set is a pair  $(R_*(F), R^*(F))$  of fuzzy sets on  $\mathfrak{X}$  such that for every  $x \in \mathfrak{X}$ ,

$$R_*(F)(x) = \inf_{y \in \mathfrak{X}} \max \{1 - R(x, y), F(y)\}$$

$$R^*(F)(x) = \sup_{y \in \mathfrak{X}} \min \{R(x, y), F(y)\}.$$

It is easily noted that this is exactly an  $(\mathcal{I}_b, \mathcal{T}_M)$ -fuzzy rough set, where  $\mathcal{I}_b$  is the Kleene–Dienes fuzzy implicator and  $\mathcal{T}_M$  is the standard *min* operation.  $\square$

Let  $\mathcal{I}$  and  $\mathcal{T}$  be a  $B$ -implicator and a  $t$ -norm, respectively. For a fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$ , we say that  $A \in \mathcal{F}(\mathfrak{X})$  is  $(\mathcal{I}, \mathcal{T})$ -definable in FAS iff  $\underline{\text{FAS}}_{\mathcal{I}}(A) = A = \overline{\text{FAS}}^{\mathcal{T}}(A)$ .

## 4 BASIC PROPERTIES OF FUZZY ROUGH SETS

Let us begin with the following simple observation.

**Property 1** For every  $B$ -implicator  $\mathcal{I}$ , every  $t$ -norm  $\mathcal{T}$ , every fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$  and every fuzzy set  $A \in \mathcal{F}(\mathfrak{X})$ ,

$$\underline{\text{FAS}}_{\mathcal{I}}(A) \subseteq A \subseteq \overline{\text{FAS}}^{\mathcal{T}}(A). \quad \blacksquare$$

By straightforward verification one can easily check the following property.

**Property 2** Let  $\mathcal{I}$  and  $\mathcal{T}$  be a  $B$ -implicator and a  $t$ -norm, respectively. Then for every fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$ ,

$$\underline{\text{FAS}}_{\mathcal{I}}(\emptyset) = \emptyset = \overline{\text{FAS}}^{\mathcal{T}}(\emptyset)$$

$$\overline{\text{FAS}}^{\mathcal{T}}(\mathfrak{X}) = \mathfrak{X}. \quad \blacksquare$$

In general, it holds that  $\underline{\text{FAS}}_{\mathcal{I}}(\mathfrak{X}) \neq \mathfrak{X}$ . To see that, consider the following example.

**Example 1** Consider a fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$ , where  $\mathfrak{X} = \{a, b, c\}$  and  $R$  is given in the table below.

	$a$	$b$	$c$
$a$	1	$\frac{1}{2}$	0
$b$	$\frac{1}{2}$	1	0
$c$	0	0	1

$$\underline{\text{FAS}}_{\mathcal{I}_m}(\mathfrak{X}) = \left\{ (a : \frac{1}{2}), (b : \frac{1}{2}), (c : 1) \right\}. \quad \square$$

However, for some classes of fuzzy implicators, we have  $\underline{\text{FAS}}_{\mathcal{I}}(\mathfrak{X}) = \mathfrak{X}$ .

**Property 3** Let  $\mathcal{I}$  be an  $S$ -implicator or an  $R$ -implicator. For every fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$ ,

$$\underline{\text{FAS}}_{\mathcal{I}}(\mathfrak{X}) = \mathfrak{X}. \quad \blacksquare$$

In rough set theory (see Pawlak (1982)) it is the well-known fact that for every approximation space  $\text{AS} = (\mathfrak{X}, R)$  and every  $A, B \subseteq \mathfrak{X}$  such that  $A \subseteq B$ ,  $\underline{\text{AS}}(A) \subseteq \underline{\text{AS}}(B)$  and  $\overline{\text{AS}}(A) \subseteq \overline{\text{AS}}(B)$ . While considering fuzzy rough sets, we can prove the following analogon of this property.

**Property 4** Let  $\mathcal{I}$  be an  $r$ -monotonic  $B$ -implicator and let  $\mathcal{T}$  be a  $t$ -norm. Then for every fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$  and every  $A, B \in \mathcal{F}(\mathfrak{X})$ ,  $A \subseteq B$  implies:

$$\begin{aligned} \overline{\text{FAS}}^{\mathcal{T}}(A) &\subseteq \overline{\text{FAS}}^{\mathcal{T}}(B) \\ \underline{\text{FAS}}_{\mathcal{I}}(A) &\subseteq \underline{\text{FAS}}_{\mathcal{I}}(B). \end{aligned} \quad \blacksquare$$

Given a fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$ , let us consider now fuzzy rough approximations of fuzzy sets  $\underline{\text{FAS}}_{\mathcal{I}}(A)$  and  $\overline{\text{FAS}}^{\mathcal{T}}(A)$  for  $A \in \mathcal{F}(\mathfrak{X})$ . Property 1 immediately implies:

$$\underline{\text{FAS}}_{\mathcal{I}}\left(\underline{\text{FAS}}_{\mathcal{I}}(A)\right) \subseteq \underline{\text{FAS}}_{\mathcal{I}}(A) \subseteq \overline{\text{FAS}}^{\mathcal{T}}\left(\underline{\text{FAS}}_{\mathcal{I}}(A)\right) \quad (3)$$

$$\underline{\text{FAS}}_{\mathcal{I}}\left(\overline{\text{FAS}}^{\mathcal{T}}(A)\right) \subseteq \overline{\text{FAS}}^{\mathcal{T}}(A) \subseteq \overline{\text{FAS}}^{\mathcal{T}}\left(\overline{\text{FAS}}^{\mathcal{T}}(A)\right). \quad (4)$$

The question arises for which fuzzy implicators and  $t$ -norms (3)–(4) have the form of equalities, as for (crisp) rough approximations (see Pawlak (1982)).

**Property 5** Let  $\mathcal{I}$  be any  $R$ -implicator or any  $S$ -implicator based on an involutive fuzzy negator  $\mathcal{C}$ . Then for every fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$  and every fuzzy set  $A \in \mathcal{F}(\mathfrak{X})$ ,

$$\underline{\text{FAS}}_{\mathcal{I}}\left(\underline{\text{FAS}}_{\mathcal{I}}(A)\right) = \underline{\text{FAS}}_{\mathcal{I}}(A). \quad \blacksquare$$

However, the above equality does not generally hold for  $QL$ -implicators.

**Example 2** Let  $\text{FAS} = (\mathfrak{X}, R)$  be defined as in Example 1. Consider a  $QL$ -implicator  $\mathcal{I}_p(x, y) = 1 - x + x^2 * y$  for all  $x, y \in [0, 1]$  (based on  $\mathcal{T}_P$ ,  $\mathcal{S}_P$  and  $\mathcal{C}_s$ ). By simple calculations we get

$$\begin{aligned} \underline{\text{FAS}}_{\mathcal{I}_p}(\mathfrak{X}) &= \left\{ (a : \frac{3}{4}), (b : \frac{3}{4}), (c : 1) \right\} \\ \underline{\text{FAS}}_{\mathcal{I}_p}\left(\underline{\text{FAS}}_{\mathcal{I}_p}(\mathfrak{X})\right) &= \left\{ (a : \frac{11}{16}), (b : \frac{11}{16}), (c : 1) \right\}. \end{aligned}$$

Clearly,  $\underline{\text{FAS}}_{\mathcal{I}_p}\left(\underline{\text{FAS}}_{\mathcal{I}_p}(\mathfrak{X})\right) \neq \underline{\text{FAS}}_{\mathcal{I}_p}(\mathfrak{X})$ . □

Moreover, we have the following property.

**Property 6** Let  $\mathcal{T}$  be a continuous  $t$ -norm and let  $\mathcal{I}_{\mathcal{T}}$  be an  $R$ -implicator based on  $\mathcal{T}$ . Then for every fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$  and every  $A \in \mathcal{F}(\mathfrak{X})$ ,

$$\overline{\text{FAS}}^{\mathcal{T}}\left(\underline{\text{FAS}}_{\mathcal{I}_{\mathcal{T}}}(A)\right) = \underline{\text{FAS}}_{\mathcal{I}_{\mathcal{T}}}(A) \quad (5)$$

$$\underline{\text{FAS}}_{\mathcal{I}_{\mathcal{T}}}\left(\overline{\text{FAS}}^{\mathcal{T}}(A)\right) = \overline{\text{FAS}}^{\mathcal{T}}(A). \quad (6)$$

■

$\mathcal{I}_g$ ) is an  $R$ -implicator based on the  $t$ -norm  $\mathcal{T}_B$  ( $\mathcal{T}_P$  and  $\mathcal{T}_M$ , respectively). Therefore, by Property 6, the pairs  $(\mathcal{I}_a, \mathcal{T}_B)$ ,  $(\mathcal{I}_\Delta, \mathcal{T}_P)$  and  $(\mathcal{I}_g, \mathcal{T}_M)$  satisfy (5) and (6) for all  $\text{FAS} = (\mathfrak{X}, R)$  and  $A \in \mathcal{F}(\mathfrak{X})$ .

In general, however, neither (5) nor (6) holds for a  $t$ -norm  $\mathcal{T}$  and a  $QL$ -implicator based on  $\mathcal{T}$ . Also, (5) is not satisfied for some  $S$ -implicators (based on  $t$ -conorm  $S$  and a negator  $\mathcal{C}$ ), and  $t$ -norms  $\mathcal{T}_S$  given by  $\mathcal{T}_S(x, y) = \mathcal{C}(S(\mathcal{C}(x), \mathcal{C}(y)))$  for all  $x, y \in [0, 1]$ .

**Example 3** Let  $\text{FAS} = (\mathfrak{X}, R)$  be defined as in Example 1. For the Early Zadeh implicator  $\mathcal{I}_m$  and the fuzzy set  $A = \{(a : 0), (b : 1), (c : 0)\}$  we have

$$\begin{aligned}\underline{\text{FAS}}_{\mathcal{I}_m}(A) &= \{(a : 0), (b : \frac{1}{2}), (c : 0)\} \subset \{(a : \frac{1}{2}), (b : \frac{1}{2}), (c : 0)\} = \overline{\text{FAS}}^{\mathcal{T}_M}(\underline{\text{FAS}}_{\mathcal{I}_m}(A)) \\ \overline{\text{FAS}}^{\mathcal{T}_M}(A) &= \{(a : \frac{1}{2}), (b : 1), (c : 0)\} \supset \{(a : \frac{1}{2}), (b : \frac{1}{2}), (c : 0)\} = \underline{\text{FAS}}_{\mathcal{I}_m}(\overline{\text{FAS}}^{\mathcal{T}_M}(A)).\end{aligned}$$

Moreover, the Kleene–Dienes–Lukasiewicz implicator  $\mathcal{I}_*$  is an  $S$ -implicator based on  $\mathcal{S}_P$  and  $\mathcal{C}_s$ . Then, for every  $x, y \in [0, 1]$ ,  $\mathcal{T}_S(x, y) = x * y$ . It is easy to check that

$$\begin{aligned}\underline{\text{FAS}}_{\mathcal{I}_*}(A) &= \{(a : 0), (b : \frac{1}{2}), (c : 0)\} \subset \{(a : \frac{1}{4}), (b : \frac{1}{2}), (c : 0)\} = \overline{\text{FAS}}^{\mathcal{T}_S}(\underline{\text{FAS}}_{\mathcal{I}_*}(A)) \\ \overline{\text{FAS}}^{\mathcal{T}_S}(A) &= \{(a : \frac{1}{2}), (b : 1), (c : 0)\} \supseteq \{(a : \frac{1}{2}), (b : \frac{3}{4}), (c : 0)\} = \underline{\text{FAS}}_{\mathcal{I}_*}(\overline{\text{FAS}}^{\mathcal{T}_S}(A)).\end{aligned} \quad \square$$

**Property 7** For every continuous  $t$ -norm  $\mathcal{T}$ , every fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$  and every  $A \in \mathcal{F}(\mathfrak{X})$ ,

$$\overline{\text{FAS}}^{\mathcal{T}}(\overline{\text{FAS}}^{\mathcal{T}}(A)) = \overline{\text{FAS}}^{\mathcal{T}}(A). \quad \blacksquare$$

Properties 5, 6 and 7 immediately imply the following corollary.

**Corollary 1** Let  $\mathcal{T}$  be a continuous  $t$ -norm and let  $\mathcal{I}_{\mathcal{T}}$  be an  $R$ -implicator based on  $\mathcal{T}$ . Then for every fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$  and every  $A \in \mathcal{F}(\mathfrak{X})$ , both  $\underline{\text{FAS}}_{\mathcal{I}_{\mathcal{T}}}(A)$  and  $\overline{\text{FAS}}^{\mathcal{T}}(A)$  are  $(\mathcal{I}_{\mathcal{T}}, \mathcal{T})$ -definable in  $\text{FAS}$ .  $\square$

Finally, let us consider fuzzy rough approximations of complementation of fuzzy sets.

**Property 8** For every fuzzy approximation space  $\text{FAS} = (\mathfrak{X}, R)$  and every  $A \in \mathcal{F}(\mathfrak{X})$ ,

(a) if  $\mathcal{I}$  is an  $S$ -implicator based on a  $t$ -conorm  $S$  and a continuous and involutive fuzzy negator  $\mathcal{C}$ , then

$$\underline{\text{FAS}}_{\mathcal{I}}(co_{\mathcal{C}}A) = co_{\mathcal{C}}\overline{\text{FAS}}^{\mathcal{T}}(A) \quad (7)$$

$$\overline{\text{FAS}}^{\mathcal{T}}(co_{\mathcal{C}}A) = co_{\mathcal{C}}\underline{\text{FAS}}_{\mathcal{I}}(A), \quad (8)$$

where  $\mathcal{T}(x, y) = \mathcal{C}(S(\mathcal{C}(x), \mathcal{C}(y)))$  for all  $x, y \in [0, 1]$ ,

(b) if  $\mathcal{I}$  is a  $QL$ -implicator based on a  $t$ -norm  $\mathcal{T}$ , a  $t$ -conorm  $S$  and a continuous fuzzy negator  $\mathcal{C}$ , then

$$\underline{\text{FAS}}_{\mathcal{I}}(co_{\mathcal{C}}A) \subseteq co_{\mathcal{C}}\overline{\text{FAS}}^{\mathcal{T}}(A) \quad (9)$$

$$\overline{\text{FAS}}^{\mathcal{T}}(co_{\mathcal{C}}A) \subseteq co_{\mathcal{C}}\underline{\text{FAS}}_{\mathcal{I}}(A). \quad (10)$$

Thus, (7)–(8), the strict fuzzy counterparts of the properties of rough sets, hold, for instance, for the pairs  $(\mathcal{I}_a, \mathcal{T}_B)$ ,  $(\mathcal{I}_*, \mathcal{T}_M)$  and  $(\mathcal{I}_b, \mathcal{T}_M)$ . The weaker conditions (9)–(10) are satisfied for  $(\mathcal{I}_m, \mathcal{T}_M)$ .  $\blacksquare$

Let  $co_s$  be the standard fuzzy complementation and let  $\mathcal{I}_{\mathcal{T}}$  be an  $R$ -implicator based on a continuous  $t$ -norm  $\mathcal{T}$ . In general, there is no link between  $\underline{\text{FAS}}_{\mathcal{I}_{\mathcal{T}}}(co_sA)$  and  $co_s\overline{\text{FAS}}^{\mathcal{T}}(A)$ , for any  $\text{FAS} = (\mathfrak{X}, R)$  and any  $A \in \mathcal{F}(\mathfrak{X})$ .

on  $\mathcal{T}_P$ ) and the fuzzy sets

$$A = \{(a : \frac{1}{2}), (b : 1), (c : 0)\}, \quad B = \{(a : 0), (b : \frac{1}{2}), (c : 0)\},$$

we get

$$\overline{\text{FAS}}^{\mathcal{T}_P}(co_s A) = \{(a : \frac{1}{2}), (b : \frac{1}{4}), (c : 1)\} \supseteq \{(a : \frac{1}{2}), (b : 0), (c : 1)\} = co_s \underline{\text{FAS}}_{\mathcal{I}_g}(A),$$

$$\overline{\text{FAS}}^{\mathcal{T}_P}(co_s B) = \{(a : 1), (b : \frac{1}{2}), (c : 1)\} \subseteq \mathfrak{X} = co_s \underline{\text{FAS}}_{\mathcal{I}_g}(B). \quad \square$$

## 5 CONCLUSIONS AND FUTURE WORK

In this paper we have proposed a very general definition of fuzzy rough sets. Taking into account three classes of fuzzy implicators, three corresponding classes of fuzzy rough sets have been defined, namely  $\mathbb{S}$ ,  $\mathbb{R}$  and  $\mathbb{Q}$ . We have investigated several basic properties of these classes, essentially in the context of well-known rough equalities.

The obtained results show that the classes  $\mathbb{R}$  and  $\mathbb{S}$  exhibit essential similarities to crisp rough sets and might be viewed as the most straightforward fuzzy generalizations of Pawlak's original concept. Moreover, since the Lukasiewicz implicator  $\mathcal{I}_a$  is both an  $R$ -implicator (based on  $\mathcal{T}_B$ ) and an  $S$ -implicator (based on  $\mathcal{S}_B$  and  $\mathcal{C}_s$ , where  $\mathcal{T}_B \equiv \mathcal{T}_{\mathcal{S}_B}$ ), fuzzy rough sets determined by  $(\mathcal{I}_a, \mathcal{T}_B)$  belong both to the class  $\mathbb{R}$  and  $\mathbb{S}$ . Consequently, these very fuzzy rough sets satisfy ALL properties considered here, which means, in turn, that  $(\mathcal{I}_a, \mathcal{T}_B)$ -fuzzy rough sets may be viewed as the very "close" ones to crisp rough sets.

The established results are the starting point for further studies. In particular, it would be worthwhile to consider a problem of *general postulates* (axioms) representing desirable features of fuzzy rough sets. This context might be viewed as a general criterion of applicability of particular fuzzy implicators and triangular norms for constructing fuzzy rough sets. Another intended direction of our investigations concerns the problem of reasoning in hybrid fuzzy-rough systems. Our question is how to combine inference patterns widely applied in fuzzy logics with reasoning methods traditionally developed within the framework of rough set theory. Finally, it is certainly worthwhile to consider links between fuzzy rough sets and modal logics. As it was pointed out in Orłowska (1998), fuzzy rough sets offer a nice tool for developing modal-like logic with fuzzy modalities applied to fuzzy propositions. These topics will be investigated in our future work.

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