A deductive System based on the Force Implication

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Abstract

ABSTRACT : The Force Implication previously introduced in 1995, was proven to be useful in Command Problems. The operator is here investigated from a new theoretical point of view. Among our main results hereafter exhibited, appear the transitivity and the compatibility properties. Particularly, it is shown how to build a deductive monotone system based on the Force Implication.

KEYWORDS : Implication, deductive system.

1 Introduction

Non standard logics often have in common to go back to sensible experience, to some kind of pragmatism, based on the fact that reasoning also arises from experience learning and then can reach universal forms. [Da C97; Got93; Bou993]

In this trend, logicians, like Ackermann [Ack56], think the heart of logic lies in the "conditional" connective representing the "If....then" of the current language. Many studies on "conditional" elements have been achieved [Dub91; Goo91; Ngu93].

In this point of view, the so-called "material implication" of classical logic is not natural from the point of human reasoning in the daily life [Zad83].

We believe with Da Costa [DaC97] that, when asserting "If A, Then B", truth of A is in some way pertinent with respect to truth of B, that A and B should get something in common, their meanings cannot be completely heterogeneous.

Therefore, previous motivations which has led to introduce the Force Implication were related to both fundamental and applicative aspects :

- from basic considerations, there was a claim in Approximate Reasoning and Logic for modeling the ‘if - then’ statement by something different from the so-called material implication.
- from the applicative aspects, the largely and successfully used operator of Mamdani [Mam74] (or similar operators) in Command and Control, is actually not an implication operator (namely because of its symmetry property and because of values it takes for boolean data)

Facing this problem, the aim was to find a kind of non symmetric generalization of the Mamdani operator.

Hence Vincent and Dujet introduced [Vin95] the Force Implication and showed in a problem of command of a car, that it works better than using the Mamdani operator as well as the Lukasiewicz one.

This paper is intended to go deeper in the theoretical investigations of the Force Implication and to show how to build a monotone deductive system based on this operator.

The first part presents the axiomatic of the Force Implication, the second part defines the logic framework chosen in order to get coherent desired properties as transitivity and compatibility with the inferential rule of Modus Ponens. The third part is a recall about the application of the Force Implication to the command of a car.

The last part is devoted to the study of the Force Implication versus monotone deductive systems.

2 Axiomatic of the Force Implication

Many operators have been defined and studied. They differ by the properties they want to express [Hal88]. Let us recall that any implication operator [Tha89] has to satisfy at least 3 axioms :

. Truth functionality
. Transitivity
. Non symmetry

Moreover, our concern is the search of a graded operator, indicating to which extent the relation "A implies B" (which nevertheless remains a binary relation, as underlined by Toth [Tot87]) is effectively observed, is realized, actually happens, in a given context.

Hence, as for the question of a graded equality, we have to choose between different points of view, following the positivist attitude or not.

That is to say, in the case of a graded equality E, are we to choose to model the reflexivity property by

$\forall x\ E(x,x)=1$

or by $\forall x\ E(x,x)=x$?

which means, in the second case, that a kind of "local" existence of x is taken into consideration, and that x is considered to be equal to itself only to the degree of its "partial" existence.

We adapt this second point of view, pondering that A can imply B only to a degree which is equal to its degree of "partial" existence.

From this ontological thinking, joined to the preceding considerations, it is possible to define the axiomatic of the so-called Force Implication.

Definition of Force Implication

A Force Implication is an operator in the unit interval I = [0,1], denoted by $a \xrightarrow{F} b$, satisfying the following axioms :

. Idempotence
. Truth functionality
. Non Symmetry
. Transitivity

Transitivity here is to be taken in the sense of Meyer meaning a $\wedge$ - transitivity, where $\wedge$ is connective corresponding to the set intersection.

Therefore it is needed to precise the logic framework which is chosen.

Let us point out that the first axiom puts the Force Implication in the family of operators looking as generalizations of the Boolean conjunction. The different operator family have been studied by Dujet and Vincent in [Duj94], and besides it is a non symmetric generalization.

3 Logical Framework

3.1 Definitions

The t-norms T_∞ and S_∞ are of particular interest for theoretical set-operations because of the simple characterization of fuzzy partitions of a set they allow. See Dumitrescu [Dum94]

Hence our chosen logical framework is the residuated lattice $(I,\wedge,\vee,\neg,0,1)$ where I is the unit real interval, and the connectives $\wedge$, $\vee$, $\neg$ are defined as follows :

$a \wedge b = T_\infty(a,b) = \text{Max}\,(a+b-1,0)$
$a \vee b = S_\infty(a,b) = \text{Min}\,(a+b,1)$
$\neg a = 1 - a$

It is possible to define as usually a material implication, $a \rightarrow b = \neg a \vee b$
as well as an equivalence $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$,

and we add the Force Implication : $a \xrightarrow{F} b$, defined as follows :

$$\boxed{a \xrightarrow{F} b = a(1 - |a - b|)} \qquad (1)$$

Hereafter can be shown the graphs of the Mamdani, Lukasiewicz and Force Implication Operators.

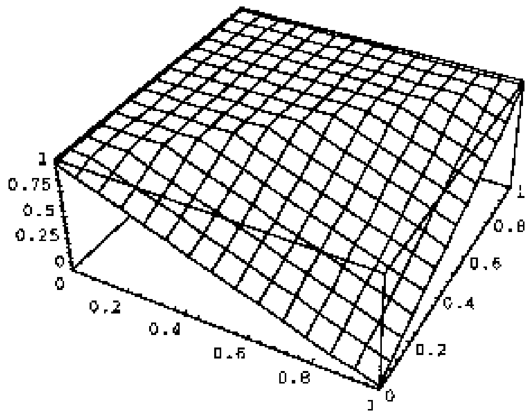

Figure 1 : Lukasiewicz implication operator

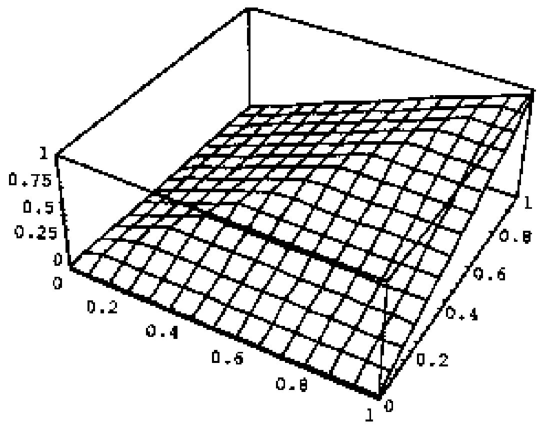

Figure 2 : Mamdani implication operator

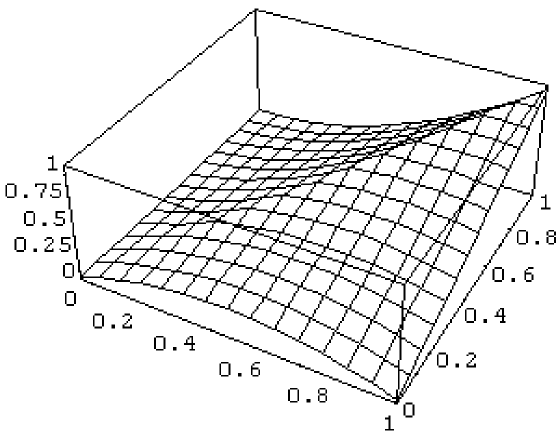

Figure 3 : Force Implication Operator

3.2 Properties of the Force Implication

Immediate properties

- $1 \xrightarrow{F} b = b \quad \forall b \in [0,1]$
- $0 \xrightarrow{F} b = 0 \quad \forall b \in [0,1]$
- $a \xrightarrow{F} b \leq a \quad \forall (a,b) \in [0,1]^2$ (this is consistent with our will to reinforce the role of the antecedent)
- if $a \xrightarrow{F} b = \alpha$ and $a \wedge a' \xrightarrow{F} b = \beta$,then there does not exist any relation between a et b independently of a'.

Expression by means of t-norms

The formula (1) in 3.1 may be rewritten as follows : $a \xrightarrow{F} b = a \,.\, T_\infty(S_\infty(a,\neg b),S_\infty(\neg a,b))$
A straight forward generalization is suggested by using

$$a \xrightarrow{F} b = T_1(a\ ,\ T(S\ (a,\neg b),S\ (\neg a,b)))$$

when T and S are dual.

But, for instance, trying $T = T_1 = \text{Min}$, we get an operator $\wedge$-transitive for the Lukasiewicz $\wedge$, but not compatible with modus ponens.

Further more, working in a lattice (or Heyting Algebra) equipped with a relation of indistinguishability [Hohle, 1993] denoted by I, and moreover, noticing that
I(a,b)=1-d(a,b), where d denotes a distance, is a particular case of this kind of relation, a second step of generalization is achieved by stating

$$\boxed{a \xrightarrow{F} b = T1(a,I(a,b))}$$

3.3 The transitivity property

Let us recall that an implication operator $\rightarrow$ is said to be T-transitive if $a \rightarrow c \geq T(a \rightarrow b, b \rightarrow c)$ for the t-norm T.

Proposition

In the logic system $(I,\wedge,\vee,\neg,0,1)$, with $\wedge = T_\infty$ and $\vee = S_\infty$ equipped with the Force Implication $a \xrightarrow{F} b = a(1 - |a - b|)$, the Force Implication is $\wedge$ - transitive

Proof : easy

3.4 Compatibility with modus ponens

Looking at the modus ponens rule :

$$\frac{\begin{array}{l} A \\ A \xrightarrow{F} B \end{array}}{B}$$

It is wanted that the conjunction of A and $A \xrightarrow{F} B$ leads to conclusion B, in the context of our logic, where conjunction is modeled by T_∞ and implication is the Force Implication given by formula (1) in 3.1

Hence, we have to check that

$$\forall y \in [0,1] \quad y = \sup_{x\in[0,1]} T_\infty(x, x(1-|x-y|))$$

which effectively holds.

So, the logic system $(I,\wedge,\vee,\neg,0,1)$ equipped with the Force Implication would be coherent.

As it happens, our force implication operator, in the form defined in formula (1) en 3.1 is not compatible with modus ponens for classical AND and OR operators such as MIN and MAX. The product as a t-norm does not ensure compatibility. Nevertheless, we get compatibility, when using the inferential conjunction rule for generalized modus ponens, with the well known t-norm T and t-conorm S of Lukasiewicz, that is to say :

$$T(u,v) = \max(u+v-1,0) \text{ and } S(u,v) = \min(u+v,1)$$

We can visualize the difference between the Lukasiewicz AND operator and the more classical Min operator [Fod93].

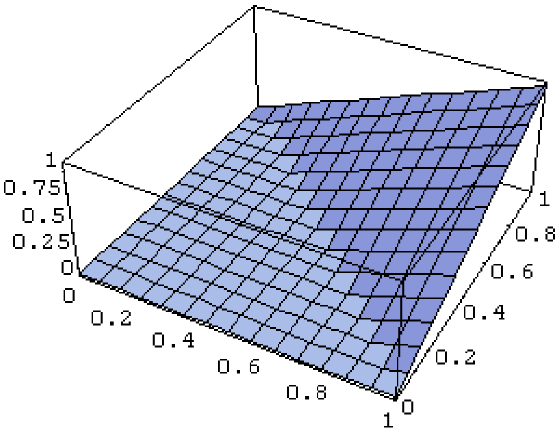

Figure 4 : Lukasiewicz AND operator

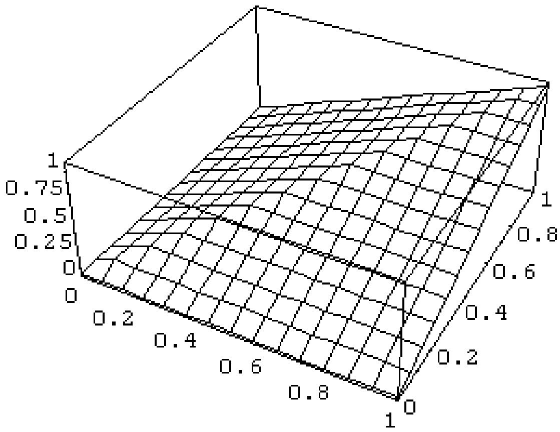

Figure 5 : MIN operator

So appears a natural environment for the use of our force-implication, and t-norm and t-conorm will now refer to Lukasiewicz operators.

4 Deductive System versus Force Implication

In the chosen logic $(I,\wedge,\vee,\neg,\xrightarrow{F})$, the connective $\xrightarrow{F}$ is now interpreted as a deduction operator, (but a graded one).

So, it is possible to state next definition.

Definition

p ||→ q means that proposition q is deduced from proposition p to the degree $p \xrightarrow{F} q$

It is sufficient, of course, to study the properties of this operator in [0,1].

• Reflexivity

p ||→ p means proposition p is deduced from proposition p to the degree $p \xrightarrow{F} p$

As we have already noticed $p \xrightarrow{F} p = p$ (idempotence axiom of $\xrightarrow{F}$) , then the reflexivity holds.

• Equivalence property

This property states that if h is deduced from p and q is equivalent to p, then h can be deduced from q.
In our logic, the following interpretation, can be given :

$$(h \xrightarrow{F} q) \geq (h \xrightarrow{F} p)$$

provided the condition : q is equivalent to p.
This last condition means here :

$$(\neg q \vee p) \wedge (\neg p \vee q) = 1$$

and leads obviously (because of the definition of connectives ∧ and ∨ in our framework) to the conditions :

$$q \geq p \text{ and } p \geq q$$

hence q = p, so equivalence property holds.

• Right weakness

The significance of this property can be stated : if g can be deduced from f and if h is a necessary consequence of g, then h can be deduced from f. So we have the following scheme

$$\frac{f \,||\!\!\rightarrow g \qquad (g \rightarrow h)=1}{f \,||\!\!\rightarrow h}$$

The sense of this affirmation is that :

$f \xrightarrow{F} h \geq f \xrightarrow{F} g$ under the hypotheses that, $g \leq h$ and $f \xrightarrow{F} g \geq f$.

• Cutting property

This property states that the following scheme holds :

$$\frac{f \wedge g \,||\!\!\rightarrow h \qquad f \,||\!\!\rightarrow g}{f \,||\!\!\rightarrow h}$$

The property gives the possibility to deduce h from f when the hypotheses are that h can be deduced from f and g and that besides, g can be deduced from f.

In our case, it can be shown that $(f \xrightarrow{F} h) \geq f^2$
under the following hypotheses :

$$(f \wedge g \xrightarrow{F} h) \geq f \wedge g \quad \text{and} \quad (f \xrightarrow{F} g) \geq f^2$$

• Cautious monotony

The property can be formulated by :

$$\frac{f \,||\!\!\rightarrow g \qquad f \,||\!\!\rightarrow h}{f \wedge g \,||\!\!\rightarrow h}$$

It says that if g can be deduced from f and at the same time h can be deduced from f, then h can be deduced from f∧g.

More precisely, it can be shown that under the hypotheses

$$(f \xrightarrow{F} g) \geq f^2 \quad \text{and} \quad (f \xrightarrow{F} h) \geq f^2$$

the inequality $(f \wedge g \xrightarrow{F} h) \geq (f \wedge g)^2$ holds.

In the classical frame we know that a deductive system in which these 5 crude properties hold and in which we have the transitivity property is a monotone deductive system [Kay97].

$$\frac{f \,\|\!\rightarrow g \qquad g \,\|\!\rightarrow h}{f \,\|\!\rightarrow h}$$

Of course, the conclusion cannot be the same here, but the transitivity and the properties we have just identified, make evident the importance of the force implication operator used in the frame of human reasoning.

Conclusion

This theoretical extensive study of the Force-Implication shows how important such a notion can become, not only in the mathematical field but also in the so many aspects of every day life and in the applications where the human element, specially from the reasoning point of view, takes a large place.

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