DETECTING MULTIPATH SIGNALS WITH THE MATCHED-LAG FILTER

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ABSTRACT

A detection problem is considered for a single broadband source of unknown waveform and emission time. The signal travels to the receiver along multipath with unknown delays and temporal separation exceeding the inverse bandwidth of the signal. The received noise has uncertain variance. The travel times of the multipath are impractical to predict because of uncertainties in the environment. The presence or absence of the signal is estimated from the auto-correlation function. Instead of stochastically modeling the multipath in terms of their received auto-correlation function, receivers are constructed which constrain the signal-related lags in the auto-correlation function to have physically possible arrangements. For simple cases, this approach, called a matched-lag filter, yields probabilities of detection that are 1.35 times greater (for a false-alarm probability of 0.001) than conventional filters which base their decision on the signal-to-noise ratio in the auto-correlation function.

1. INTRODUCTION

A detection problem is considered where either noise or signal plus noise is present at a receiver. It is assumed that the emitted signal travels along two or more paths because of reflections or refraction within the environment. Assume the transmitted waveform, amplitude, and emission time are unknown, as are the travel times of the multipath. The lack of information about a library of transmitted waveform shapes appears to preclude the use of detection methods based on the matched filter [11] or those discussed for wireless communication systems [12]. It is assumed that it is impractical to accurately model the travel times of the multipath because of insufficient information about the environment. Thus it would not be possible to used matched field processing [3] or other methods to model the multipath. Such conditions may occur for the propagation of acoustic or electromagnetic waves on land where the locations of boundaries such as the ground, trees, rocks, buildings etc. are often unknown [12, 7], or in shallow water with complicated bathymetry, or with electromagnetic propagation through the ionosphere [15]. Such conditions may also occur in the Earth when acoustic signals from earthquakes or nuclear blasts propagate to receivers along many paths. Further assume that the signal has a wide bandwidth, so that some multipath arrive at intervals exceeding the inverse bandwidth of the signal. Assume that the variance of the noise is imperfectly known, either because one does not

know when the signal is on or off, or because the noise is not stationary, or because one estimates the variance from the data. This assumption makes it difficult to use signal detectors which assume that the variance of the noise is known.

These kinds of signals might be detected with receivers which base their decision on the received power [14]. This paper discusses additional techniques for detecting these signals based on the data from the auto-correlation function. This function provides the standard gain obtained with a matched filter [11], since the signals from each multipath are assumed to be attenuated and delayed replicas of one other. Perhaps the most distinguishing feature of the receiver discussed here is that it inherently incorporates only physically possible lags at which the signals occur in the auto-correlation function. In other words, the lags and amplitudes of signal-related peaks are treated deterministically rather than stochastically. There are examples of the detection of multipath which are modeled stochastically, such that the auto-correlation function of the multipath signals is assumed to be known ahead of time [15, 4, 17, 10]. This assumption allows multipath signals to occur at any lags in the auto-correlation function, but indeed there are many lag arrangements that are unphysical. Take, for example, three paths arriving at a receiver. There are at most $3(3-1)/2 = 3$ signal-related peaks at positive lag in the auto-correlation function [7]. The lags, τ, of these peaks must satisfy the lag equations,

$$\tau[m, n] = t[m] - t[n] \; ; \; n < m \; , \tag{1}$$

[7] where the travel time for path m is $t[m]$. The reader may verify that it is impossible to have signals at lags 2, 4, and 5 where samples are taken at times having integer values. So detecting the signals described in the first paragraph can involve more than looking at signal-to-noise ratios. It is shown here how to incorporate the physically possible lag locations into the design of a receiver in a deterministic way, and thus increase the probability of detection compared with receivers which base their decision solely on signal-to-noise ratios and a stochastic model for the travel times of the multipath. The receiver developed here which bases its decision for a signal's presence on both the signal-to-noise ratio and physically possible lags in the auto-correlation function is called a "matched-lag filter."

In most situations, the auto-correlation function of the emitted signal has an unknown shape. This problem is too difficult to deal with in an introductory paper dealing with the concept of a matched-lag filter. Instead, the matched-lag filter will be evaluated in an ideal situation where it is assumed that the emitted signal has an auto-correlation function that is like a delta function.

2. LIKLIHOOD RATIO

Under hypothesis H_0, the data, $r(k)$, at the receiver consist of $k = 1, 2, 3, \cdots, K$ mutually uncorrelated Gaussian random variables, $e(k)$, with mean zero and variance ρ^2. Under hypothesis H_1, the data contain N delayed and attenuated replicas of an emitted signal, $s(k)$, plus additive noise,

$$r(k) = \sum_{n=1}^{N} a(n)s(k - t(n)) + e(k) \ , \tag{2}$$

where the travel time of the nth path is $t(n)$. Travel time is measured in units of the sample number at the receiver. The auto-correlation function for non-negative lags is,

$$R(p) \equiv \frac{1}{K} \sum_{k=1+p}^{K} r(k)r(k - p) \ ; \ p \geq 0 \ . \tag{3}$$

2.1. Probability Density Function Under Hypothesis H_0

Under H_0, it is straightforward to show that auto-correlation function lags have mean zero, except for lag zero which will not be used. The lags are also uncorrelated. When there are many terms in the summation indices of the auto-correlation function, the correlates are also Gaussian because of the central limit theorem [8]. When the lag, p, is small compared with the greatest lag, $K - 1$, computed for the correlation function, the variance of the correlates is approximately stationary with value [16],

$$\sigma_0^2 = \frac{(\rho^2)^2}{K} \ . \tag{4}$$

The joint probability density function (pdf) of the positive lags in the auto-correlation function is,

$$f_0(\vec{R}) = (2\pi\sigma_0^2)^{-Q/2} \exp\left[-\frac{1}{2\sigma_0^2} \sum_{q=1}^{Q} R^2(q)\right] \ , \tag{5}$$

where there are Q positive lags used for the receiver and $Q << K - 1$. The travel time difference between the last significant and first multipath is assumed to be less than or equal to Q.

2.2. Lag Structure With Multipath

Under H_1, it is assumed for simplicity that N paths arrive at the receiver, each with amplitude a in Eq. (2). Their arrival intervals are assumed to be greater than or equal to the sample interval. Arrival times are assumed to coincide with a sample time. The emitted signal is assumed to have an auto-correlation function that is white and its energy is,

$$\mathcal{E} \equiv \sum_{k=1}^{K} s^2(k) \ , \tag{6}$$

and $\overline{s(k)s(k - p)}$ is zero for p not equal to zero where the overline denotes an expected value. A resolved signal has a value of A in the auto-correlation function where [16],

$$A = \frac{a^2 \mathcal{E}}{K} \ . \tag{7}$$

The number of signal-related positive lags, P', in the auto-correlation function is bounded by,

$$N - 1 \leq P' \leq P \ , \tag{8}$$

where the maximum number of resolved signals at positive lag is,

$$P = N(N - 1)/2 \ , \tag{9}$$

[7]. One obtains less than P resolved signal lags when a signal lag has two or more pairs of travel time differences which are equal to each other.

The P positive lags in the auto-correlation function satisfy the lag equations where $1 \leq n < m \leq N$ in Eq. (1). The structure of signal related lags can be obtained by generating all the possible arrangements of $N - 1$ relative travel times,

$$t[m] - t[1] \ , \ m = 2, 3, 4, \cdots N \tag{10}$$

among Q positive lags. There are,

$$B_1 \equiv \binom{Q}{N - 1} \ , \tag{11}$$

such arrangements. The bth arrangement of $N - 1$ relative travel times determines the P'_b positive lags,

$$\psi_b(p) \ ; \ p = 1, 2, 3, \cdots P'_b \ , \tag{12}$$

where signals occur. Because there can be contributions from two or more signals at the same lag, we define the number of signals at lag $\psi_b(p)$ as $\eta_b(p)$, where $\eta_b(p)$ is called the redundancy function. It is always greater than or equal to one. The bth lag-redundancy arrangement is defined to consist of the $2P'_b$ elements in the set $\{\psi_b(n), \eta_b(n)\} \ , \ n = 1, 2, 3, \cdots, P'_b$.

Some of the B_1 lag-redundancy arrangements may be the same. For example, the two travel time sets, $t \in \{0, 1, 3\}$ and $t \in \{0, 2, 3\}$ both yield the same positive auto-correlation function lags, $\psi(n) = 1, 2,$ and 3, and redundancy functions, $\eta(n) = 1, 2,$ and 1 for $n = 1, 2,$ and 3 respectively. In other words, lags 1 and 3 are resolved, and lag 2 has contributions from two signals. The total number of unique lag-redundancy sets is [16],

$$\breve{B}_1 = \frac{B_1 - B_s}{2} + B_s \ , \tag{13}$$

where,

$$B_s = \frac{2}{N}(Q - N + 2) \binom{\frac{Q}{2}}{\frac{N-2}{2}} \ ; \text{ for } N \text{ and } Q \text{ even} \ . \tag{14}$$

The liklihood ratio for the auto-correlation function lags will require using the set of unique lag-redundancy arrangements, the bth of which is denoted $\{\breve{\psi}_b(n), \breve{\eta}_b(n)\} \ , \ n = 1, 2, 3, \cdots, \breve{P}'_b$.

2.3. Probability Density Function Under Hypothesis H_1

Under H_1, the variance of the noise in an auto-correlation function is,

$$\sigma_1^2 = \sigma_0^2(1 + c_1 l^2) \ , \tag{15}$$

where the time-averaged signal-to-noise intensity ratio over K samples at the receiver is,

$$l^2 = \frac{AN}{\rho^2} \ , \tag{16}$$

and,

$$c_1 \equiv 2\left(1 + \frac{N-1}{4Q}\right) \ , \tag{17}$$

[16].

The joint pdf of the positive lags in the auto-correlation function is,

$$f_1(\vec{R}) = \sum_{b=1}^{\breve{B}_1} f_2(\vec{R}|\breve{\psi}_b) f_{\breve{\psi}_b}(\breve{\psi}_b) \ , \tag{18}$$

where the conditional probability is,

$$\begin{aligned}
f_2(\vec{R}|\breve{\psi}_b) &= (2\pi\sigma_1^2)^{-Q/2} \\
&\quad \exp\left[\frac{-1}{2\sigma_1^2}\sum_{p=1}^{\breve{P}_b'}\left(R(\breve{\psi}_b(p)) - \breve{\eta}_b(p)A\right)^2 - \right. \\
&\quad \left. \frac{1}{2\sigma_1^2}\sum_{p=\breve{P}_b'+1}^{Q} R^2(\breve{\Psi}_b(p))\right] \ ,
\end{aligned} \tag{19}$$

where $\breve{\Psi}_b(p)$ are the lags where no signals occur at positive lag. The probability of obtaining a particular arrangement of signal lags, $f_3(\breve{\psi}_b)$, will be taken to be the one of most ignorance, so it will be uniformly distributed as,

$$f_{\breve{\psi}_b}(\breve{\psi}_b) = \frac{1}{\breve{B}_1} \ . \tag{20}$$

2.4. Likelihood Ratio

It may be difficult to accurately estimate the variance of the noise, as for example is the case even if one has stationary noise consisting of K uncorrelated samples of a zero mean Gaussian random variable with true variance $\rho^2 = 1$. The 95% confidence limits for the sample standard deviation from K samples is,

$$1 \pm \frac{2^{3/2}}{K^{1/4}} \ ,$$

which, for $K = 1000$ is 1 ± 0.5 [1]. For this reason, the performance of a receiver will be investigated for the case where the variance of the noise at the receiver is imperfectly known using an average likelihood ratio, $\overline{\Lambda} = \overline{f_1}/\overline{f_0}$, where the expected value is taken over the uncertainty in the noise [6]. Since the probability of the data depends on the variance of the noise, hypotheses H_0 and H_1 are composite.

Using the Neyman-Pearson criterion, the decision on whether the data consist only of noise or of signal plus noise is made by choosing an acceptable probability of false-alarm, Q_0, and solving for a threshold value, Λ_0, in,

$$Q_0 = \int_{\Lambda_0}^{\infty} P_0(\overline{\Lambda}) d\overline{\Lambda} \ . \tag{21}$$

The probability of detection is given by,

$$Q_d = \int_{\Lambda_0}^{\infty} P_1(\overline{\Lambda}) d\overline{\Lambda} \ , \tag{22}$$

where the pdfs of $\overline{\Lambda}$ under hypotheses H_0 and H_1 are $P_0(\overline{\Lambda})$ and $P_1(\overline{\Lambda})$ respectively. The results are summarized using a receiver operating curve[13, 2].

It can be shown that an expression for the average liklihood ratio for the positive auto-correlation function lags is [16],

$$\overline{\Lambda_1(\vec{R})} = \frac{\overline{f_1(\vec{R}|y)}}{\overline{f_0(\vec{R}|x)}} \ , \tag{23}$$

where the overline denotes expectation over the uncertainty of the noise and x and y parameterize the noise uncertainty according to,

$$\begin{aligned}
x &\equiv \sigma_0^{-2} \\
y &\equiv \sigma_1^{-2},
\end{aligned} \tag{24}$$

and it is assumed that x and y are uniformly distributed on the intervals $[\breve{x}, \hat{x}]$ and $[\breve{y}, \hat{y}]$ respectively. To complete Eq. (23) we need the average pdf under hypothesis H_0,

$$\begin{aligned}
\overline{f_0(\vec{R}|x)} &\equiv \int_{\breve{x}}^{\hat{x}} p_x(x) f_0(\vec{R}|x) dx = \frac{p_x(x)}{(2\pi)^{Q/2}}\mu^{-(\frac{Q}{2}+1)} \\
&\quad \left[\gamma\left(\frac{Q}{2}+1, \mu\hat{x}\right) - \gamma\left(\frac{Q}{2}+1, \mu\breve{x}\right)\right] \ ,
\end{aligned} \tag{25}$$

where,

$$\mu \equiv \frac{1}{2}\sum_{l=1}^{Q} R^2(l) \ , \tag{26}$$

and the incomplete gamma function [5] is,

$$\gamma(a, b) \equiv \int_0^b e^{-t} t^{a-1} dt \ . \tag{27}$$

We also need the average joint pdf of the data under H_1,

$$\begin{aligned}
\overline{f_1(\vec{R}|y)} &\equiv \int_{\breve{y}}^{\hat{y}} p_y(y) f_1(\vec{R}|y) dy \\
&= \frac{p_y(y)}{(2\pi)^{Q/2}}\frac{1}{\breve{B}_1}\sum_{b=1}^{\breve{B}_1}\mu_b^{-(\frac{Q}{2}+1)} \\
&\quad \left[\gamma\left(\frac{Q}{2}+1, \mu_b\hat{y}\right) - \gamma\left(\frac{Q}{2}+1, \mu_b\breve{y}\right)\right] \ , \tag{28}
\end{aligned}$$

where,

$$\mu_b \equiv \frac{1}{2}\sum_{l=1}^{Q} R_{11}^2(l) + \frac{A^2}{2}\sum_{p=1}^{\breve{P}_b'}\breve{\eta}_b^2(p) - A\sum_{p=1}^{\breve{P}_b'}\breve{\eta}_b(p) R_{11}(\breve{\psi}_b(p)) \ . \tag{29}$$

2.5. Example

Consider a case where the standard deviation of the noise at the receiver is known within 50% and consider,

$$\begin{aligned}
K &= 40 \\
Q &= 7 \\
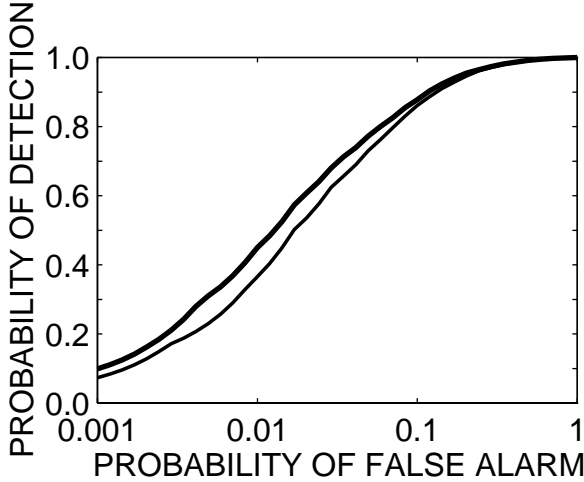10\log_{10}\frac{A^2}{\sigma_1^2} &= 1 \text{ dB} \ , \tag{30}
\end{aligned}$$

Figure 1: Receiver operating curves for the values from Eq. (30). The thick curve is for the matched-lag filter based on the averaged likelihood ratio in Eq. (23) which uses only the physically possible lag-redundancy arrangements. The thin curve is for the "conventional" receiver which uses unphysical lag-redundancy arrangements.

(thick curve, Fig. 1). All examples in this paper generate receiver operating curves from 10,000 simulations.

The author has not come across a receiver which bases its decision for the presence of a signal from the lags of the auto-correlation function, though such a receiver may have been discussed. In the absence of the lag equations in Eq. (1), we consider a conventional receiver that bases its decision for the presence of a signal on the signal-to-noise ratios of the positive lags. Assuming there are N multipath, the conventional receiver design will take all $P \equiv N(N-1)/2$ signal-related peaks to be resolved among the Q positive lags. The conventional receiver would allow for all,

$$B_{\mathbf{u}} \equiv \left(\begin{array}{c} Q \\ \frac{N(N-1)}{2} \end{array} \right) \, . \tag{31}$$

arrangements of these signals, instead of using only the physically possible ones given by Eq. (13). The subscript u stands for unphysical, even though some of the B_u arrangements are physical. The probability of detection for the matched-lag filter exceeds that for this conventional receiver for all false-alarm probabilities (Fig. 1). For example, for a false-alarm probability of 0.001, the probability of detection for the matched-lag filter is about 1.35 times that for the conventional receiver.

3. DISCUSSION

The matched-lag filter was constructed under the assumption that it is impossible to model the travel times of multipath, leading to the assumption that the probability of each physically possible lag-redundancy arrangement is equally likely. In cases where it is possible to place bounds on the relative travel times of multipath, the *a priori* probability density function of the lag-redundancy arrangement could be modified to fit expectations to improve the probabilities of detection [9].

Finally, it should be realized that the matched-lag filter is more efficient to implement than the conventional type because the number of physically possible lag-redundancy arrangements is fewer than the number of unphysical arrangements for most cases.

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