PERFORMANCE ANALYSIS FOR DIRECTION FINDING IN NON-GAUSSIAN NOISE

Brian M. Sadler, Richard J. Kozick, and Terrence Moore

Army Research Laboratory, Adelphi, MD 20783; Bucknell University, Lewisburg, PA 17837;
American University, Washington, DC 20016

ABSTRACT

We consider narrowband angle of arrival estimation in non-Gaussian (NG) noise channels, such as arises in some indoor and outdoor mobile communications channels. We develop a general expression for the Cramer-Rao bound (CRB) for direction finding using arrays for deterministic signals plus iid non-Gaussian noise, generalizing the Gaussian CRB. The CRBs for the noise and direction parameters decouple. The CRB for direction finding is expressed as a product of two terms that depend on the noise distribution, and the signal, respectively. We illustrate the results for a Gaussian mixture pdf, and present simulation results comparing five direction finding algorithms. An approach based on the expectation-maximization (EM) algorithm, that simultaneously estimates the noise parameters, the signal directions, and the signal waveforms, is shown to achieve the CRB over a wide SNR range.

1. INTRODUCTION

We consider narrowband angle of arrival (AOA) estimation in non-Gaussian (NG) noise channels. Impulsive NG noise arises due to man-made interference in indoor and outdoor mobile communications channels. Measurements of outdoor urban channels reveal automobile ignition noise exceeding typical thermal noise levels, especially below 500 MHz, e.g., see Parsons [1], with implications for military and other mobile radio channels. Similar indoor measurements show significant interference in a much wider frequency range due to various mechanical switching and other devices, e.g., see Blackard et al [2].

We develop Cramer-Rao bounds (CRBs) for direction finding using arrays for the general case of deterministic signals plus iid NG noise. We then consider the specific case of an L term Gaussian mixture (GM) noise model. The use of the GM pdf noise model is motivated for several reasons. This model includes thermal Gaussian noise, which is always present in electronic systems. The GM can well approximate a very large variety of finite variance symmetric pdfs, and the EM algorithm can be used in practice to estimate the pdf parameters. GM also includes an approximation to Middleton's canonical class A model, which has been studied extensively over the past two decades. Array processing methods based on GM modeling are therefore a natural extension of Gaussian methods [3].

2. SIGNAL AND NOISE MODEL

The complex envelope of narrowband array samples for an m element array is modeled by

$$\begin{aligned} \mathbf{y}(t) &= \sum_{i=1}^{n} \mathbf{a}(\theta_i)x_i(t) + \mathbf{e}(t) \\ &= \mathbf{A}(\mathbf{\Theta})\mathbf{x}(t) + \mathbf{e}(t), \quad t = 1, \ldots, N, \end{aligned} \tag{1}$$

where $\mathbf{a}(\theta) = [a_1(\theta), \ldots, a_m(\theta)]^T$ is the array response, there are n sources whose directions are $\mathbf{\Theta} = [\theta_1, \ldots, \theta_n]^T$, $\mathbf{A}(\mathbf{\Theta}) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_n)]$ is $m \times n$, $\mathbf{x}(t)$ is the vector of complex signal amplitudes, and $\mathbf{e}(t)$ is the additive noise with variance σ_e. (The superscripts $T, *$, and H denote the transpose, complex conjugate, and conjugate-transpose operations, respectively.) We assume the signals are deterministic (conditional model). We model the complex noise samples as zero-mean iid in space and time, with complex pdf given by $p(e)$, and assume the real and imaginary parts are identically distributed and uncorrelated. We also assume $p(e)$ is continuous with continuous first and second derivatives, and is symmetric around the (zero) mean.

We assume $m > n$ (more sensors than sources), and the number of sources is known. We note that the standard methods of estimating the number of sources in Gaussian noise, such as AIC and MDL, may perform poorly in NG noise. We have developed an approach for estimating the number of sources that is based on GM noise modeling, the EM algorithm, and subspace processing. This work will be described elsewhere. Estimates of the signals may be obtained after estimating $\mathbf{\Theta}$. However, simple least-squares (LS) is generally not appropriate in NG noise, and alternative weighted LS schemes can provide significant improvement [3]. Thus, the unknown parameters in (1) are the source directions, the signal waveforms, and the noise pdf parameters.

3. CRAMER-RAO BOUND

Consider the complex scalar case

$$y(t) = s(t) + e(t), \quad t = 1, \ldots, N, \tag{2}$$

where deterministic signal $s(t)$ is parameterized by vector θ, and additive iid non-Gaussian noise $e(t)$ is parameterized by vector λ. Under our assumptions, Ghogho and Swami [5] have shown that the Fisher information matrix (FIM) is block diagonal in θ and λ. Consequently, the achievable accuracy in estimating θ is the same whether λ is known or not. We focus on the CRB for θ, slightly generalizing the real-valued case treated by Swami [6]. Let $\mathbf{y} = [y(1), \cdots, y(N)]$. The CRB for any unbiased estimator of θ is given by $\mathrm{CRB}(\hat{\theta}) \geq J^{-1}$, where J is the FIM with elements

$$J_{mn} = E\left[\frac{\partial \ln p(\mathbf{y}|\theta)}{\partial \theta_m}\frac{\partial \ln p(\mathbf{y}|\theta)}{\partial \theta_n}\right]. \quad (3)$$

The pdf of $y(t)$ is $p_y(y) = p_e(y-s)$, which leads to

$$J_{mn} = I(\lambda)\sum_{t=1}^{N}\frac{\partial s(t)}{\partial \theta_m}\frac{\partial s(t)}{\partial \theta_n}, \quad (4)$$

where

$$I(\lambda) = \int_{-\infty}^{\infty}\frac{(p_e'(e))^2}{p_e(e)}de. \quad (5)$$

The integral in (5) is over the complex plane. Using the circular symmetry of $p_e(e)$ this can be written

$$I(\lambda) = \pi\int_{0}^{\infty}\frac{(p_e'(x))^2}{p_e(x)}xdx. \quad (6)$$

Thus the FIM for θ consists of the product of two parts, which depend only on the noise pdf and the signal, respectively. $I(\lambda)$ is the Fisher information for location of the pdf $p_e(e)$. For $p(x)$ Gaussian with variance σ_e then $I(\lambda) = \sigma_e/2$, and we recover the Gaussian FIM in (4), and hence the Gaussian CRB. Also, $I(\lambda)$ achieves a minimum over all symmetric pdfs when $p_e(e)$ is normal.

These results extend to the array case under the assumption of iid noise in both space and time, yielding the following.

Proposition: Based on the assumptions above, the CRB for $\mathbf{\Theta}$ is given by

$$\mathrm{CRB}(\mathbf{\Theta}) = \frac{B_c}{I(\lambda)}, \quad (7)$$

where

$$B_c = \left\{\sum_{t=1}^{N}\mathrm{Re}\left[X^H(t)D^H[I - A(A^HA)^{-1}A^H]DX(t)\right]\right\}^{-1}, \quad (8)$$

and $I(\lambda)$ is given by (6).

Here, $X(t) = \mathrm{diag}(x_1(t), \cdots, x_n(t))$, and D is a matrix with ith column given by $d\mathbf{a}(\theta)/d\theta$ evaluated at $\theta = \theta_i$. From the preceeding we note that the FIM is block diagonal in θ and λ, so the achievable accuracy in direction finding is the same regardless of whether the noise parameters are known or not. Again, for $p_e(e)$ Gaussian, then $I(\lambda) = \sigma_e/2$ and we recover the iid Gaussian CRB (see Theorem 4.1 of Stoica and Nehorai [4]). Thus, (7) generalizes the Gaussian case to iid non-Gaussian noise, and may be readily evaluated for various NG pdf families such as generalized Gaussian or generalized Cauchy, e.g., see Kassam [8].

4. GAUSSIAN MIXTURE NOISE

Consider the L-term Gaussian mixture (GM)

$$f_e(e) = \sum_{l=1}^{L}\frac{\lambda_l}{\pi\sigma_l}\exp\left(-\frac{|e|^2}{\sigma_l}\right), \quad (9)$$

a spherically-symmetric, bivariate pdf for the complex random variable $e = e_r + je_i$. In (9), λ_l represents the probability that e is chosen from the l^{th} term in the mixture pdf, with $\sum_{l=1}^{L}\lambda_l = 1$, and the total variance is given by $\sigma_e = \sum_{l=1}^{L}\lambda_l\sigma_l$. For the case of $L = 2$ terms, a typical model for impulsive noise has $\sigma_2 \gg \sigma_1$ with $\lambda_2 < \lambda_1$, so that large noise samples with variance σ_2 occur with frequency λ_2 in a background of Gaussian noise with variance σ_1.

Note that addition of further Gaussian terms results in a new Gaussian mixture of the form (9), and the model always includes additive Gaussian noise. The GM can well approximate a very large variety of finite variance symmetric pdfs, and the EM algorithm can be used in practice to estimate the pdf parameters. GM also includes a good approximation to Middleton's canonical class A model, which is physically motivated.

In principle, the number of mixture terms L can be determined to fit the observed data, but we have found that choosing $L \leq 4$ terms in (9) provides very good detection and estimation performance in a variety of cases, including infinite variance distributions such as Cauchy noise [3, 7]. Therefore we will assume that L is fixed and known.

Using (9) we find that

$$I(\lambda) = \int_0^{\infty}\frac{\sum_{l=1}^{L}\sum_{q=1}^{L}\frac{4\lambda_l\lambda_q}{\pi\sigma_l^2\sigma_q^2}\exp\{-(\frac{1}{\sigma_l}+\frac{1}{\sigma_q})|x|^2\}}{\sum_{r=1}^{L}\frac{\lambda_r}{\pi\sigma_r}\exp\{-\frac{1}{\sigma_r}|x|^2\}}x^3dx. \quad (10)$$

The integral does not easily admit a closed form, but is readily evaluated numerically.

An example is shown in Figure 1. We show CRBs for angle estimation of a single source, with N = 10,000, a uniform linear array with half-wavelength spacing, and $m = 8$. The source signal was a realization of a narrowband white Gaussian process. The noise was generated via an $L = 2$ GM, with $\sigma_2/\sigma_1 = 100$, for various values of λ_1, with the total variance set to unity. Note $\lambda_2 = 1 - \lambda_1$, and $\lambda_1 = 1$ corresponds to the additive Gaussian noise case. As λ_1 increases, a larger portion of the noise power is concentrated into impulses, with a corresponding decrease in the CRB (i.e., better estimates are possible).

5. SIMULATION EXAMPLE

A simulation example is presented in this section that compares the direction finding accuracy of several algorithms with the CRB in GM noise. The results indicate that an array processing technique based on the EM algorithm [3] achieves accuracy that is close to the CRB over a wide range of source power levels. The EM-based algorithm in [3] is an iterative procedure for maximum-likelihood array processing when the noise is modeled with a GM distribution.

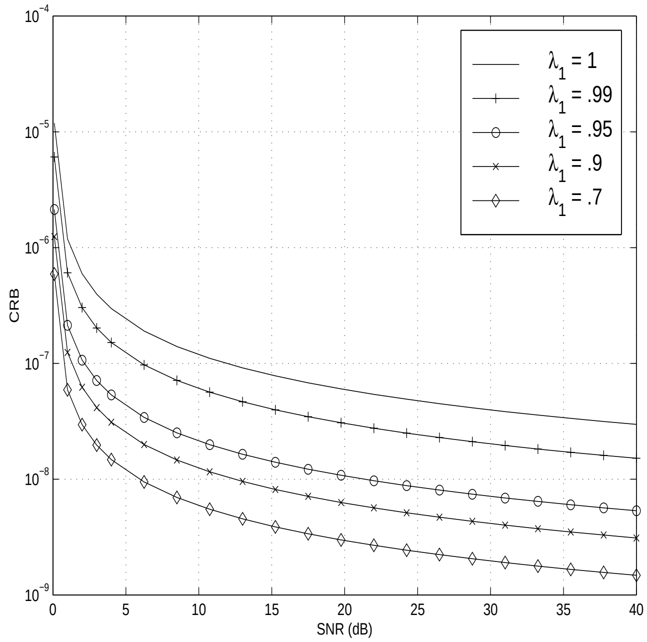

Figure 1: CRB for AOA estimation in $L = 2$ term Gaussian mixture noise, parameterized by $\lambda_1 = 1 - \lambda_2$, with $\sigma_2/\sigma_1 = 100$. $\lambda_1 = 1$ corresponds to the Gaussian noise case.

Let us consider a narrowband array processing scenario in which $m = 10$ elements are arranged in a line with uniform spacing of half-wavelength between elements. Assume that each element is omnidirectional, so the array response vector is $\mathbf{a}(\theta) = [1, e^{j\pi \sin\theta}, \ldots, e^{j(m-1)\pi\sin\theta}]^T$ where θ is the elevation angle of the source with respect to the array broadside. We will parameterize the source location in terms of $u = \sin\theta$. This simulation considers $n = 1$ source located at $u_1 = \sin\theta_1 = 0.1$, with average signal power $10\log_{10}\left[\sum_{t=1}^{N} |x_1(t)|^2/N\right]$ that varies from 0 dB to 20 dB. The number of array snapshots is $N = 100$, and the additive noise in the observations is iid in space and time with a GM distribution (9) with $L = 2$ terms and parameters $\lambda_1 = 0.95, \lambda_2 = 0.05, \sigma_1 = 1, \sigma_2 = 1000$.

The performance of five methods for direction finding are evaluated in this environment:

1. The standard MUSIC algorithm, which is based on the eigendecomposition of the sample covariance matrix $\hat{\mathbf{R}}_{yy} = \sum_{t=1}^{N} \mathbf{y}(t)\mathbf{y}(t)^H$.
2. Linear beamforming, which estimates the source location as the angle that maximizes the spatial power spectrum $P(\theta) = \mathbf{a}(\theta)^H \hat{\mathbf{R}}_{yy}\mathbf{a}(\theta)/[\mathbf{a}(\theta)^H\mathbf{a}(\theta)]$. This is the maximum likelihood estimator for the case of a single source in Gaussian noise.
3. Robust covariations-based MUSIC (ROC-MUSIC), which uses fractional lower-order moments instead of a covariance matrix. ROC-MUSIC is developed in [9] in the context of array processing in non-Gaussian noise that is modeled with an alpha-stable process.
4. Pre-processing with a data-adaptive, zero memory nonlinearity (DA-ZMNL) [10], which limits the influence of impulsive noise samples while adapting the "cutoff" to avoid clipping the signal amplitude. The data that results from DA-ZMNL pre-processing is used to form a covariance matrix, which is then used in the MUSIC algorithm or linear beamforming. For the case in this simulation, MUSIC and beamforming yield identical performance when applied to the DA-ZMNL covariance matrix.
5. An iterative processing scheme based on the EM algorithm, which is an iterative scheme for maximum likelihood estimation in GM noise. This method requires more computation than the previous four methods, but it provides superior performance. An outline of the EM algorithm for array processing is provided in [3]. The initial estimates of source directions for the EM algorithm are obtained from the DA-ZMNL processing described in item 4 above. Iterations of the EM algorithm then improve the parameter estimates. The EM algorithm produces estimates of the source locations, the signal waveforms, and the GM parameters. The EM algorithm is formulated using a GM model for the noise. We use 4-terms in the GM noise model, because we have found that this produces good detection and estimation performance in a variety of non-Gaussian noise environments [3],[7], including Cauchy noise. In this example, the 4-term GM model for the noise contains more terms than the actual noise, which is generated with a 2-term GM distribution.

The direction finding performance is also compared with the CRB, which is computed using (7), (8), and (10).

Figure 2 contains the simulation results, which are based on 400 Monte Carlo runs for each value of average signal power. The vertical axis in Figure 2 is the root-mean-square (RMS) error in the estimate of the source location parameterized by $u = \sin\theta$. Notice that ordinary MUSIC and BEAMFORMING perform poorly, since they do not suppress the effects of the impulsive noise. ROC-MUSIC provides some improvement compared with MUSIC and BEAMFORMING, but the accuracy of ROC-MUSIC is relatively far from the CRB.

The DA-ZMNL closely approximates the locally optimum (weak signal) nonlinearity for this highly impulsive noise environment. Thus in Figure 2, it is not surprising that the DA-ZMNL accuracy is close to the CRB for low values of the signal power. However, the accuracy of the DA-ZMNL degrades as the signal power increases. This is because the DA-ZMNL increases the "cutoff" of the nonlinearity in order to avoid clipping the signal, which also allows more of the impulsive noise to pass through the DA-ZMNL. Notice that when the average signal power equals 20 dB, the DA-ZMNL performs at approximately the same level as ordinary MUSIC and BEAMFORMING.

The EM algorithm achieves accuracy that is very close to the CRB over the entire range of signal power levels in Figure 2. Furthermore, the EM algorithm performance equals the CRB when the average signal power equals 20 dB. A novel feature of the EM algorithm [3] is that it provides estimates of the signal waveforms using a nonlinear beamforming operation that suppresses impulsive noise.

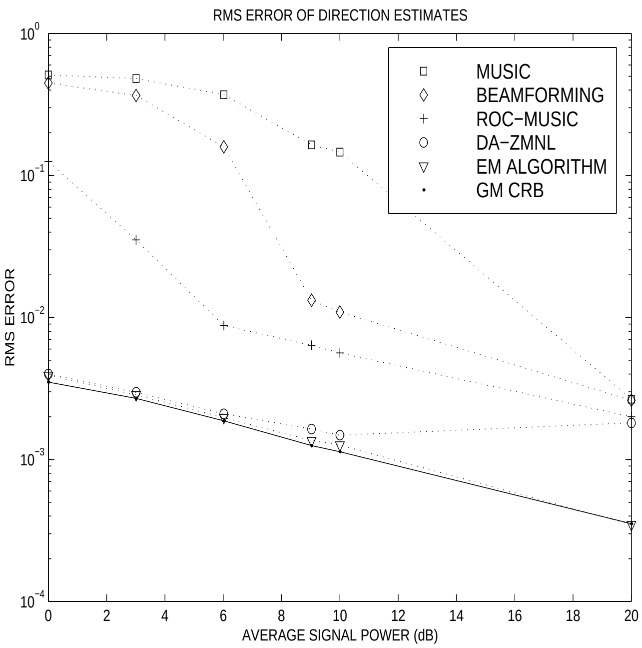

Figure 2: Root-mean-square (RMS) error of source location estimates for various algorithms in Gaussian mixture noise. The CRB is indicated by the solid line. The noise distribution is an $L = 2$ term Gaussian mixture with $\lambda_1 = 0.95, \lambda_2 = 0.05, \sigma_1 = 1, \sigma_2 = 1000$.

We have observed results similar to Figure 2 in simulation experiments involving multiple sources that are spaced by less than one beamwidth.

5.1. Cauchy noise

We have investigated the performance of the EM algorithm for array processing when the additive, complex noise samples follow a bivariate isotropic Cauchy pdf [11] given by

$$f_e(e) = \frac{1}{2\pi b^2}\left(\frac{1}{1+\frac{|e|^2}{b^2}}\right)^{\frac{3}{2}}, \qquad (11)$$

where b is a scale parameter. An example is presented in [3] in which the EM algorithm achieves direction finding performance that is close to the CRB for Cauchy noise. The CRB for source location parameters in the case of Cauchy noise (11) is derived in [11], and it has the form of (7) with $I(\lambda)^{-1} = 5b^2/3$.

In the example presented in [3], the EM algorithm used a 4-term GM distribution to approximate the Cauchy noise with pdf (11). For each set of array snapshots $\mathbf{y}(1), \ldots, \mathbf{y}(N)$, the EM algorithm yielded GM parameters $\hat{\lambda}_l, \hat{\sigma}_l, l = 1, \ldots, 4$ to model the Cauchy noise samples. It is interesting to compare the Cauchy CRB with the GM CRB corresponding to the GM parameters produced by the EM algorithm. This is equivalent to comparing the scaling factor $I(\lambda)^{-1} = 5b^2/3$ for Cauchy noise with the scaling factor for GM $I(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3, \hat{\sigma}_4)^{-1}$ computed with (10).

The result of this comparison is as follows for Cauchy noise with $b = 1$ in (11). The Cauchy CRB scaling factor is $I(\lambda)^{-1} = 5/3 = 1.6667$, and the statistics of GM CRB scaling factors in 500 runs are as follows:

Mean = 1.2695, Median = 1.1908
Standard Deviation = 0.8098

The GM CRBs tend to be slightly smaller than the Cauchy CRB, which is reasonable since the GM distribution is a finite variance approximation to the infinite variance Cauchy distribution. The GM approximation must underestimate the “variance” of the Cauchy distribution, which reduces the GM CRB scaling factor.

6. REFERENCES

[1] D. Parson, The Mobile Radio Propagation Channel (Wiley, 1996).

[2] K. L. Blackard, T. S. Rappaport, C. W. Bostian, “Measurements and models of radio frequency impulsive noise for indoor wireless communications,” IEEE J. Sel. Areas in Comm., Vol. 11, No. 7, p. 991–1001, 1993.

[3] R. J. Kozick, B. M. Sadler, R. S. Blum, “Array processing in impulsive noise: a nonlinear beamformer based on the EM algorithm,” Proc. of the 1998 Conf. on Inf. Sci. and Sys., March 1998.

[4] P. Stoica, A. Nehorai, “MUSIC, maximum likelihood, and Cramer-Rao bound,” IEEE Trans. ASSP, Vol. 37, No. 5, pp. 720–741, 1989.

[5] M. Ghogho, A. Swami, “Fast computation of the exact FIM for deterministic signals in colored noise,” IEEE Trans. SP, to appear 1998.

[6] A. Swami, “Cramer-Rao bounds for deterministic signals in additive and multiplicative noise,” Sig. Proc., Vol. 53, pp. 231–244, 1996.

[7] R.J. Kozick, R.S. Blum, B.M. Sadler, “EM-Based Approaches to Adaptive Signal Detection on Fading Channels with Impulsive Noise,” Proc. of the 1997 Conf. on Inf. Sci. and Sys., pp. 112-117, March 19-21, 1997.

[8] S. A. Kassam, Signal detection in non-Gaussian noise (Springer-Verlag, 1988).

[9] P. Tsakalides, C.L. Nikias, “The robust covariation-based MUSIC (ROC-MUSIC) algorithm for bearing estimation in impulsive noise environments”, IEEE Trans. SP, vol. 44, no. 7, pp. 1623-1633, July 1996.

[10] B.M. Sadler, “Detection in Correlated Impulsive Noise Using Fourth-Order Cumulants”, IEEE Trans. SP, vol. 44, no. 11 , pp. 2793, Nov. 1996.

[11] P. Tsakalides and C.L. Nikias, “Maximum likelihood localization of sources in noise modeled as a stable process”, IEEE Trans. SP, vol. 43, no. 11, pp. 2700-2713, Nov. 1995.