NEW TIME-FREQUENCY SYMBOL CLASSIFICATION [*]

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ABSTRACT

We propose *new* time-frequency (TF) symbols as the narrowband Weyl symbol (WS) smoothed by an appropriate kernel. These new symbols preserve time and frequency shifts on a random process. Choosing specific smoothing kernels, we can obtain various *new* symbols (e.g. Levin symbol and Page symbol). We link a quadratic form of the signal to the new symbols and Cohen's class of quadratic time-frequency representations, and we derive a simple kernel constraint for unitary symbols. We also propose an affine class of symbols in terms of the wideband Weyl symbol (P_0WS). These symbols preserve scale changes and time shifts. Furthermore, we generalize the smoothed versions of the WS and P_0WS to analyze random processes undergoing generalized frequency shifts or generalized time shifts.

1. INTRODUCTION

The narrowband Weyl symbol (WS) has been successfully used in the time-frequency (TF) analysis of linear time-varying systems and nonstationary processes [4, 10, 15]. The WS of a linear operator $\mathcal{L}$ is defined as[1]

$$\text{WS}_{\mathcal{L}}(t, f) = \int K_{\mathcal{L}}(t + \frac{\tau}{2}, t - \frac{\tau}{2}) e^{-j2\pi f \tau} d\tau \quad (1)$$

$$= \sum_n \mu_n \text{WD}_{u_n}(t, f), \quad (2)$$

where $K_{\mathcal{L}}(t, \tau) = \sum_n \mu_n u_n(t) u_n^*(\tau)$ is the kernel of the operator $\mathcal{L}$ defined on $L_2(\mathbb{R})$ [5], μ_n and $u_n(t)$ are the eigenvalues and eigenfunctions, respectively, of $K_{\mathcal{L}}(t, \tau)$, and WD_{u_n} is the Wigner distribution (WD) of $u_n(t)$ [6]. Based on (2), we say that the unitary operator symbol WS is *associated with* the unitary WD quadratic time-frequency representation (QTFR). The WS can be interpreted as a time-varying spectrum of a *random process*. It is a useful TF analysis tool as it preserves constant time shifts and frequency shifts on a random process $x(t)$ [4],

$$y(t) = x(t - \tau) \Rightarrow \text{WS}_{\mathbf{R}_y}(t, f) = \text{WS}_{\mathbf{R}_x}(t - \tau, f), \quad (3)$$

$$y(t) = x(t)e^{j2\pi t\nu} \Rightarrow \text{WS}_{\mathbf{R}_y}(t, f) = \text{WS}_{\mathbf{R}_x}(t, f - \nu), \quad (4)$$

where $\mathbf{R}_z$ is the autocorrelation operator of $z(t)$. The WS also preserves scale changes on the random process [12],

$$y(t) = \sqrt{|a|}x(at) \Rightarrow \text{WS}_{\mathbf{R}_y}(t, f) = \text{WS}_{\mathbf{R}_x}(at, f/a). \quad (5)$$

The Weyl correspondence is a unitary mapping between the operator $\mathcal{L}$ and its WS [14, 4, 10, 15]. In particular, the WS (and its 2-D Fourier transform (FT), the spreading function, $\text{SF}_{\mathcal{L}}(\tau, \nu)$) provide an important formulation of a quadratic form of the process $x(t)$,

$$\int (\mathcal{L}x)(t) x^*(t) dt = \iint \text{WS}_{\mathcal{L}}(t, f) \text{WD}_x(t, f) \, dt \, df \quad (6)$$

$$= \iint \text{SF}_{\mathcal{L}}(\tau, \nu) \text{AF}_x^*(\tau, \nu) \, d\tau \, d\nu,$$

where $\text{AF}_x(\tau, \nu)$ is the ambiguity function (AF) of $x(t)$ [6]. This important relationship in (6) is useful for a TF concentration measure definition [15] and for TF detection applications [11].

The wideband Weyl symbol (P_0WS) in [16, 7, 8] is defined as

$$P_0\text{WS}_{\mathcal{B}}(t, f) = f \iint \Gamma_{\mathcal{B}}(f\lambda(\alpha)e^{\frac{\alpha}{2}}, f\lambda(\alpha)e^{-\frac{\alpha}{2}})\lambda(\alpha)e^{j2\pi f t\alpha} d\alpha, \quad (7)$$

$$= \sum_n \beta_n P_{0_{Q_n}}(t, f), \quad f > 0 \quad (8)$$

where $\Gamma_{\mathcal{B}}(f, \nu) = \sum_n \beta_n Q_n(f) Q_n^*(\nu)$ is the kernel of the operator $\mathcal{B}$ on $L_2(\mathbb{R}^+)$, β_n and $Q_n(f)$ are the eigenvalues and eigenfunctions, respectively, of $\Gamma_{\mathcal{B}}(f, \nu)$, $P_{0_{Q_n}}$ is the unitary Bertrand P_0-distribution of $Q_n(f)$ [1], and $\lambda(\alpha) = \frac{\alpha/2}{\sinh \alpha/2}$. In (8), the unitary P_0WS is *associated with* the unitary Bertrand P_0-distribution. The P_0WS is an important operator symbol that satisfies the time-shift covariance in (3) and scale covariance in (5). It also satisfies hyperbolic time-shift covariance [8]. A quadratic form of the process $x(t)$ (with FT $X(f)$) can be expressed in the frequency domain in terms of the P_0WS and the unitary Bertrand P_0-distribution

$$\int_0^\infty (\mathcal{B}X)(f)X^*(f)df = \iint_0^\infty P_0\text{WS}_{\mathcal{B}}(t, f) P_{0_X}(t, f) dt df. \quad (9)$$

In [14], it was noted that if one chooses a different QTFR than the WD in (6), then one can have an alternative mapping between a linear operator and the symbol corresponding to the QTFR. In this paper, we propose a class of operator symbols to (i) satisfy the desirable TF shift covariance properties (3) and (4), and (ii) provide alternative formulations of the quadratic form in (6) using corresponding QTFRs in Cohen's class [2, 6]. We express this *new* class[2] of TF symbols in terms of a kernel function[2] that is related to the kernel of a Cohen's class QTFR. If the associated Cohen's class QTFR is unitary, then the symbol kernel satisfies a simple constraint for unitarity. We show that the conventional WS in (1) is a member of this class, and that all other members can be written as smoothed versions of the WS. Other members include the Kohn-Nirenberg symbol [4], the α-generalized WS [10], as well as the *new* Levin symbol, the *new* Page symbol, and the *new* pseudo WS proposed in this paper.

We also propose new operator symbols based on an affine class kernel formulation; they satisfy the time-shift covariance property in (3) and the scale covariance property in (5). This affine class of symbols can be written both in terms of the WS in (1) and the P_0WS in (7). We derive the symbol kernel constraint necessary for these *new* affine symbols to provide alternative formulations of the quadratic form in (9). Specific members of this class of symbols include the conventional WS in (1), the P_0WS in (7), and new

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[1]Unless otherwise specified, integrals and sums range from $-\infty$ to ∞.

[2]In this paper, by the term "QTFR class" ("TF symbol class"), we mean the group of all QTFRs (TF symbols) that satisfy a given set of properties. Also, by the term "kernel", we mean the function that uniquely characterizes a specific member of a group. Here, we talk about three different types of kernels: (a) kernel of an operator (cf. $K_{\mathcal{L}}$ in (1)), (b) kernel of a symbol (cf. $\theta_{\text{TS}}^{(C)}$ in (10)), and (c) kernel of a QTFR (cf. $\Psi_{\text{T}}^{(C)}$ in (13)).

Symbol Name	Unitary Time-Frequency Symbols	Smoothing Kernel, $\Theta_{\mathrm{TS}}^{(C)}(\tau,\nu)$	Reference
Narrowband WS	$\mathrm{WS}_{\mathcal{L}}(t,f) = \int K_{\mathcal{L}}(t+\frac{\tau}{2},t-\frac{\tau}{2})\,e^{-j2\pi f\tau}d\tau$	1	[4, 10, 15]
Kohn-Nirenberg symbol	$\mathrm{KS}_{\mathcal{L}}(t,f)=\int K_{\mathcal{L}}(t,t-\tau)e^{-j2\pi\tau f}d\tau$	$\exp(-j\pi\tau\nu)$	[4]
Levin symbol	$\mathrm{LS}_{\mathcal{L}}(t,f)=\int_{-\infty}^{0}K_{\mathcal{L}}(t,t-\tau)e^{-j2\pi f\tau}d\tau+\int_{0}^{\infty}K_{\mathcal{L}}(t+\tau,t)e^{-j2\pi f\tau}d\tau$	$\exp(j\pi|\tau|\nu)$	*new*
Page symbol	$\mathrm{PS}_{\mathcal{L}}(t,f)=\int_{-\infty}^{0}K_{\mathcal{L}}(t+\tau,t)e^{-j2\pi f\tau}d\tau+\int_{0}^{\infty}K_{\mathcal{L}}(t,t-\tau)e^{-j2\pi f\tau}d\tau$	$\exp(-j\pi|\tau|\nu)$	*new*
α-Generalized WS	$\mathrm{WS}_{\mathcal{L}}^{(\alpha)}(t,f) = \int K_{\mathcal{L}}\left(t+(\frac{1}{2}-\alpha)\tau,t-(\frac{1}{2}+\alpha)\tau\right)e^{-j2\pi f\tau}d\tau$	$\exp(-j2\pi\alpha\tau\nu)$	[10, 12]

Table 1: *Various unitary smoothed versions of the narrowband WS in (11). $K_{\mathcal{L}}(t,\tau)$ is the kernel of the operator $\mathcal{L}$ defined on $L_2(\mathbb{R})$.*

symbols corresponding to the unitary Bertrand P_κ-distributions [1]. Finally, we extend the symbol class formulation concept to *new* hyperbolic symbols preserving scale changes and hyperbolic frequency shifts, to *new* power symbols preserving scale changes and power time shifts, to new exponential symbols preserving frequency shifts and exponential time shifts, and to *new* generalized warped symbols preserving generalized time shifts or generalized frequency shifts.

2. CLASS OF TF SHIFT COVARIANT SYMBOLS

We propose a *new* class of operator symbols which preserve time and frequency shifts (like the WS in (3) and (4)), an important property for the TF analysis of random processes. We define these symbols as smoothed versions of the conventional WS in (1) [9]

$$\mathrm{TS}_{\mathcal{L}}^{(C)}(t,f) = \iint \theta_{\mathrm{TS}}^{(C)}(t-\hat{t},f-\hat{f})\mathrm{WS}_{\mathcal{L}}(\hat{t},\hat{f})\,d\hat{t}\,d\hat{f} \qquad (10)$$

$$= \iint \Theta_{\mathrm{TS}}^{(C)}(\tau,\nu)\mathrm{SF}_{\mathcal{L}}(\tau,\nu)e^{-j2\pi(\tau f-t\nu)}\,d\tau\,d\nu, \ (11)$$

where[3] $\theta_{\mathrm{TS}}^{(C)}(t,f)=$2-D FT$\{\Theta_{\mathrm{TS}}^{(C)}(\tau,\nu)\}$ is a 2-D kernel function that uniquely characterizes the operator symbol, $\mathrm{TS}_{\mathcal{L}}^{(C)}(t,f)$. The *new* symbol class provides a new quadratic form to (6):

$$\int(\mathcal{L}x)(t)x^*(t)\,dt = \iint\mathrm{TS}_{\mathcal{L}}^{(C)}(t,f)\mathrm{T}_x^{(C)}(t,f)\,dt\,df. \qquad (12)$$

The QTFR $\mathrm{T}_x^{(C)}$ that corresponds to the "T-symbol" $\mathrm{TS}_{\mathcal{L}}^{(C)}$ in (12) is a member of Cohen's class of TF shift covariant QTFRs [2, 6]

$$\mathrm{T}_x^{(C)}(t,f)=\iint\Psi_{\mathrm{T}}^{(C)}(\tau,\nu)\mathrm{AF}_x(\tau,\nu)e^{-j2\pi(\tau f-t\nu)}\,d\tau\,d\nu, \quad (13)$$

where $\Psi_{\mathrm{T}}^{(C)}(\tau,\nu)$ is a 2-D kernel that uniquely characterizes the QTFR, $\mathrm{T}_x^{(C)}(t,f)$. For (12) to hold, the QTFR $\mathrm{T}^{(C)}$ in (13) has a kernel $\Psi_{\mathrm{T}}^{(C)}$ that is related to the symbol kernel $\Theta_{\mathrm{TS}}^{(C)}$ in (11) as

$$\Theta_{\mathrm{TS}}^{(C)}(\tau,\nu)=1/\Psi_{\mathrm{T}}^{(C)*}(\tau,\nu)=\Psi_{\mathrm{T}}^{(C)}(\tau,\nu)/|\Psi_{\mathrm{T}}^{(C)}(\tau,\nu)|^2. \quad (14)$$

For example, the WS in (1) is a member of the symbol class in (10)-(11) when the symbol kernel $\Theta_{\mathrm{WS}}^{(C)}(\tau,\nu) = 1$. The Cohen's class QTFR $\mathrm{T}^{(C)}$ in the quadratic form (12) must be the QTFR with kernel $\Psi_{\mathrm{T}}^{(C)}(\tau,\nu)=1/\Theta_{\mathrm{WS}}^{(C)*}(\tau,\nu)=1$, which corresponds to the WD (cf. quadratic form in (6)). Note that the quadratic form in (12) can also be written as

$$\int(\mathcal{L}x)(t)\,x^*(t)dt=\iint\mathrm{TSF}_{\mathcal{L}}^{(C)}(\tau,\nu)\mathrm{TAF}_x^*(\tau,\nu)\,d\tau\,d\nu$$

where $\mathrm{TSF}_{\mathcal{L}}^{(C)}(\tau,\nu)=\Theta_{\mathrm{TS}}^{(C)}(\tau,\nu)\mathrm{SF}_{\mathcal{L}}(\tau,\nu)=$2-D FT$\{\mathrm{TS}_{\mathcal{L}}^{(C)}(t,f)\}$ and $\mathrm{TAF}_x(\tau,\nu)=\Psi_{\mathrm{T}}^{(C)}(\tau,\nu)\mathrm{AF}_x(\tau,\nu)=$2-D FT$\{\mathrm{T}_x^{(C)}(t,f)\}$.

2.1. Unitary TF Shift Covariant Symbols

The formulation in (10) provides a *new* mapping between a linear operator $\mathcal{L}$ and its *new* symbol, $\mathrm{TS}_{\mathcal{L}}^{(C)}(t,f)$. We can show that this mapping is unitary if the symbol is unitary, i.e. if the symbol satisfies the relation [9]

$$\iint\mathrm{TS}_{\mathcal{L}}^{(C)}(t,f)\mathrm{TS}_{\mathcal{M}}^{(C)*}(t,f)dtdf=\sum_n\sum_m\mu_n\gamma_m^*|\int u_n(t)e_m^*(t)dt|^2(15)$$

where μ_n and $u_n(t)$ (γ_m and $e_m(t)$) are the eigenvalues and eigenfunctions, respectively, of the kernel of the operator $\mathcal{L}$ ($\mathcal{M}$). We can further show [9] that (15) is satisfied if and only if the symbol kernel in (11) satisfies the constraint $|\Theta_{\mathrm{TS}}^{(C)}(\tau,\nu)|=1$. The associated Cohen's class QTFR $\mathrm{T}^{(C)}$ in (12) is also unitary [6] and its kernel satisfies $|\Psi_{\mathrm{T}}^{(C)}(\tau,\nu)|=1$. Thus, we say that this unitary symbol is *associated with* the unitary QTFR. Let us denote the unitary symbol as $\mathrm{UTS}_{\mathcal{L}}^{(C)}(t,f)$ and its associated unitary QTFR in (12) as $\mathrm{UT}_x^{(C)}(t,f)$. Using the relation in (14) and the unitarity QTFR constraint $|\Psi_{\mathrm{UT}}^{(C)}(\tau,\nu)|=1$, we can show that the unitary symbol kernel equals the unitary Cohen's class QTFR kernel, i.e. $\Theta_{\mathrm{UTS}}^{(C)}(\tau,\nu)=\Psi_{\mathrm{UT}}^{(C)}(\tau,\nu)$. This leads to the useful relation

$$\mathrm{UTS}_{\mathcal{L}}^{(C)}(t,f)=\sum_n\mu_n\mathrm{UT}_{u_n}^{(C)}(t,f) \qquad (16)$$

between the unitary symbol satisfying (15) and its *associated* QTFR (cf. (2)) where μ_n and $u_n(t)$ are the eigenvalues and eigenfunctions of the kernel of $\mathcal{L}$ in (1)-(2).

2.2. Examples of Unitary TF Shift Covariant Symbols

Some examples of unitary symbols in (10)-(11) satisfying the constraint $|\Theta_{\mathrm{TS}}^{(C)}(\tau,\nu)|=1$ are summarized in Table 1 and below:

● **Narrowband Weyl symbol:** $\mathrm{WS}_{\mathcal{L}}(t,f)$ is defined in (1) with kernel $\Theta_{\mathrm{WS}}^{(C)}(\tau,\nu) = 1$ or $\theta_{\mathrm{WS}}^{(C)}(t,f)=\delta(t)\delta(f)$.

● **Kohn-Nirenberg symbol:** The Kohn-Nirenberg symbol of an operator associated with the Rihaczek distribution is defined as [4]

$$\mathrm{KS}_{\mathcal{L}}(t,f)=\int K_{\mathcal{L}}(t,t-\tau)e^{-j2\pi\tau f}d\tau \qquad (17)$$

when $\Theta_{\mathrm{KS}}^{(C)}(\tau,\nu)=e^{-j\pi\tau\nu}$. The Kohn-Nirenberg symbol is also known as the Zadeh's transfer function [10].

● **Levin symbol:** We define the *new* symbol associated with the Levin distribution as

$$\mathrm{LS}_{\mathcal{L}}(t,f) = \int_{-\infty}^{0}K_{\mathcal{L}}(t,t-\tau)e^{-j2\pi f\tau}d\tau$$
$$+\int_{0}^{\infty}K_{\mathcal{L}}(t+\tau,t)e^{-j2\pi f\tau}d\tau.$$

The smoothing kernel for the Levin symbol is $\Theta_{\mathrm{LS}}^{(C)}(\tau,\nu)=e^{j\pi|\tau|\nu}$.

● **Page symbol:** We define the *new* symbol, $\mathrm{PS}_{\mathcal{L}}(t,f)$, associated with the Page distribution as

$$\mathrm{PS}_{\mathcal{L}}(t,f) = \int_{-\infty}^{0}K_{\mathcal{L}}(t+\tau,t)e^{-j2\pi f\tau}d\tau$$
$$+\int_{0}^{\infty}K_{\mathcal{L}}(t,t-\tau)e^{-j2\pi f\tau}d\tau$$

with smoothing kernel $\Theta_{\mathrm{PS}}^{(C)}(\tau,\nu)=e^{-j\pi|\tau|\nu}$.

● α **-Generalized WS:** The symbol associated with the α-generalized WD [6] is defined in [10, 12] as

$$\mathrm{WS}_{\mathcal{L}}^{(\alpha)}(t,f)=\int K_{\mathcal{L}}\left(t+(\frac{1}{2}-\alpha)\tau,t-(\frac{1}{2}+\alpha)\tau\right)e^{-j2\pi f\tau}d\tau. (18)$$

Its smoothing kernel is $\Theta_{\mathrm{WS}^{(\alpha)}}^{(C)}(\tau,\nu)=e^{-j2\pi\alpha\tau\nu}$. Note that when $\alpha = 0$, $\mathrm{WS}_{\mathcal{L}}^{(\alpha)}(t,f)$ reduces to the conventional WS in (1), and when $\alpha=1/2$, it simplifies to the Kohn-Nirenberg symbol in (17).

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[3]We use the superscript (C) to link the symbol in (10) to the TF shift covariance of Cohen's QTFR class formulation.

2.3. Example of Non-Unitary TF Shift Covariant Symbol

When $|\Theta_{\text{TS}}^{(C)}(\tau,\nu)| \neq 1$, the symbol $\text{TS}_{\mathcal{L}}^{(C)}$ in (10)-(11) is not unitary. As a consequence, the symbol does not satisfy the unitary relation in (15) and cannot be written as in (16). However, the symbol still satisfies the covariance properties in (3)-(4) and provides an alternative quadratic form of a random process in (12). An example of a non-unitary symbol follows:

• **Pseudo WS:** We define the *new* pseudo WS that corresponds in (12) to the pseudo WD with non-zero window $\eta(t)$ as

$$\text{PWS}_{\mathcal{L}}(t,f) = \int \frac{1}{\eta^*(\frac{\tau}{2})\eta(-\frac{\tau}{2})} K_{\mathcal{L}}(t+\frac{\tau}{2}, t-\frac{\tau}{2}) e^{-j2\pi\tau f} d\tau.$$

Since the kernel of the pseudo WD is $\Psi_{\text{PWD}}^{(C)}(\tau,\nu) = \eta(\frac{\tau}{2})\eta^*(-\frac{\tau}{2})$ in (13), then, using (14), the corresponding symbol kernel of the PWS in (11) is $\Theta_{\text{PWS}}^{(C)}(\tau,\nu) = 1/(\eta^*(\frac{\tau}{2})\eta(-\frac{\tau}{2}))$.

3. CLASS OF AFFINE TF SYMBOLS

The class of symbols in (10) does not necessarily preserve scale changes on a random process as in (5). Thus, we propose a *new* class of TF symbols covariant to time shifts in (3) and scale changes in (5) of a random process. We denote these new symbols with the superscript (A) for affine. We define them as affine smoothed versions of the WS in (1) or of the P_0WS in (7),

$$\text{TS}_{\mathcal{B}}^{(A)}(t,f) = \iint_0^\infty \theta_{\text{TS}}^{(A)}\left(f(t-\hat{t}), -\frac{\hat{f}}{f}\right) \text{WS}_{\hat{\mathcal{B}}}(\hat{t},\hat{f}) d\hat{t} d\hat{f}, \ f > 0 \quad (19)$$

$$= \iint_0^\infty \widehat{\theta}_{\text{TS}}^{(A)}\left(f(t-\hat{t}), -\frac{\hat{f}}{f}\right) P_0\text{WS}_{\mathcal{B}}(\hat{t},\hat{f}) d\hat{t} d\hat{f}, \quad (20)$$

where $\widehat{\theta}_{\text{TS}}^{(A)}(c,b) = -b \iint \theta_{\text{TS}}^{(A)}(\hat{c}, \frac{a}{2}\frac{e^a+1}{e^a-1}b)\lambda(a)e^{j2\pi ab(\hat{c}-c)} da d\hat{c}$ is a 2-D kernel characterizing the symbol $\text{TS}^{(A)}$, $\widehat{\mathcal{B}} = \mathcal{F}^{-1}\mathcal{B}\mathcal{F}$, the FT operator $\mathcal{F}$ is $(\mathcal{F}x)(f) = X(f)$ and $(\mathcal{F}^{-1}\mathcal{F}x)(t) = x(t)$.

We can show that if the affine symbol in (19) is unitary, i.e. satisfies (15), then its corresponding kernel satisfies the constraint $\iiint \theta_{\text{TS}}^{(A)*}(c, b\beta)\theta_{\text{TS}}^{(A)}(\hat{c},\beta) e^{j2\pi\zeta\beta(c-\hat{c})} dc d\hat{c} d\beta = \delta(b-1), \forall \zeta$. The unitary affine symbols denoted $\text{UTS}_{\mathcal{B}}^{(A)}(t,f)$ provide a new formulation of the quadratic form in (9),

$$\int_0^\infty (\mathcal{B}X)(f) X^*(f) df = \iint_0^\infty \text{UTS}_{\mathcal{B}}^{(A)}(t,f) \text{UT}_X^{(A)*}(t,f) \, dt df.$$

Here, $\text{UT}_X^{(A)}(t,f)$ is the symbol's associated unitary affine QTFR [3], $\text{UT}_X^{(A)}(t,f) = \iint_0^\infty \psi_{\text{UT}}^{(A)}(f(t-\hat{t}), -\frac{\hat{f}}{f}) \text{WD}_X(\hat{t},\hat{f}) d\hat{t} d\hat{f}$, with QTFR kernel $\psi_{\text{UT}}^{(A)}(c,b) = \theta_{\text{UTS}}^{(A)}(c,b)$ in (19).

3.1. Examples of Unitary Affine Symbols

• **Narrowband Weyl symbol:** When the symbol kernel $\theta_{\text{WS}}^{(A)}(c,b) = \delta(c)\delta(b+1)$ in (19), the conventional WS in (1) is obtained.

• **P_0-Weyl symbol:** $\text{TS}_{\mathcal{B}}^{(A)}$ in (20) simplifies to the P_0WS in (7) when the kernel $\widehat{\theta}_{P_0\text{WS}}^{(A)}(c,b) = \delta(c)\delta(b+1)$.

• **α-Generalized Weyl symbol:** The α-generalized WS in (18) is obtained from (19) when $\theta_{\text{WS}^{(\alpha)}}(c,b) = e^{-j2\pi c(b+1)/\alpha}/\alpha$.

• **Bertrand P_κ-symbols:** We propose the *new* symbols associated with the unitary Bertrand P_κ-distributions [1] and define them as
$$P_\kappa S_{\mathcal{B}}(t,f) = \int \Gamma_{\mathcal{B}}(f\lambda_\kappa(\alpha), f\lambda_\kappa(-\alpha))$$
$$\cdot e^{j2\pi tf[\lambda_\kappa(\alpha) - \lambda_\kappa(-\alpha)]} \mu(\alpha) \, d\alpha, \ \kappa \neq 0, 1$$
where $\lambda_\kappa(\alpha) = [\kappa(e^{-\alpha}-1)/(e^{-\kappa\alpha}-1)]^{1/(\kappa-1)}$ and $\mu(\alpha) = (\lambda_\kappa(\alpha)\lambda_\kappa(-\alpha))^{1/2}\left(\frac{d}{d\alpha}(\lambda_\kappa(\alpha) - \lambda_\kappa(-\alpha))\right)^{1/2}$.

4. OTHER CLASSES OF TF SYMBOLS

4.1. Hyperbolic Smoothed Symbols

We define the class of hyperbolic (note the superscript (H)) smoothed symbols for $t > 0$ as

$$\text{HTS}_{\mathcal{Y}}^{(H)}(t,f) = \int_0^\infty \int \theta_{\text{HTS}}^{(H)}(\ln(\frac{t}{\hat{t}}), ft - \hat{f}\hat{t}) \text{HWS}_{\mathcal{Y}}(\hat{t},\hat{f}) d\hat{t} d\hat{f} \quad (21)$$

where the operator $\mathcal{Y}$ is defined on $L_2(\mathbb{R}^+)$ and $\theta_{\text{HTS}}^{(H)}(c,b) = \theta_{\text{TS}}^{(C)}(t_r c, b/t_r)$ in (19) with a fixed time reference $t_r > 0$. Here, the hyperbolic Weyl symbol $\text{HWS}_{\mathcal{Y}}(t,f)$ is defined as [7]

$$\text{HWS}_{\mathcal{Y}}(t,f) = t\int K_{\mathcal{Y}}(te^{\zeta/2}, te^{-\zeta/2}) e^{-j2\pi tf\zeta} d\zeta, \ t > 0. \quad (22)$$

This class of symbols preserves hyperbolic frequency shifts and scale changes on a random process $x(t)$, i.e.

$$y(t) = x(t)e^{j2\pi c\ln\frac{t}{t_r}} \Rightarrow \text{HTS}_{\mathbf{R}_y}^{(H)}(t,f) = \text{HTS}_{\mathbf{R}_x}^{(H)}(t, f-\frac{c}{t}),$$

$$y(t) = \sqrt{|a|}x(at) \Rightarrow \text{HTS}_{\mathbf{R}_y}^{(H)}(t,f) = \text{HTS}_{\mathbf{R}_x}^{(H)}(at, f/a).$$

We can show that the symbols in (21) are unitary if and only if $|\Theta_{\text{HTS}}^{(H)}(\zeta,\beta)| = 1$ where $\Theta_{\text{HTS}}^{(H)}(\zeta,\beta) = $ 2-D FT$\{\theta_{\text{HTS}}^{(H)}(c,b)\}$. A quadratic form can be expressed in terms of these unitary hyperbolic symbols and the associated unitary QTFRs of the hyperbolic class [13]. We can obtain special examples of $\text{HTS}_{\mathcal{Y}}^{(H)}(t,f)$ by choosing the corresponding symbol kernel as follows.

• **Hyperbolic WS:** The $\text{HTS}_{\mathcal{Y}}^{(H)}(t,f)$ in (21) simplifies to the hyperbolic WS in (22) [7] when $\Theta_{\text{HWS}}^{(H)}(\zeta,\beta) = 1$.

• **Hyperbolic Kohn-Nirenberg symbol:** When $\Theta_{\text{HKS}}^{(H)}(\zeta,\beta) = e^{-j\pi\zeta\beta}$, then the $\text{HTS}_{\mathcal{Y}}^{(H)}(t,f)$ reduces to the new hyperbolic Kohn-Nirenberg symbol,

$$\text{HKS}_{\mathcal{Y}}(t,f) = \int te^{-\zeta/2}K_{\mathcal{Y}}(t, te^{-\zeta}) e^{-j2\pi\zeta tf} d\zeta. \quad (23)$$

• **Hyperbolic Levin symbol:** When $\Theta_{\text{HLS}}^{(H)}(\zeta,\beta) = e^{j\pi|\zeta|\beta}$, we obtain the new hyperbolic Levin symbol as

$$\text{HLS}_{\mathcal{Y}}(t,f) = \int_{-\infty}^0 te^{-\zeta/2}K_{\mathcal{Y}}(t, te^{-\zeta}) e^{-j2\pi\zeta tf} d\zeta$$
$$+ \int_0^\infty te^{\zeta/2}K_{\mathcal{Y}}(te^\zeta, t) e^{-j2\pi\zeta tf} d\zeta.$$

• **Hyperbolic Page symbol:** When $\Theta_{\text{HPS}}^{(H)}(\zeta,\beta) = e^{-j\pi|\zeta|\beta}$, we define the new hyperbolic Page symbol as

$$\text{HPS}_{\mathcal{Y}}(t,f) = \int_{-\infty}^0 te^{\zeta/2}K_{\mathcal{Y}}(te^\zeta, t) e^{-j2\pi\zeta tf} d\zeta$$
$$+ \int_0^\infty te^{-\zeta/2}K_{\mathcal{Y}}(t, te^{-\zeta}) e^{-j2\pi\zeta tf} d\zeta.$$

• **Hyperbolic α-generalized WS:** We obtain the new hyperbolic α-generalized WS, $\text{HWS}_{\mathcal{Y}}^{(\alpha)}(t,f)$, as

$$\text{HWS}_{\mathcal{Y}}^{(\alpha)}(t,f) = \int te^{\alpha\zeta}K_{\mathcal{Y}}(te^{(\alpha-\frac{1}{2})\zeta}, te^{(\alpha+\frac{1}{2})\zeta}) e^{-j2\pi\zeta tf} d\zeta$$

when $\Theta_{\text{HWS}^{(\alpha)}}^{(H)}(\zeta,\beta) = e^{-j2\pi\alpha\zeta\beta}$. Note that this symbol reduces to the HWS in (22) when $\alpha=0$, and to the HKS in (23) when $\alpha=1/2$.

4.2. Power smoothed symbols

We define the new power class of TF symbols for $f > 0$ as

$$\text{PTS}_{\mathcal{Q}}^{(\kappa)}(t,f) = \iint_0^\infty \widehat{\theta}_{\text{PTS}}^{(A)}\left(\frac{f^\kappa}{\kappa}(\frac{t}{f^{\kappa-1}} - \frac{\hat{t}}{\hat{f}^{\kappa-1}}), -(\frac{\hat{f}}{f})^\kappa\right)$$
$$\cdot \text{PP}_0\text{WS}_{\mathcal{Q}}^{(\kappa)}(\hat{t},\hat{f}) d\hat{t}d\hat{f}, \quad (24)$$

where $\text{PP}_0\text{WS}_{\mathcal{Q}}^{(\kappa)}(t,f)$ is the κth power P_0WS [7] with symbol kernel $\widehat{\theta}_{\text{PP}_0\text{WS}}^{(A)}(c,b) = \delta(c)\delta(b+1)$, $\mathcal{Q}$ is defined on $L_2(\mathbb{R}^+)$, and the power symbol kernel $\widehat{\theta}_{\text{PTS}}^{(A)}(c,b) = \widehat{\theta}_{\text{TS}}^{(A)}(c,b)$ in (20). Note that when $\kappa=1$, the $\text{PTS}_{\mathcal{Q}}^{(\kappa)}$ reduce to the affine smoothed symbols in (20).

The class in (24) preserves power time shifts and scale changes on a random process $X(f)$, i.e.

$$y(t) = \left(\mathcal{F}^{-1}[X(f)e^{-j2\pi\zeta(\frac{f}{f_r})^\kappa}]\right)(t) \Rightarrow$$
$$\text{PTS}_{\mathbf{R}_y}^{(\kappa)}(t,f) = \text{PTS}_{\mathbf{R}_x}^{(\kappa)}\left(t - \frac{\zeta\kappa}{f_r}(\frac{f}{f_r})^{\kappa-1}, f\right),$$

$$y(t){=}\sqrt{|a|}x(at) \Rightarrow \mathrm{PTS}_{\mathbf{R}_y}^{(\kappa)}(t,f) = \mathrm{PTS}_{\mathbf{R}_x}^{(\kappa)}(at,\ f/a)$$

where $f_r > 0$ is a fixed reference frequency.

4.3. Exponential smoothed symbols

We define the new class of exponential smoothed symbols as

$$\mathrm{ETS}_{\mathcal{Q}}^{(E)}(t,f){=}\iint\widehat{\theta}_{\mathrm{ETS}}^{(A)}\left(f_r e^{f/f_r}\big(\frac{t}{e^{f/f_r}}-\frac{\hat{t}}{e^{\hat{f}/f_r}}\big),\ -e^{(\hat{f}-f)/f_r}\right)$$
$$\cdot\mathrm{EP_0WS}_{\mathcal{Q}}(\hat{t},\hat{f})d\hat{t}d\hat{f} \qquad (25)$$

where $\mathcal{Q}$ is defined on $L_2(\mathbb{R})$, the exponential symbol kernel is $\widehat{\theta}_{\mathrm{ETS}}^{(A)}(c,b){=}\widehat{\theta}_{\mathrm{TS}}^{(A)}(c,b)$ in (20) and $\mathrm{EP_0WS}_{\mathcal{Q}}(t,f)$ is the exponential $\mathrm{P_0WS}$ [7] with kernel $\widehat{\theta}_{\mathrm{EP_0WS}}^{(A)}(c,b){=}\delta(c)\delta(b+1)$. The class of exponential symbols, $\mathrm{ETS}_{\mathcal{Q}}^{(E)}(t,f)$, preserves exponential time shifts and constant frequency shifts on a random process,

$$y(t){=}\big(\mathcal{F}^{-1}[X(f)e^{-j2\pi\zeta e^{f/f_r}}]\big)(t) \Rightarrow$$
$$\mathrm{ETS}_{\mathbf{R}_y}^{(E)}(t,f){=}\mathrm{ETS}_{\mathbf{R}_x}^{(E)}(t-\frac{\zeta}{f_r}e^{f/f_r},f),$$

$$y(t) = x(t)e^{j2\pi\nu t} \Rightarrow \mathrm{ETS}_{\mathbf{R}_y}^{(E)}(t,f) = \mathrm{ETS}_{\mathbf{R}_x}^{(E)}(t,f-\nu).$$

4.4. Generalized frequency-shift covariant symbols

We propose a new class of TF symbols, $\mathrm{GTS}_{\mathcal{Y}}^{(GC)}(t,f)$, that preserve generalized frequency shifts on a random process. The shift depends on a one-to-one warping function $\xi(b)$

$$y(t){=}x(t)e^{j2\pi c\xi(\frac{t}{t_r})}\Rightarrow\mathrm{GTS}_{\mathbf{R}_y}^{(GC)}(t,f){=}\mathrm{GTS}_{\mathbf{R}_x}^{(GC)}(t,f-\frac{c}{t_r}\xi'(\frac{t}{t_r}))$$

where $\xi'(b) = \frac{d}{db}\xi(b)$. The new symbols are defined as

$$\mathrm{GTS}_{\mathcal{Y}}^{(GC)}(t,f){=}\iint_p^q\theta_{\mathrm{GTS}}^{(GC)}\left(\xi(\frac{t}{t_r})-\xi(\frac{\hat{t}}{t_r}),\ \frac{t_r f}{\xi'(\frac{t}{t_r})}-\frac{t_r\hat{f}}{\xi'(\frac{\hat{t}}{t_r})}\right)$$
$$\cdot\mathrm{GWS}_{\mathcal{Y}}(\hat{t},\hat{f})d\hat{t}d\hat{f} \qquad (26)$$
$$= \mathrm{TS}_{\mathcal{W}_\xi\mathcal{Y}\mathcal{W}_\xi^{-1}}^{(C)}(t_r\xi(\frac{t}{t_r}),\ f/\xi'(\frac{t}{t_r})),$$

where $\mathcal{Y}$ is defined on $L_2([p,q])$, $[p,q]$ is determined by the domain of the warping function $\xi(b)$, $\theta_{\mathrm{GTS}}^{(GC)}(c,b)=\theta_{\mathrm{TS}}^{(C)}(t_rc,\frac{b}{t_r})$ in (10), the warping is $(\mathcal{W}_\xi x)(t){=}x(t_r\xi^{-1}(\frac{t}{t_r}))/|\xi'(\xi^{-1}(\frac{t}{t_r}))|^{1/2}$, and $(\mathcal{W}_\xi\mathcal{W}_\xi^{-1}x)(t){=}x(t)$. The generalized Weyl symbol $\mathrm{GWS}_{\mathcal{Y}}$ is defined as [7] $\mathrm{GWS}_{\mathcal{Y}}(t,f){=}\mathrm{WS}_{\mathcal{W}_\xi\mathcal{Y}\mathcal{W}_\xi^{-1}}(t_r\xi(\frac{t}{t_r}),\ f/\xi'(\frac{t}{t_r}))$.

Depending on the smoothing kernel $\theta_{\mathrm{GTS}}^{(GC)}(c,b)$ in (26), $\mathrm{GTS}_{\mathcal{Y}}^{(GC)}$ results in the new generalized Levin symbol, the generalized Page symbol or the generalized Kohn-Nirenberg symbol [9]. Also, we can derive special cases of $\mathrm{GTS}_{\mathcal{Y}}^{(GC)}$ by choosing a specific warping function $\xi(b)$ in (26). For example, we obtain the hyperbolic smoothed symbols in Section 4.1 when $\xi(b) = \ln b$.

4.5. Generalized time-shift covariant symbols

We define a class of generalized time-shift covariant symbols as

$$\mathrm{GTS}_{\mathcal{Q}}^{(GA)}(t,f){=}\iint_c^d\widehat{\theta}_{\mathrm{GTS}}^{(A)}\left(\xi(\frac{f}{f_r})\big(\frac{f_r t}{\xi'(\frac{f}{f_r})}-\frac{f_r\hat{t}}{\xi'(\frac{\hat{f}}{f_r})}\big),\ -\frac{\xi(\frac{\hat{f}}{f_r})}{\xi(\frac{f}{f_r})}\right)$$
$$\cdot\mathrm{GP_0WS}_{\mathcal{Q}}(\hat{t},\hat{f})d\hat{t}\,\hat{f} \qquad (27)$$

where $\mathcal{Q}$ is defined on $L_2([c,d])$, $[c,d]$ is determined by the domain of the warping function $\xi(b)$, the generalized symbol kernel $\widehat{\theta}_{\mathrm{GTS}}^{(A)}(c,b) = \widehat{\theta}_{\mathrm{TS}}^{(A)}(c,b)$ in (20) and $\mathrm{GP_0WS}_{\mathcal{Q}}(t,f)$ is the generalized $\mathrm{P_0WS}$ [8]. The $\mathrm{GTS}_{\mathcal{Q}}^{(GA)}$ preserves generalized time shifts on a random process $X(f)$,

$$y(t){=}\big(\mathcal{F}^{-1}[e^{-j2\pi\zeta\xi(\frac{f}{f_r})}X(f)]\big)(t) \Rightarrow$$
$$\mathrm{GTS}_{\mathbf{R}_y}^{(GA)}(t,f){=}\mathrm{GTS}_{\mathbf{R}_x}^{(GA)}(t-\frac{\zeta}{f_r}\xi'(\frac{f}{f_r}),f).$$

Depending on the warping function $\xi(b)$, the $\mathrm{GTS}^{(GA)}$ in (27) can be simplified to a specific class of affine symbols. For example, we obtain the $\mathrm{TS}^{(A)}$ in (20) when $\xi(b) = b$, the $\mathrm{PTS}^{(\kappa)}$ in (24) when $\xi(b) = b^\kappa$ and the $\mathrm{ETS}^{(E)}$ in (25) when $\xi(b) = e^b$.

5. CONCLUSION

In this paper, we proposed new classes of smoothed versions of the narrowband Weyl symbol and the wideband Weyl symbol. These new symbols preserve important changes on a random process. For example, a smoothed version of the narrowband WS preserves constant TF shifts on a random process and belongs to the new class of TF shift covariant symbols. The new symbols formulate the quadratic form of a random process with corresponding QT-FRs. We showed that a symbol kernel is identical to its associated QTFR kernel if the QTFR is unitary. We also proposed generalized formulations of smoothed symbols. We provided special examples of generalized smoothed symbols (e.g. hyperbolic smoothed symbols, power smoothed symbols, and exponential smoothed symbols). For example, we derived the hyperbolic Weyl symbol, hyperbolic Kohn-Nirenberg symbol, hyperbolic Levin symbol and hyperbolic Page symbol as hyperbolic smoothed symbols.

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