A NEW NORMALIZED RELATIVELY STABLE LATTICE STRUCTURE

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ABSTRACT

This paper proposes a new lattice filter structure that has the following properties. When the filter is Linear Time Invariant (LTI), it is equivalent to the celebrated Gray Markel Lattice. When the lattice parameters vary with time it sustains arbitrary rate of time variations without sacrificing a prescribed degree of stability, provided that the lattice coefficients are magnitude bounded in a region where all LTI lattices have the same degree of stability. We also show that the resulting LTV lattice obeys an energy contraction condition. This structure thus generalizes the normalized Gray-Markel lattice which has similar properties but only with respect to stability as opposed to relative stability.

1. INTRODUCTION

Consider the Linear Time Invariant (LTI) normalized Gray-Markel lattice of fig. 1 with

$$\{p_i, q_i, r_i, s_i\} = \{\alpha_i, \hat{\alpha_i}, \hat{\alpha_i}, -\alpha_i\}, 1 \leq i \leq n \quad (1)$$

and

$$\hat{\alpha}_i = \sqrt{1 - |\alpha_i|^2}. \quad (2)$$

It is known that this lattice is stable iff for all i,

$$|\alpha_i| < 1. \quad (3)$$

In addition under these conditions it is also All Pass, [1]. This lattice realization has the following added attractive property. If one permits the lattice coefficients α_i to vary with time, as would be the case for example in adaptive implementations, then the resulting linear time varying (LTV) normalized lattice remains exponentially asymptotically stable (eas) as long as for arbitrary $1 \geq \epsilon > 0$, [3],

$$|\alpha_i(k)| < 1 - \epsilon, \quad 1 \leq i \leq n. \quad (4)$$

The remarkable fact about this result is that as long as the reflection coefficients strictly obey the conditions for LTI stability arbitrary rates of time variations can be sustained without the loss of exponential stablility. Further under these conditions the time varying lattice is also all pass, i.e. for all square summable inputs the input and output energies are equal.

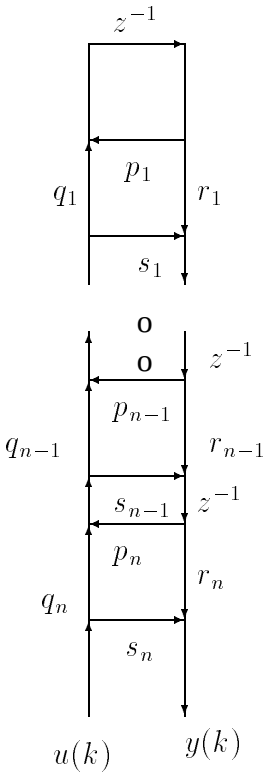

Figure 1: A Lattice structure

In practice robustness dictates that mere exponential asymptotic stability should be replaced by stability with a margin. Thus for example one would like all zero input state trajectories to decay at an exponential rate no slower than $1/\rho^k$ for some $\rho > 1$. In such a case we call the filter ρ-stable.

More precisely, we will call a linear time varying (LTV) system with scalar input $u(k)$, scalar output $y(k)$, $n \times 1$ state

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$x(k)$ and a state variable realization (SVR)

$$x(k+1) = A(k)x(k) + b(k)u(k) \quad (5)$$
$$y(k) = c(k)x(k) + d(k)u(k) \quad (6)$$

ρ-stable if there exist constants $\beta_1 > 0$, $0 \leq \beta_2 < 1$ such that with zero input, the state obeys for all k and initial time k_0

$$\rho^{k-k_0}\|x(k)\| \leq \beta_1\|x(k_0)\|\beta_2^{k-k_0} \quad (7)$$

where $\|\cdot\|$ denotes the standard 2-norm. If ρ is 1, we simply call the system eas, henceforth *stable*. Observe LTI ρ-stable systems have all poles inside a circle of radius $1/\rho$. Henceforth we will say that this system has an SVR $\{A(k), b(k), c(k), d(k)\}$.

Until recently the clean characterization of the ρ-stability of even the LTI Gray-Markel lattice was unavailable. In a recent paper, [6] we provide the following result. Given a $\rho > 1$, we give necessary and sufficient conditions on a given set of $0 < \delta_i < 1$ to be such that the all LTI Gray-Markel lattices with coefficients obeying

$$|\alpha_i| < \delta_i \quad 1 \leq i \leq n \quad (8)$$

are ρ-stable. Henceforth we will call sets of δ_i that obey this condition as being ρ-*compatible*. Now suppose we have some δ_i that are ρ-compatible. It is readily shown that if one permits the lattice coefficients to vary according to

$$|\alpha_i(k)| < \delta_i - \epsilon, \quad 1 \leq i \leq n \quad (9)$$

then no matter how small the ϵ, one can find rates of time variation under which the resulting LTV normalized lattice loses ρ-stability. Thus whereas it can sustain arbitrary adaptation rates without losing stability, the normalized Gray-Markel lattice can lose ρ-stability even if (9), holds.

Accordingly this paper proposes an equivalent realization of the LTI Gray-Markel lattice that has the following property. If the δ_i are ρ-compatible, this realization remains ρ-stable, as long as (8) holds for some positive ϵ, regardless of the rate with which the $\alpha_i(k)$ may vary.

Section 2 gives two key results from [6]. Section 3 defines the ρ-normalized lattice and gives a key algebraic property it obeys. Section 4 argues the ρ-stability of the time varying ρ-normalized lattice and gives a relation characterizing the dependence between its input and out put energy. Section 5 is the conclusion.

2. AN UNNORMALIZED LTI ROBUST RELATIVE STABILITY RESULT

In this Section we give two key results from [6]. The first concerns characterization of ρ-compatibility.

Theorem 1 *Consider the normalized Gray-Markel LTI Lattice of fig. 1 with (1) holding, a $\rho > 1$ and $0 < \delta_i < 1$. Define, should it exist, the sequence*

$$f_i = \frac{\rho f_{i-1} - \delta_i}{1 - \rho\delta_i f_{i-1}}, \quad 1 \leq i \leq n, \quad (10)$$

with $f_0 = 1$. Then the δ_i are ρ-compatible iff the f_0 to f_n exist and obey

$$0 < \rho f_{i-1}\delta_i < 1. \quad (11)$$

Now define the following sequence that depends on the $|\alpha_i$.

$$g_i(|\alpha_1|, \ldots, |\alpha_i|) = \frac{\rho g_{i-1}(|\alpha_1|, \ldots, |\alpha_{i-1}|) - |\alpha_i|}{1 - \rho g_{i-1}(|\alpha_1|, \ldots, |\alpha_{i-1}|)|\alpha_i|} \quad (12)$$

with $g_0 = 1$. Notice $f_i = g_i(|\delta_1|, \ldots, |\delta_i|)$. This sequence will play an important role in the sequel. It has an appealing interpretation. Consider the unnormalized Gray-Markel lattice obtained by choosing

$$\{p_i, q_i, r_i, s_i\} = \{\alpha_i, 1, 1 - \alpha_i^2, -\alpha_i\}, 1 \leq i \leq n. \quad (13)$$

Of course in the LTI case the unnormalized Gray-Markel lattice, [2] is an equivalent realization of the normalized Gray-Markel lattice. Define the i-th block of fig. 1 to be the block defined by p_i, q_i, r_i, s_i and $G_i(z^{-1}, \alpha_1, \ldots, \alpha_i)$ to be the transfer function relating the upward input to this block and the downward output from this block for the unnormalized Gray-Markel lattice. Then it is shown in [5] that with $G_0(z^{-1}) = 1$,

$$G_{i+1}(z^{-1}, \alpha_1, \ldots, \alpha_{i+1}) = \frac{z^{-1}G_i(z^{-1}, \alpha_1, \ldots, \alpha_i) - \alpha_{i+1}}{1 - z^{-1}G_i(z^{-1}, \alpha_1, \ldots, \alpha_i)\alpha_{i+1}}.$$

Thus,

$$g_i(|\alpha_1|, \ldots, |\alpha_i|) = G_i(z^{-1}, \alpha_1, \ldots, \alpha_i).$$

Henceforth, for notational convenience drop the arguments $|\alpha_i|$ from the g_l. Notice, that when $\rho = 1$, $g_i = 1$ as well. Define the matrices

$$P = \text{diag}\,\{(1-\alpha_1^2)\cdots(1-\alpha_{n-1}^2), \quad (14)$$
$$(1-\alpha_2^2)\cdots(1-\alpha_{n-1}^2), \ldots, 1\}, \quad (15)$$
$$\Lambda = \text{diag}\,\{\rho^{n-1}, \rho^{n-2}, \ldots, \rho, 1\} \quad (16)$$
$$\Gamma = \text{diag}\,\{g_{n-1}/g_0, g_{n-1}/g_1, \ldots, g_{n-1}/g_{n-2}, 1\} \quad (17)$$

In the sequel we denote $\alpha = [\alpha_1, \cdots, \alpha_n]'$.

Theorem 2 *Consider the LTI unnormalized lattice and its SVR $\{A(\alpha), b(\alpha), c(\alpha), d(\alpha)\}$ when the ith element of the state vector is the output of the delay element pointing toward the ith block. Suppose for some $\rho > 1$ a given set of $0 < \delta_i < 1$ are ρ-compatible. Then for all $|\alpha_i| \leq \delta_i$, $1 \leq i \leq n$,*

$$1 - \rho^2 g_{n-1}^2(\alpha)\alpha_n^2 > 0 \quad (18)$$

Further, the matrix

$$\Pi(\alpha) = \Lambda P(\alpha)\Gamma(\alpha) \tag{19}$$

is positive definite and obeys

$$\rho^2 A(\alpha)'\Pi(\alpha)A(\alpha) - \Pi(\alpha) \leq -Q'(\alpha)Q(\alpha) \tag{20}$$

with

$$Q(\alpha) = \sqrt{(1-\rho^2 g_{n-1}^2(\alpha)\alpha_n^2)}e_n' \tag{21}$$

and e_n is $n \times 1$ vector

$$e_n = [0, \ldots, 0, 1]'.$$

The significance of this result is as follows. A system with SVR $\{A, b, c, d\}$ is ρ-stable iff the system with SVR $\{\rho A, b, c, d\}$ is stable. It has been shown in [6] that the pair $[\rho A, Q]$ is completely observable. Thus, under (8) and the ρ-compatiblity of the δ_i, (20) acts as the Lyapunov equation that proves the ρ-stability of the LTI lattices. Further Π serves as a Lyapunov matrix and *is a function of the lattice parameters*. This Theorem plays a key role in the subsequent analysis.

3. THE ρ-NORMALIZED LATTICE

The proposed new lattice structure is defined below.

Definition 1 *Consider the g_i as defined in the previous Section, $\hat{\alpha}_i$ defined in (2) and*

$$\hat{\gamma}_i(\alpha) = \sqrt{\frac{g_i(\alpha)}{g_{i-1}(\alpha)}}. \tag{22}$$

Then the ρ-normalized lattice is as in fig. 1 with

$$\{pi, qi, ri, si\} = \{\alpha_i, \hat{\alpha}_i\hat{\gamma}_i(\alpha), \hat{\alpha}_i/\hat{\gamma}_i(\alpha), -\alpha_i\}. \tag{23}$$

Notice the $\hat{\gamma}_i(\alpha)$ depend on α_i. Further, it is shown in [6] that these exist whenever (8) holds and the δ_i are ρ-compatible. Moreover, should $\rho = 1$, then $\hat{\gamma}_i(\alpha) = 1$ and the *ρ-normalized normalized lattice reduces identically to the normalized lattice*.

We now give a structural relationship between SVR's of the ρ-normalized and the unnormalized lattice that will be useful in the next section.

Theorem 3 *Suppose the LTI unnormalized lattice has the SVR $\{A(\alpha), b(\alpha), c(\alpha), d(\alpha)\}$ defined in Theorem 2 and that $\{\tilde{A}(\alpha), \tilde{b}(\alpha), \tilde{c}(\alpha), \tilde{d}(\alpha)\}$ is the SVR for the ρ-normalized lattice when the ith element of the state vector is the output of the delay element pointing toward the ith block. Define*

$$t_i(\alpha) = \frac{1}{\hat{\alpha}_i\sqrt{\hat{\gamma}_i(\alpha)}}$$

and

$$\begin{aligned} T(\alpha) \quad = \quad & diag\,\{t_{n-1}(\alpha)t_{n-2}(\alpha)\cdots t_1(\alpha), \\ & t_{n-1}(\alpha)t_{n-2}(\alpha)\cdots t_2(\alpha), \\ & \ldots, t_{n-1}(\alpha), 1\}. \end{aligned}$$

If (8) holds and the δ_i are ρ-compatible then $T(\alpha)$ is non-singular. In this case

$$\begin{aligned} &\{T^{-1}(\alpha)A(\alpha)T(\alpha), T^{-1}(\alpha)b(\alpha), c(\alpha)T(\alpha), d(\alpha)\} \\ &\qquad = \{\tilde{A}(\alpha), \tilde{b}(\alpha), \tilde{c}(\alpha), \tilde{d}(\alpha)\}. \end{aligned}$$

Proof: We give an outline only. The last equality essentially says that the ρ-normalized lattices state is obtained by scaling the second to last state element by t_{n-1}, the previous element by $t_{n-1}t_{n-2}$ etc.. Since these are constants, the t_i can commute with the delay elements. Thus the result follows by equivalently scaling the q_i and s_i parameters.

This shows that for the same α, both the LTI ρ-normalized and unnormalized lattice represent the same transfer function whenever (8) holds and the δ_i are ρ-compatible. Also observe that in fact

$$T(\alpha) = (\Gamma(\alpha)P(\alpha))^{-1/2} \tag{24}$$

4. PROPERTIES OF THE ρ-NORMALIZED LATTICE

In this Section we give two properties of the LTV ρ-Normalized Lattice. The first is the ρ-stability result that motivates its formulation. Then we have the following theorem.

Theorem 4 *Suppose the lattice parameters vary with time and (9) holds with δ_i ρ-compatible. Then the ρ-Normalized Lattice exists and is ρ-stable with the state vector defined as in Theorem 3.*

Proof: Again only a proof outline will be given. This outline we believe illuminates the procedure by which the ρ-normalized lattice was arrived at. Assume that $\alpha(k)$ varies with time. Expanding $\Pi(\alpha(k))$ into its component matrices in (20) and pre and post-multiplying by $T(\alpha(k))$ and the transformation (24) gives

$$\rho^2\tilde{A}'(\alpha(k))\Lambda\tilde{A}(\alpha(k)) - \Lambda \leq -Q'(\alpha(k))Q(\alpha(k)) \tag{25}$$

where we have used the fact that the structure of $Q(\alpha(k))$ and $T(\alpha(k))$ makes

$$Q(\alpha(k))T(\alpha(k)) = Q(\alpha(k)).$$

Thus, as Λ is constant and positive definite, the result follows by showing that $[\tilde{A}(\alpha(k)), Q(\alpha(k))]$ is uniformly completely observable (uco), (see [7] for definition. Showing uco is a straightforward if tedious task.

Note its is this constant Lyapunov matrix Λ that is critical toρ-stability. An important property of the time varying normalized lattice is that it is all pass. However, the ρ-Normalized Lattice is not all pass under time variation. However the following input output relationship holds. We omit ther proof.

Theorem 5 *Consider the ρ-Normalized Lattice under the conditions of Theorem 4. If $u(k)$ is such that $\rho^k u(k)$, is square summable, then*

$$\sum_{k=-\infty}^{\infty} \rho^{2k} y^2(k) \leq \sum_{k=-\infty}^{\infty} \rho^{2k} u^2(k) g_n^2(\alpha(k)) \qquad (26)$$

whenever the system is at initial rest. Here $g_n(\alpha(k))$ is given in Section 2.

Thus, a suitably scaled LTV ρ-Normalized Lattice, when viewed as a mapping from $\rho^k u^k$ to $\rho^k y^k$ is *energy contractive*.

5. CONCLUSION

In this paper, a new lattice structure was developed which satisfied the following properties: in the LTI case, its transfer function is the same as that of the normalized lattice. In the LTV case it remains ρ-stable as long as the lattice coefficients are magnitude bounded in a region where all LTI lattices are ρ-stable. Though in the LTV case, the lattice is not all pass, it obeys a contractiveness property.

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