A CONTRIBUTION TO THE STABILITY TEST FOR ONE-DIMENSIONAL DISCRETE TIME LINEAR SYSTEMS

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ABSTRACT

The objective of this paper is to produce a general formulation of an order reduction procedure for testing the stability of discrete time linear systems. The order reduction procedure involves a series of iterations and, at each step of the iteration process, the aim is to derive a new polynomial of order lower than the given one. The new polynomial serves as the input to the following iteration. A specific form of the formulation is considered in which first order auxiliary polynomials are employed in the order reduction process. There follows from this a new testing procedure which is computationally more efficient than the existing ones. Moreover the current methods appear as special cases of the new test. An extension is further proposed which employs second order auxiliary polynomials within the order reduction formulation. This second order form is, however, for all practical cases the limit to which such a procedure can be put.

1. INTRODUCTION

The stability test of a transfer function of a linear system is a fundamental problem and as a result it has received considerable attention through the years. There are many contributions starting from the epoch making approaches of Routh and Hurwitz for continuous time systems [1] [2] [3] [4]. The corresponding case for discrete time systems is as old and as significant and the pivotal contributions here are related to the work of Schur, Cohn, and Marden, and the interpretation produced by Jury and Fujiwara.[5] [6] [7] [7] [8] [9]. The have been additional developments and interpretations which are contained in the publications of Vaidyanathan and Mitra [10].

The improvements on the work of Schur and Cohn have focused on the reduction of the computational burden associated with the early procedure by the brilliant use of order reduction algorithms as proposed by Marden [11]. These have led to the very efficient Jury-Marden test [7], and is also the basis of the iterations found in the Levinson-Durbin algorithm and in related work concerned with lattice filter structures [12] [13 [14]. Intimately related to these procedures are studies that are concerned with allpass structures [17] [18] and positive real behaviour relative to the unit circle [19].

2. THE GENERAL PRINCIPLES AND THEORY

Given a real polynomial

$$f_n(z) = p_0 z^n + p_1 z^{n-1} + \ldots + p_n \qquad (1)$$

to determine whether its roots lie within $|z|=1$. In principle this could be done by root finding by this is an ill-conditioned problem as indicated by Hadamard [15]. The Schur-Cohn Stability Test relies on setting up square matrices of size $2n \times 2n$ and algorithmically reduced progressively to size 2×2. At each stage to test stability their determinants are examined for positivity. A computationally improved version is found in the Shur-Cohn-Fujiwara Stability Test [7] but this is only marginally computationally better. Further simplifications of the basic Schur matrices have also been produced.

A fundamentally new direction is taken in the Jury-Marden Stability Test. The basis for this test and related current and efficient tests is order reduction by iteration. We are to employ similar techniques in this contribution.

We need some basic results.

Observation 1:

On the unit circle $f_n(z)$ and $f_n(z^{-1})$ are complex conjugates and hence

$$|f_n(z)| = |f_n(z^{-1})| \text{ and } |f_n(z)| = |z^m f_n(z^{-1})|$$

Observation 2: (Rouche's Theorem [16]).

If on the unit circle the polynomials $f(z)$ and $g(z)$ are such that

$$|f(z)| < |g(z)|$$

Then $f(z)+g(z)$ has the same number of zeros as $g(z)$.

General order recursion

Let there be real polynomials $g_1(z)$ and $g_2(z)$ each of order at most *p* and a function $f_{n-1}(z)$, such that

$$f_n(z) = g_1(z) f_{n-1}(z) + g_2(z) z^{n-p} f_{n-1}(z^{-1}) \quad (2)$$

then we can write the following expression.

$$z^p \left[g_1(z) g_1(z^{-1}) - g_2(z) g_2(z^{-1}) \right] f_{n-1}(z) =$$

$$z^p g_1(z^{-1}) f_n(z) - g_2(z^{-1}) z^n f_n(z^{-1}) \quad (3)$$

The function $f_{n-1}(z)$ is desired to be a polynomial of degree lower than the degree of the original polynomial.

The following conditions must prevail.

(i) The polynomial $g_1(z) g_1(z^{-1}) - g_2(z) g_2(z^{-1})$ must be either a factor of the right hand side above, or it must be a constant.

(ii) On the right hand side of equation (3) the coefficients of $|\alpha z + \beta| > |1 + \alpha\beta z|$ $z^i, i = 0,1,\ldots, p-1$ must all be zero to enable cancellation process to take place. This cancellation is related to the factor z^p.

(iii) The degree of $f_{n-1}(z)$ must less than the degree of $f_n(z)$

Moreover, for Rouche's Theorem to be used in the development of the test we must have on $|z| = 1$ either $|g_1(z)| < |g_2(z)|$ or $|g_1(z)| > |g_2(z)|$.

The crucial part in the above is concerned with the choice for the polynomials $g_1(z)$ and $g_2(z)$. These polynomials are taken in our approach to be real and of the same order. Their role is to enable the separation property crucial to the application of Rouche's theorem to be used as the deciding factor. There are different options for these polynomials even when the order is unity and these different possibilities give rise to different tests.

We give below the case for $p = 1$ i.e. a first order case. This specific choice encompasses as a special case the well-known Jury-Marden test. In addition by a judicious use of the degrees of freedom available in a first order polynomial an improvement is achieved in terms of the total computational burden associated with this specific test.

We need at this juncture however, to establish the following result which plays crucial significance in the development of the algorithms.

Lemma I: On the unit circle $|z| = 1$ for any real α, and β, the inequality $|\alpha z + \beta| < |1 + \alpha\beta z|$ holds, provided that either $|\alpha| > 1$ and $|\beta| > 1$ or $|\alpha| < 1$ and $|\beta| < 1$.

Proof:

Let the modulus squared of $(\alpha z + \beta)$ be

$$A = (\alpha z + \beta)(\alpha z^{-1} + \beta)$$

(4) and correspondingly the squared modulus s of $(1 + \alpha\beta z)$ be

$$B = (1 + \alpha\beta z)(1 + \alpha\beta z^{-1})$$

(5) Both of these quantities are real and positive since they are squared moduli. Their difference is given by

$$B - A = 1 + \alpha^2\beta^2 - \alpha^2 - \beta^2 = (1 - \alpha^2)(1 - \beta^2)$$

(6) Thus the above equation (6) is positive only when either $|\alpha| > 1$ and $|\beta| > 1$ or $|\alpha| < 1$ and $|\beta| < 1$.

Hence the result follows.

Lemma II: Given a real polynomial $f_n(z)$ of degree n, there exists a real polynomial $f_{n-1}(z)$ of degree less than n such that

$$f_n(z) = (\alpha z + \beta) f_{n-1}(z) + (1 + \alpha\beta z) z^{n-1} f_{n-1}(z^{-1})$$

(7)

where α, β are real coefficients.

Proof

The proof is evident from the relationship that follows directly from above

$$(\alpha^2 - 1)(\beta^2 - 1) z f_{n-1}(z) = (\alpha + \beta z) f_n(z) - (1 + \alpha\beta z) z^n f_n(z^{-1})$$

which becomes upon the insertion of the polynomial form equation (1) for $f_n(z)$

$$\begin{aligned}
&(\alpha^2 - 1)(\beta^2 - 1) z f_{n-1}(z) \\
&= (\alpha p_0 - \beta\alpha p_n) z^{n+1} \\
&+ (\alpha p_1 + \beta p_0 - \alpha\beta p_{n-1} - p_n) z^n + \ldots \\
&+ (\alpha p_{r+1} + \beta p_r - \alpha\beta p_{n-r-1} - p_{n-r}) z^{n-r} + \ldots. \\
&+ (\alpha p_n + \beta p_{n-1} - \alpha\beta p_0 - p_1) z \\
&+ (\beta p_n - p_0)
\end{aligned} \quad (8)$$

Thus we can set the constant term on the right hand side to zero to facilitate cancellation from both sides of the equation. This yields

$$\beta = \frac{p_0}{p_n} \quad (9)$$

However, as it is evident from the above relationships in equation (8), this value for β also sets the highest term coefficient on the right hand side equal to zero.

Thus without assigning a specific value to α as yet, we have a polynomial $f_{n-1}(z)$ of degree $n - l$. We can use this additional freedom to reduce the complexity further either by making the remaining highest term equal to zero or by making the linear term equal to zero. The first option yields

$$\alpha = \frac{p_n - \beta p_0}{p_1 - \beta p_{n-1}} \quad (10)$$

Thus at every iteration we can potentially reduce the degree of the polynomial to be examined at the next iteration stage, by 2.

3. THE STABILITY TEST

The above results with the aid of Rouche's Theorem, can be put into a form appropriate for a stability test as follows.

Given the real polynomial

$$f_n(z) = p_0 z^n + p_1 z^{n-1} + ... + p_n \qquad p_0 \neq p_n$$

we can express it in a form

$$f_n(z) = (\alpha z + \beta) f_{n-1}(z) + (1 + \alpha\beta z) z^{n-1} f_{n-1}(z^{-1})$$

with

$$\beta = \frac{p_0}{p_n} \qquad \alpha = \frac{p_n - \beta p_0}{p_1 - \beta p_{n-1}}$$

We note that the real α, β parameters determined from the given polynomial coefficients above, would have values such that either

$$|\alpha| > 1 \text{ and } |\beta| > 1 \text{ or } |\alpha| < 1 \text{ and } |\beta| < 1 \qquad (11)$$

in which case we shall have $|\alpha z + \beta| < |1 + \alpha\beta z|$, or the conditions given by the inequalities (11) are not satisfied, in which case we shall have $|\alpha z + \beta| > |1 + \alpha\beta z|$. These two inequalities are in effect the determining relationships, which provide the separation needed for the use of Rouche's Theorem.

For stability we need to check the following:

1) If $|\alpha| > 1$ and $|\beta| > 1$ or $|\alpha| < 1$ and $|\beta| < 1$ then $f_n(z)$ has the same number of zeros within the unit circle as $(\alpha z + \beta) f_{n-1}(z)$ (Rouche's Theorem). The following two conditions can be tested now to determine the location of the zero associated with the factor $(\alpha z + \beta)$

(a) If $|\beta| > |\alpha|$, then the given polynomial is unstable.

(b) If $|\beta| < |\alpha|$ then we can proceed to test the reduced degree polynomial $f_{n-1}(z)$ (degree=*n-2*)

2) Otherwise $f_n(z)$ has the same number of zeros within the unit circle as $(1 + \alpha\beta z) z^{n-1} f_{n-1}(z^{-1})$ (Rouche's Theorem). Again the following two conditions must be checked in order to locate the zero of the factor $(1 + \alpha\beta z)$

(a) If $|\alpha\beta| < 1$, then the given polynomial is unstable.

(b) If $|\alpha\beta| > 1$ then we can proceed to test the reduced degree polynomial $z^{n-2} f_{n-1}(z^{-1})$ (degree=*n-2*)

The singular cases corresponding to the parameters given in equation (9) and equation (10), when they assume infinite values are essentially covered by the above conditions.

4. SPECIAL CASES

Special Case (i) (The Jury-Marden Test)

For $\alpha = 0, \beta = \lambda$ we have

$$f_n(z) = f_{n-1}(z) + \lambda z^n f_{n-1}(z^{-1})$$

(12) There exists a real number λ that makes $f_{n-1}(z)$ a polynomial of degree $n - 1$ namely

$$\lambda = \frac{p_0}{p_n} \qquad (13)$$

This is the parameter that decides on the basis of Rouche's Theorem, which of the two components in equation (13) above is to be considered at the next iteration of the algorithm. The procedure is essentially Jury's Test.

Special Case (ii) (A variant of the Jury-Marden Test)

For $\alpha = 1, \beta = 0$ and a slight readjustment of the parameters to make this case appear in the same sense as case (i) we have

$$f_n(z) = z f_{n-1}(z) + \lambda z^{n-1} f_{n-1}(z^{-1}) \qquad (14)$$

There exists a real number λ, which makes $f_{n-1}(z)$ a polynomial of order $n - 1$ namely

$$\lambda = \frac{p_n}{p_0} \qquad (15)$$

This case may be considered as a variant of the Jury-Marden Test.

Special Case (iii)

For $\alpha = 1$ we have the following result.

There exist real numbers β and λ which make $f_{n-1}(z)$ a polynomial of degree $n - 1$ such that

$$f_n(z) = (z + \beta) f_{n-1}(z) + \lambda(1 + \beta z) z^{n-1} f_{n-1}(z^{-1})$$

We form the reverse polynomial as indicated in the earlier cases

$$z^n f_n(z^{-1}) = (1 + \beta z) z^{n-1} f_{n-1}(z^{-1}) + \lambda(z + \beta) f_{n-1}(z)$$

and hence by eliminating $f_{n-1}(z^{-1})$ from the above we obtain

$$(1 - \lambda^2)(z + \beta) f_{n-1}(z) = f_n(z) - \lambda z^n f_n(z^{-1}) \qquad (16)$$

It follows therefore that for $f_{n-1}(z)$ to be a polynomial a cancellation must occur between the two sides, namely the factor $(z + \beta)$.

In effect implies that at $z = -\beta$ the right hand side must disappear. This yields the condition below.

$$\lambda = \frac{f_n(-\beta)}{\beta^n f_n(-\beta^{-1})} \qquad (17)$$

The conditions set out earlier are now satisfied, and hence the order recursion algorithm for the required stability test can now be implemented.

This recursion as set out by equation (16) and equation (17) is interesting from the theoretical point of view. It has a free parameter to be selected but it is computationally more intensive than the previous recursions.

5. THE SECOND ORDER CASE

The polynomials $g_1(z)$ and $g_2(z)$ were chosen for the above approaches to be first order. The question arises, however, whether a higher order choice may be even more efficient.

It should be observed that at some stage in the order reduction process it becomes necessary to check the location of the zeros of these polynomials as it is evident in steps 1 and 2 of the new stability test.

Therefore, it is not feasible to have $g_1(z)$ and $g_2(z)$ of order higher than the second because only then can we have simple closed form formulae for the roots of a polynomial.

A specific choice for $g_1(z)$ and $g_2(z)$ for second order polynomial reductions is then as follows.

$$g_1(z) = a_0 z^2 + a_1 z + a_2$$

and

$$g_1(z) = b_0 z^2 + b_1 z + b_2$$

where

$$a_0 a_2 = b_0 b_2 \text{ and } a_0 a_1 + a_1 a_2 = b_0 b_1 + b_1 b_2$$

At every step of the iteration process of the order reduction process there are eight subsidiary tests to be carried out to enable the appropriate component in equation (1) to be chosen on the basis of Rouche's Theorem. In addition there is a corresponding number of quadratics to be checked where their roots lie before proceeding to the next iteration.

The subsidiary tests involved in the above procedure for the second order case can be simplified under certain conditions, and the complexity therefore can be reduced further. It is expected that the full range of the alternative open to the user will be explored in another publication.

Comments and Conclusions:

1) The computational savings in the general first order recursion are associated with the removal of the multiplications of the intermediate stages in the earlier recursions. The worst case computational complexity is one half of that corresponding to the Jury-Marden Test. Thus the triangular array of the Jury-Marden Test is of complexity $O(n(n-1)/2)$, while the first order general case presented above is of complexity $O(n(n-1)/4)$.

2) In Signal Processing terms, the first order iteration procedure can have implications and consequences on the design of lossy lattice filters. This issue will be explored in the future.

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