ASYMPTOTIC NON-NULL DISTRIBUTION OF THE GENERALIZED COHERENCE ESTIMATE

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ABSTRACT

The use of the generalized coherence estimate as a statistic for detection of a common signal in multiple independent channels of additive gaussian noise has been studied in several recent papers. This work has relied on simulations to evaluate detector performance because the distribution of the generalized coherence estimate with signal present is unknown. This paper derives an asymptotic expression for the non-null distribution of the estimate as the length of the sample sequences approaches infinity, develops an asymptotic performance analysis based on this distribution, and compares the receiver operating characteristics derived from this theoretical approach to those obtained using simulations with large sample sequence lengths.

1. INTRODUCTION

The generalized coherence (GC) estimate has been studied as a statistic for detection of a common signal on $M \geq 2$ noisy channels [1, 2]. The GC estimate has been shown to provide a natural geometrical generalization of the magnitude-squared coherence (MSC) estimate, a widely used statistic for detection of a common signal on two noisy channels [3]. Recently it has been observed that the GC estimate arises as the test statistic in the uniformly most powerful invariant matched subspace detector for a class of multiple channel detection problems [4, 5].

The GC-based detector described in [1] is a significance test: it does not assume an explicit signal model, but relies on the ability of the GC estimate to discern deviations from the H_0 hypothesis that all M channels contain independent gaussian noise. Explicit knowledge of the probability distribution of the GC estimate under these H_0 conditions makes it possible to calculate detection thresholds corresponding to given probabilities of false alarm.

Previous research has had to rely on Monte Carlo simulations to evaluate the performance of GC-based multiple-channel detectors because no analytical expressions for the distribution of the GC estimate under signal-present hypotheses have been known. This paper sketches a derivation of the asymptotic distribution of the GC estimate when the channels contain a white gaussian signal in independent, additive white gaussian noise and the length of the sample sequences approaches infinity. This expression is used to develop theoretical performance predictions for GC-based detectors using long sample sequences. Receiver operating characteristic (ROC) curves derived from theory are compared to empirically determined ROC curves for various scenarios.

2. GENERALIZED COHERENCE

Suppose that the M-channel detector is to operate in a scenario

$$\begin{aligned} H_0 &: \quad z_k(\cdot) = n_k(\cdot) \\ H_1 &: \quad z_k(\cdot) = s(\cdot) + n_k(\cdot) \end{aligned}$$

where $s(\cdot)$ denotes a common signal with spectral density $S_s(\cdot)$ and the noise n_k, $k = 1, \ldots, M$ on each channel is independent and complex gaussian with spectral density $S_{n_k}(\cdot)$. For a white signal in additive white noise the vectors $\mathbf{z}$ obtained by sampling the processes $z_k(\cdot)$ can be modeled as independent realizations of a complex random M-vector $\mathbf{Z} = [Z_1 \ \cdots \ Z_M]^T$.

Suppose that N independent observations $\mathbf{z}_\alpha$ of the random vector $\mathbf{Z}$ are available. The GC estimate, introduced in [1], is defined as

$$\hat{\gamma}^2(A) \doteq 1 - \frac{\det A}{\prod_{k=1}^{M} A_{kk}} \tag{1}$$

where

$$A \doteq \frac{1}{N} \sum_{\alpha=1}^{N} \mathbf{z}_\alpha \mathbf{z}_\alpha^H, \tag{2}$$

$\mathbf{z}^H$ denotes the complex conjugate transpose of $\mathbf{z}$, and A_{kk} is the k^{th} diagonal element of the matrix A.

Prior work has focused on the GC estimate without significant attention to the underlying entity being estimated. In analogy with the well known magnitude-squared coherence coefficient, the *generalized coherence coefficient* of a complex random vector $\mathbf{Z}$ with covariance matrix Σ may be defined as

$$\gamma^2 \doteq 1 - \frac{\det \Sigma}{\prod_{k=1}^{M} \Sigma_{kk}}. \tag{3}$$

It can be shown that the GC estimate (1) provides a consistent estimate of γ^2.

The GC coefficient γ^2 is a measure of the degree of correlation of the components of $\mathbf{Z}$ and has the properties $0 \leq \gamma^2 \leq 1$ for all $\mathbf{Z}$, $\gamma^2 = 0$ if and only if all components of $\mathbf{Z}$ are uncorrelated, and $\gamma^2 = 1$ if any two components are perfectly correlated. The GC coefficient can be expressed in terms of the signal-to-noise ratios (SNRs) on the M channels as follows. The SNR on the k^{th} channel at frequency ω is defined as

$$\text{SNR}_k(\omega) = \frac{S_s(\omega)}{S_{n_k}(\omega)}.$$

For a white signal in white noise, the spectral densities (and therefore the SNRs) are independent of frequency. Substituting the expression for the SNR into (3), it is possible to express the GC coefficient in terms of the SNRs on the channels as

$$\gamma_M^2(\text{SNR}_1, \ldots, \text{SNR}_M) = \frac{\sum_{i=2}^{M} C(i, \mathcal{S})}{\prod_{i=1}^{M}(1 + \text{SNR}_i)} \tag{4}$$

where $\mathcal{S}$ denotes the set $\mathcal{S} = \{\text{SNR}_1, \text{SNR}_2, \ldots, \text{SNR}_M\}$ and $C(i, \mathcal{S})$ denotes the sum of all the i-tuples from $\mathcal{S}$. For example $C(2, \mathcal{S}) = \text{SNR}_1\text{SNR}_2 + \text{SNR}_1\text{SNR}_3 + \ldots + \text{SNR}_{M-1}\text{SNR}_M$.

For equal SNRs on all channels, the GC coefficient for M channels can be written in terms of the SNRs as

$$\gamma_M^2(\text{SNR}) = \frac{\sum_{i=0}^{M-2} \binom{M}{i} \text{SNR}^{M-i}}{(1 + \text{SNR})^M}.$$

In the following section, these relations will be used to express the asymptotic distribution in terms of the SNRs on the channels.

3. ASYMPTOTIC DISTRIBUTION

The goal of this section is to establish that, for a complex zero-mean white gaussian signal in complex zero-mean white gaussian noise, the conditional distribution of $\hat{\gamma}^2$ given $\gamma^2 > 0$ is asymptotically normal with mean $E[\hat{\gamma}^2|\gamma^2] = \gamma^2$ and variance

$$\text{var}(\hat{\gamma}^2|\gamma^2, R) = \frac{(1 - \gamma^2)^2(\text{tr}(R^2) - M)}{N} \tag{5}$$

where R denotes the matrix of correlation coefficients between the channels. Under these conditions, $\mathbf{Z}$ is a zero-mean complex gaussian random M-vector [6] with covariance matrix $\Sigma = E[\mathbf{Z}\mathbf{Z}^H]$. Defining

$$V = f(A) \doteq 1 - \hat{\gamma}^2(A) = \frac{\det A}{\prod_{k=1}^{M} A_{kk}}$$

it clearly suffices to establish that V is asymptotically normal with the appropriate mean and variance. To achieve this, V is shown in the following paragraphs to be an affine function of a complex random vector Φ that is asymptotically normal.

Denote $T = \sqrt{N}(A - \Sigma)$. Calculating the characteristic function of T and applying the central limit theorem for complex random variables [7] shows that the asymptotic distribution of T is complex gaussian. Its mean is $E[T] = \sqrt{N}(E[A] - \Sigma) = \sqrt{N}(\Sigma - \Sigma) = 0$. To calculate the covariances of the elements of T, first observe that the covariances of the elements of A are given as

$$\begin{aligned} \text{cov}(A_{ij}, A_{kl}) &= E[(A_{ij} - E[A_{ij}])(A_{kl} - E[A_{kl}])^*] \\ &= \frac{1}{N}\Sigma_{ik}\Sigma_{lj} \end{aligned} \tag{6}$$

which is obtained by expanding the product and substituting the fourth moment of a zero-mean complex gaussian random vector, given in [7] as

$$E[Z_i Z_j^* Z_k^* Z_l] = \Sigma_{ij}\Sigma_{lk} + \Sigma_{ik}\Sigma_{lj}.$$

Now, since

$$\begin{aligned} \text{cov}(T_{ij}, T_{kl}) &= NE[(A_{ij} - \Sigma_{ij})(A_{kl} - \Sigma_{kl})^*] \\ &= NE[A_{ij}A_{kl}^*] - N\Sigma_{ij}\Sigma_{kl}^*, \end{aligned}$$

substituting (6) yields

$$\text{cov}(T_{ij}, T_{kl}) = \Sigma_{ik}\Sigma_{lj}. \tag{7}$$

Define Φ to be the vector obtained by arranging the elements of T in row-major order. Then Φ is asymptotically zero-mean gaussian, as desired, and its covariance matrix Ψ is determined by equation (7).

It remains to show that, in the limit $N \to \infty$, V is an affine function of Φ. The Taylor expansion of $V = f(A)$ at $A = \Sigma$ is

$$f(A) = f(\Sigma) + f'(A)|_{A=\Sigma}(A - \Sigma) + \ldots$$

In this expression, the notation $(A - \Sigma)$ is understood to represent a column vector formed from the matrix elements in row-major order and $f'(A)$ is a row vector of partial derivatives of f with respect to the elements of A, again taken in row-major order. Assuming the higher order terms are negligible for sufficiently large N yields the desired expression for V as an affine function of Φ:

$$V = f(A) = f(\Sigma) + \Gamma\Phi \tag{8}$$

where $\Gamma = f'(A)|_{A=\Sigma}$.

The asymptotic mean of $\hat{\gamma}^2 = 1 - V$ can be calculated by taking the mean on both sides of (8) as

$$\begin{aligned} E(\hat{\gamma}^2|\gamma^2) &= E[1-V] \\ &= 1 - \frac{\det \Sigma}{\prod_{k=1}^{M} \Sigma_{kk}} = \gamma^2. \end{aligned}$$

Standard results on gaussian variates suggest that the asymptotic variance of V can be obtained using equation (8) and the covariance matrix Ψ of Φ calculated above. Following [8], the computation is accomplished using the total differential $df(A)$ at $A = \Sigma$:

$$\begin{aligned} \mathrm{var}(V) &= E[\{df(A)|_{A=\Sigma}\}^2] \\ &= E\left[\left\{\sum_{i=1}^{M}\sum_{j=1}^{M} \frac{\partial f(A)}{\partial A_{ij}} dA_{ij}\bigg|_{A=\Sigma}\right\}^2\right] \\ &= E\left[\left\{\sum_{i=1}^{M}\sum_{j=1}^{M} \frac{\partial f(A)}{\partial A_{ij}}\bigg|_{A=\Sigma} (A_{ij} - \Sigma_{ij})\right\}^2\right] \end{aligned}$$

Substituting $\gamma^2 = 1 - f(\Sigma)$ and $\hat{\gamma}^2 = 1 - V$ allows this equation to be simplified into the expression for the asymptotic variance of the GC estimate given in (5). For equal SNR on all channels, the variance simplifies to

$$\mathrm{var}(\hat{\gamma}^2|\gamma^2) = \frac{1}{N}(1-\gamma^2)^2 M(M-1)\rho^2$$

where ρ denotes the correlation coefficient between any two channels.

The preceding results establish that the conditional distribution of the GC estimate given true GC value $\gamma^2 > 0$ is asymptotically gaussian with mean γ^2 and variance given in (5).

4. A COMMENT ON BIAS

While the GC estimate is asymptotically unbiased, simulations have shown that a better approximation of the distribution for finite values of N is achieved if the asymptotic distribution is corrected by subtraction of the estimate bias. Although a closed-form expression for the bias of the GC estimate given a non-zero value of γ^2 is not known, many problems of practical interest involve SNRs sufficiently small that it is reasonable to approximate the bias by assuming $\gamma^2 = 0$. This can be calculated as

$$\begin{aligned} \mathrm{bias}(\hat{\gamma}^2|\gamma^2 = 0) &= E[\hat{\gamma}^2 - \gamma^2] \\ &= 1 - \frac{\prod_{k=1}^{M-1}(N-k)}{N^{M-1}}. \end{aligned} \tag{9}$$

5. DETECTION PERFORMANCE

As with the other coherence estimates, the GC estimate can be used as a nonparametric detection statistic because the distribution is known under the H_0 hypotheses. This allows the calculation of detection thresholds for a constant false alarm rate (CFAR).

ROC curves for a multiple-channel detector based on the GC estimate can be obtained by using the relationships between the SNR, the GC coefficient, the H_0 distribution of the GC estimate, and the asymptotic distribution of the GC estimate as derived in the previous section.

Figure 1 shows ROC curves derived from the asymptotic distribution compared to the result of Monte Carlo simulations. In the figure, the asymptotic distribution was used with the bias correction factor given in (9). It can be observed that even for small vector lengths the two curves agree well as long as the SNR is not too small.

6. CONCLUSIONS

This paper has derived the asymptotic non-null distribution of the GC estimate and shown it to depend on the true value of the GC coefficient and the structure of the covariance matrix Σ. Through equation (4), the asymptotic distribution may be expressed as a function of the SNRs on the channels.

The distribution has been applied to obtain ROC curves for a detector based on the GC estimate. The curves obtained agree with results from simulations.

The results presented here shed light on one of the major open questions surrounding generalized coherence that was posed in [1]. The authors note that many important practical and theoretical questions related to generalized coherence and its estimate remain open.

7. REFERENCES

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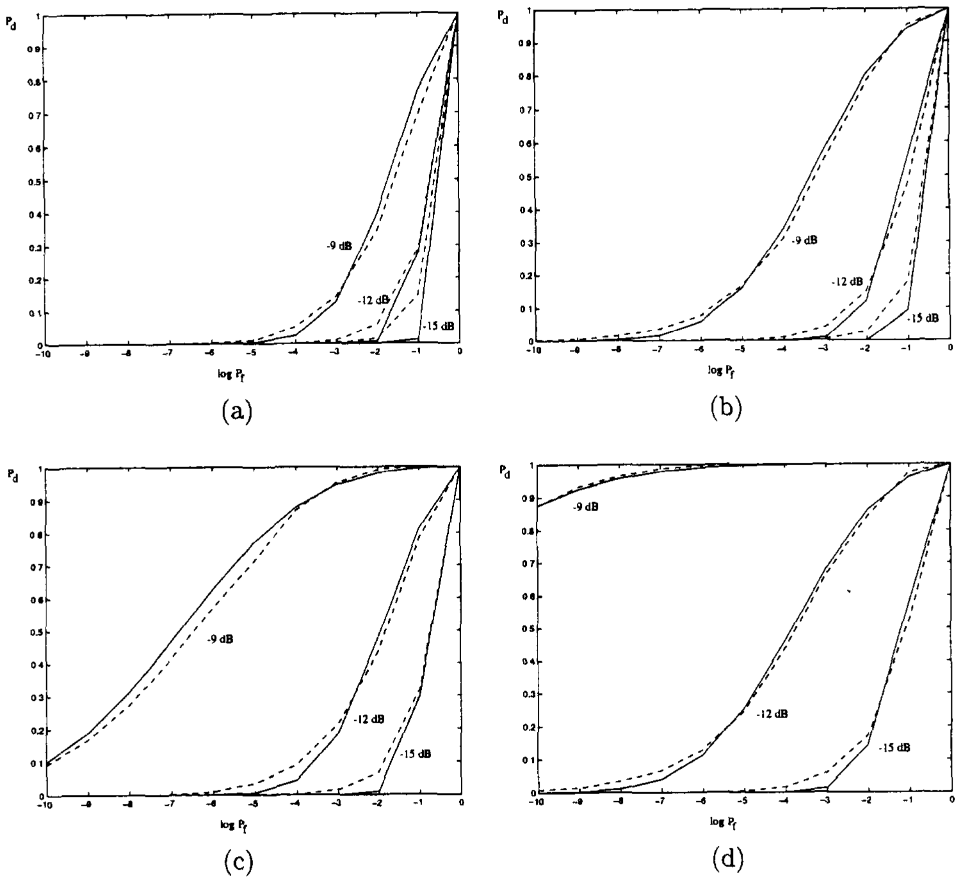

Figure 1: Comparison of empirically determined (dashed lines) and theoretically derived asymptotic ROC curves (solid lines) for a three-channel GC-based detector in detecting a white gaussian signal in white gaussian noise. SNRs were equal on all channels. Empirical results were obtained from Monte Carlo simulations of size $N_{MC} = 1000$. Curves are shown for SNRs of -9dB, -12dB, and -15dB and sample sequence lengths (a) $N = 128$, (b) $N = 256$, (c) $N = 512$, and (d) $N = 1024$.

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