MULTIPLE FREQUENCY ESTIMATION IN ADDITIVE AND MULTIPLICATIVE COLORED NOISES

Martial Coulon and Jean-Yves Tourneret

ENSEEIHT / GAPSE, 2 Rue Camichel, BP 7122
31071 Toulouse Cedex, France
coulon@len7.enseeiht.fr tournere@len7.enseeiht.fr

ABSTRACT

This paper addresses the problem of estimating sinusoidal frequencies in additive and multiplicative colored noises. Specific Yule-Walker equations yield second-order statistic-based estimates. The frequency estimates are shown to be asymptotically normally distributed. Their asymptotic covariance is derived.

1. INTRODUCTION AND PROBLEM FORMULATION

Additive noise models have been intensively considered in many signal processing applications. Indeed, these models allow to approximate a large class of physical mechanisms contaminated by measurement noise. In particular, the problem of estimating the frequencies of single or multiple sinusoids contaminated by Gaussian or non-Gaussian additive noise has received much attention in the literature [5] [6]. Unfortunately, most estimation procedures can fail dramatically, when the signal is contaminated by non-additive noise components. Multiplicative noise (MN) has been shown to be a suitable modelling for some non-linear noise effects. For instance, many problems encountered in random communication models (fading channels), Sonar, or Doppler systems concern harmonic signals corrupted by multiplicative and/or additive noise. Frequency estimation in MN context has then been intensively studied under many different hypotheses. For example, the frequency estimation problem for a single sinusoid in MN was addressed in [1], [7]. In [1], the MN was modeled as a parametric model, whose parameters can be estimated from the observation autocorrelation function. In [7], the authors proposed to use cyclic statistics for the single frequency estimation problem. The frequency estimation for multiple sinusoids is of course more complicated. Stationary or cyclostationary approaches were developed for this kind of problem [4] [8]. This paper studies a new frequency estimation algorithm for a sum of random sinusoids embedded in multiplicative and additive colored noises. The noise probability density functions (pdf) and parameters are assumed unknown. We consider the following model:

$$\begin{aligned} y(n) &= e(n)\sum_{j=1}^{p} A_j \cos(\omega_j n + \Phi_i) + u(n) \quad (1) \\ &= e(n)x(n) + u(n) \end{aligned}$$

where:

1) $x(n)$ is the sum of p random harmonics. The angular positions ω_j are deterministic constants in the interval $[0, 2\pi)$.The Φ_i's are uniformly distributed random variables in $[0, 2\pi)$, which are independent of $e(n)$ and $u(n)$. The amplitudes A_j are unknown and deterministic constants.

2) the MN $e(n)$ is modelled by an MA(q) process driven by an i.i.d. sequence with unknown pdf:

3) the additive noise $u(n)$ is an MA$(q\prime)$ process with unknown pdf (note that the additive noise is white if $q\prime = 0$).

The modeling of the additive and multiplicative noises $u(n)$ and $e(n)$ by a parametric MA model can be justified by the fact that for any continuous spectral density $S(f)$, an MA process can be found with a spectral density arbitrary close to $S(f)$ ([3], p. 132). The algorithm developed in this paper is based on second-order statistics. However, it is interesting to note that it can be generalized to higher-order statistics (HOS). Indeed, the HOS-based algorithm can be used when the additive noise is Gaussian, without specific structure (not necessary MA). We emphasize that the proposed estimator does not require any knowledge regarding the distribution of the processes $e(n)$, $x(n)$ and $u(n)$. However, the MN $e(n)$ is assumed to have non-zero mean.

2. AR PARAMETER ESTIMATION

Let $\mu_s = E[s(n)]$ denote the mean of the stationnary random process $s(n)$, $M_k^s(\rho)$ and $C_k^s(\rho)$ its kth-order moment and cumulant computed at lag $\rho = (\rho_1, \rho_2 \ldots, \rho_{k-1})$. This paper assumes that the multiplicative noise $e(n)$ has non-zero mean, i.e. $\mu_e \neq 0$. In this case, the AR parameters are estimated from the data using appropriate second- or higher-order statistics.

For brevity, the study is conducted with covariances, assuming that additive noise $u(n)$ is a (Gaussian or non-Gaussian) $MA(q')$ process. However, it could be generalized to higher-order cumulants with the same assumptions on $u(n)$, as well as for any possibly non-MA Gaussian colored additive noise.

Since the processes $e(n)$, $x(n)$, and $u(n)$ are independent, it follows that

$$C_2^y(\rho) = C_2^{ex}(\rho) + C_2^u(\rho), \; \forall \rho$$

where ex denotes the multiplicative process $e(n)x(n)$. Using the MA structure of the process $u(n)$, we obtain $C_2^u(\rho) = 0, |\rho| > q'$. Moreover,

$$C_2^{ex}(\rho) \;\; = \;\; M_2^{ex}(\rho) - \mu_{ex}^2 = M_2^e(\rho)M_2^x(\rho) - \mu_e^2\mu_x^2$$

$$= M_2^e(\rho)\left(C_2^x(\rho)+\mu_x^2\right)-\mu_e^2\mu_x^2 = M_2^e(\rho)C_2^x(\rho).$$

Since $e(n)$ is an MA(q) process, $M_2^e(\rho) = \mu_e^2, \forall|\rho| > q$. Thus,

$$C_2^{ex}(\rho) = \mu_e^2 C_2^x(\rho), \forall|\rho| > q$$

Moreover, it is well known that [5] $C_2^x(\rho) = \sum_{j=1}^{p} \frac{A_j^2}{2}\cos(\omega_j\rho)$. Consequently, for $|\rho| > \max(q, q')$, the covariances of the MN process $y(n)$ are

$$C_2^y(\rho) = \mu_e^2 \sum_{j=1}^{p} \frac{A_j^2}{2}\cos(\omega_j\rho) \quad (2)$$

Let $(a_k)_{k=0,\ldots,2p}$ denote the coefficients of the polynomial

$$A(z) = \prod_{j=1}^{p}\left(1-2\cos(\omega_j)z^{-1}+z^{-2}\right) \triangleq \sum_{k=0}^{2p} a_k z^{-k}$$

with $a_0 = 1$. The covariances $C_2^x(\rho)$ satisfy the following equation:

$$\sum_{j=0}^{2p} a_j C_2^x(\rho - j) = 0, \forall\rho \quad (3)$$

Consequently,

$$\sum_{j=0}^{2p} a_j C_2^y(\rho - j) = \mu_e^2 \sum_{j=0}^{2p} a_j C_2^x(\rho - j) = 0, \quad (4)$$

This last equation is valid only if $|\rho - j| > \max(q, q')$ for any $j \in \{0, \ldots, 2p\}$. Therefore, the validity condition (denoted condition A) is

$$\text{condition A: } \rho > \rho_0 \triangleq \max(2p+q, 2p+q')$$

Eq. (4) shows that the coefficient vector $\underline{a} = (a_1, \ldots, a_{2p})^T$ satisfies the following equation (provided that condition A is verified):

$$\underline{\mathbf{C}}_2^y(m,\rho).\underline{a} = -\underline{\mathbf{c}}_2^y(m,\rho) \quad (5)$$

for $m \geq 2p$, where $\underline{\mathbf{C}}_2^y(m,\rho)$ is the Toeplitz matrix whose first row is $[C_2^y(\rho-1), C_2^y(\rho-2), \ldots, C_2^y(\rho-2p)]$ and first column $[C_2^y(\rho-1), C_2^y(\rho), \ldots, C_2^y(\rho+m-1)]^T$, and

$$\underline{\mathbf{c}}_2^y(m,\rho) = [C_2^y(\rho), \ldots, C_2^y(\rho+m)]^T$$

The coefficient vector $\underline{a}$ is then defined by

$$\underline{a} = -\underline{\mathbf{C}}_2^{y\#}(m,\rho)\underline{\mathbf{c}}_2^y(m,\rho) \quad (6)$$

where $(.)^\#$ denotes the Moore-Penrose pseudo-inverse. Several remarks are now appropriate:

(1) Eq. (5) shows that the condition $\mu_e \neq 0$ is required. Indeed, when $\mu_e = 0$, $\underline{\mathbf{C}}_2^y(m,\rho) = 0$ and $\underline{\mathbf{c}}_2^y(m,\rho) = 0$.

(2) Eq. (5) can be obtained for orders $k > 2$, leading to $\underline{\mathbf{C}}_k^y(m,\rho_1,\ldots,\rho_{k-1}).\underline{a} = -\underline{\mathbf{c}}_k^y(m,\rho_1,\ldots,\rho_{k-1})$, providing that appropriate relations between ρ_1, ρ_2,..., ρ_{k-1}, p and q are satisfied.

(3) Eq. (5) was derived assuming that the MA orders q and q' were known. However, if these orders are unknown, eq. (5) can be easily modified by replacing q and q' by upper bounds $\overline{q}$ and $\overline{q}'$.

In practice, theoretical covariances (or cumulants) are unknown and should be replaced by sample covariances (or cumulants). Once the estimate $\widehat{\underline{a}}$ is obtained, the angular position estimates $\widehat{\omega}_j$, $j = 1, \ldots, p$, are defined as the p closest zeros of the polynomial $\widehat{A}(z) = \sum_{k=0}^{2p} \widehat{a}_k z^{-k}$ to the unit circle.

3. ASYMPTOTIC PROPERTIES OF THE FREQUENCY ESTIMATES

This section studies the asymptotic properties of the frequency estimates $\{\widehat{\omega}_j\}$. It has been shown in [6] that the covariance matrix of $\{\widehat{\omega}_j\}$ can be obtained from that of $\{\widehat{a}_j\}$ using the following expression:

$$\widehat{\omega}_j - \omega_j = F(\widehat{a}_j - a_j) + O(N^{-1})$$

where the matrix F has a complicated form which depends upon the model parameters. Therefore, we can restrict attention to the problem of the asymptotic behavior of $\widehat{\underline{a}} = \{\widehat{a}_j\}_{j=1,\ldots,2p}$. Let $\widehat{C}_2^y(m,\rho)$, and $\widehat{\underline{c}}_2^y(m,\rho)$ denote the vector obtained by replacing the true cumulants in $\mathbf{C}$ by their usual estimates computed from N samples. Suppose the cumulants of the processes $e(n)$ and $u(n)$ are absolutely sommable, i.e.

$$\sum_{\rho_1,\ldots,\rho_{k-1}=-\infty}^{\infty} \left|C_k^s(\rho_1,\ldots\rho_{k-1})\right| < \infty, \; k = 2,3,\ldots \quad (7)$$

for $s = e$ and u. In this case, Appendix A proves that the normalized estimation error vector $\sqrt{N}\left(\widehat{C}_2^y(\rho_1) - C_2^y(\rho_1), \ldots, \widehat{C}_2^y(\rho_l) - C_2^y(\rho_l)\right)^T$ is asymptotically normally distributed for any $\rho = (\rho_1, \ldots, \rho_l)$, with zero mean and covariance matrix $\mathbf{\Sigma}_\rho^y$:

$$\sqrt{N}\begin{pmatrix} \widehat{C}_2^y(\rho_1) - C_2^y(\rho_1) \\ \vdots \\ \widehat{C}_2^y(\rho_l) - C_2^y(\rho_l) \end{pmatrix} \sim \mathcal{N}(0, \Sigma_\rho^y)$$

where the matrix Σ_ρ^y is given by (12) (see Appendix). It follows that the vector estimate $\widehat{\underline{c}}_2^y(m,\rho)$ and the matrix estimate $\widehat{\underline{\mathbf{C}}}_2^y(m,\rho)$ are asymptotically Gaussian:

$$\begin{aligned} \sqrt{N}\left(\widehat{\underline{\mathbf{c}}}_2^y(m,\rho) - \underline{\mathbf{c}}_2^y(m,\rho)\right) &\sim \mathcal{N}(0, \Sigma_{\underline{\mathbf{c}}}) \\ \sqrt{N}\left(\widehat{\underline{\mathbf{C}}}_2^y(m,\rho) - \underline{\mathbf{C}}_2^y(m,\rho)\right) &\sim \mathcal{N}(0, \Sigma_{\underline{\mathbf{C}}}) \end{aligned} \quad (8)$$

where the matrices $\Sigma_{\underline{\mathbf{c}}}$ and $\Sigma_{\underline{\mathbf{C}}}$ are independent of N, and can be computed using formula (12). Since the pseudo-inverse function is differentiable at $\underline{\mathbf{C}}_2^y(m,\rho)$ for large m, it follows from standard results on asymptotic theory that

$$\sqrt{N}\left(\widehat{\underline{\mathbf{C}}}_2^{y\#}(m,\rho) - \underline{\mathbf{C}}_2^{y\#}(m,\rho)\right) \sim \mathcal{N}(0, \Sigma_{\underline{\mathbf{C}}\#}) \quad (9)$$

where the matrix $\Sigma_{\underline{\mathbf{C}}\#}$ can be computed from the matrix $\Sigma_{\underline{\mathbf{C}}}$. Consequently, eq's (6), (8) and (9) yield

$$\sqrt{N}\left(\widehat{\underline{a}} - \underline{a}\right) \sim \mathcal{N}(0, \Sigma_{\underline{a}}) \quad (10)$$

where the matrix $\Sigma_{\underline{a}}$ is a function of $\Sigma_{\underline{c}}$ and $\Sigma_{\underline{C}\#}$. Finally, it follows that the frequency vector estimate is asymptotically normally distributed:

$$\sqrt{N}\,(\widehat{\underline{\omega}} - \underline{\omega}) \sim \mathcal{N}(0, \Sigma_{\underline{\omega}}) \qquad (11)$$

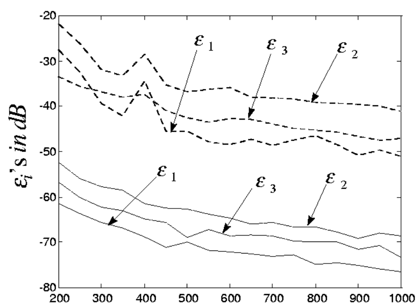
with $\Sigma_{\underline{\omega}} = F\Sigma_{\underline{a}}F^T$. It is worth noting that the theoretical results (10) and (11) are similar to that obtained for sinusoids in additive noise (see [6], for instance).

4. SIMULATION RESULTS

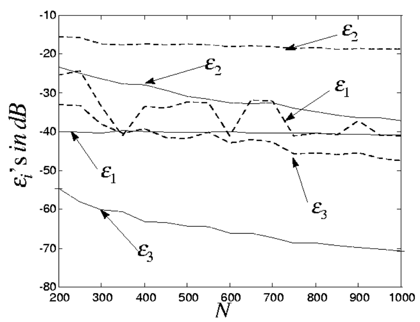
Many simulations have been performed to validate the theoretical results. Let $\widehat{\omega}_i(k)$ denote the estimate of ω_i obtained from the k^{th} Monte-Carlo run. Let M denote the number of runs. The quadratic error ε_i on the angular position ω_i is defined as

$$\varepsilon_i = \frac{1}{4M\pi^2}\sum_{k=1}^{M}\left(\widehat{\omega}_i(k) - \omega_i\right)^2$$

. The constant term $\frac{1}{4\pi^2}$ allows to obtain a normalized error (since the ω_i's are in the interval $[0, 2\pi)$). Fig.1 shows the quadratic errors ε_i's as a function of the number of samples N for different signal to noise ratios $SNR_{x,e} = \frac{\sigma_x^2}{\sigma_e^2}$. The process $x(n)$ is a sum of 3 sinusoids with angular positions $\omega_1 = 2\pi \times 0.1$, $\omega_2 = 2\pi \times 0.2$, and $\omega_3 = 2\pi \times 0.45$, and constant amplitudes $A_1 = 1$, $A_2 = 3$, $A_3 = 5$. The multiplicative noise $e(n)$ is an $MA(2)$ process with parameters $[1; 0.3; -0.2]$ driven by an exponentially distributed sequence such that $\mu_e = 0.5$. The additive noise $u(n)$ is a zero-mean Gaussian $MA(1)$ process. The number of runs is $M = 100$. The higher the $SNR_{x,e}$, the better the estimation. Indeed, when the variance σ_e^2 is low, the MN $e(n)$ is approximatively constant, and the estimation is easier. Fig. 2 shows the quadratic errors ε_i's obtained for the same process with angular positions $\omega_1 = 2\pi\times 0.1$, $\omega_2 = 2\pi\times 0.12$, and $\omega_3 = 2\pi \times 0.45$. The estimation accuracy clearly decreases with respect to fig.1, since two frequencies are closer.



Quadratic errors ε_i's (in dB) for different number of samples. Solid line: $SNR_{x,e} = 10dB$ - dashed line: $SNR_{x,e} = 0dB$. $\omega_1 = 2\pi \times 0.1$, $\omega_2 = 2\pi \times 0.2$, $\omega_3 = 2\pi \times 0.45$.



Quadratic errors ε_i's (in dB) for different number of samples. Solid line: $SNR_{x,e} = 10dB$ - dashed line: $SNR_{x,e} = 0dB$. $\omega_1 = 2\pi \times 0.1$, $\omega_2 = 2\pi \times 0.11$, $\omega_3 = 2\pi \times 0.45$.

5. CONCLUSION

This paper studied the estimation of sinusoidal frequencies in multiplicative and additive noise environment. The multiplicative noise process was modelled by an MA process. Specific Yule-Walker equations based on second-order statistics were then shown to be an efficient tool for frequency estimation. The estimation could be generalized to higher-order statistic-based estimates, in presence of Gaussian colored noise. The frequency estimates were shown to be asymptotically normally distributed, and an explicit expression of the covariance was derived. Moreover, it can be noted that this problem can be generalized to the estimation of a sum of sinusoids corrupted by independent MA multiplicative noises.

6. APPENDIX

6.1. PROOF OF THE ASYMPTOTIC NORMALITY OF $\left(\widehat{C}_2^y(\rho_k) - C_2^y(\rho_k)\right)_{k=1,\ldots,l}$

In order to prove the asymptotic normallity of the cumulant vector estimate, it is sufficient to show that the higher order cumulants of any linear combination of its components, denoted $X_N = \sqrt{N}\sum_{k=1}^{l}\alpha_k\left(\widehat{C}_2^y(\rho_k) - C_2^y(\rho_k)\right)$, are asympotically zero [2], [5], for any arbitrary scalars $\{\alpha_k\}$, not all zero. It is also sufficient to prove that the higher order cumulants of $\widetilde{X}_N = \sqrt{N}\sum_{k=1}^{l}\alpha_k\widehat{C}_2^y(\rho_k)$ tend to zero. Let Δ_N denote the mth-order cumulant of $\widetilde{X}_N$. Using the multi-linearity of the cumulants, it follows that

$$\Delta_N = \left(\sqrt{N}\right)^m \sum_{k_1,\ldots,k_m}^{l} \alpha_{k_1}\ldots\alpha_{k_m} Cum\left(\widehat{C}_2^y(\rho_{k_1}),\ldots,\widehat{C}_2^y(\rho_{k_m})\right)$$

$$= \left(\sqrt{N}\right)^m \sum_{k_1,\ldots,k_m}^{l} \alpha_{k_1}\ldots\alpha_{k_m}\frac{1}{N-\rho_{k_1}}\cdots\frac{1}{N-\rho_{k_m}}\,Cum$$

$$\left(\sum_{n=1}^{N-\rho_{k_1}} y(n)y(n+\rho_{k_1}) - \widehat{\mu}_y^2, \ldots, \sum_{n=1}^{N-\rho_{k_m}} y(n)y(n+\rho_{k_m}) - \widehat{\mu}_y^2\right)$$

where $\widehat{\mu}_y$ is the sample mean of $y(n)$. Consider the term

$$Cum\left(\sum_{n=1}^{N-\rho_{k_1}} y(n)y(n+\rho_{k_1}),\ldots,\sum_{n=1}^{N-\rho_{k_m}} y(n)y(n+\rho_{k_m})\right)$$
$$=\sum_{n_1=1}^{N-\rho_{k_1}}\cdots\sum_{n_m=1}^{N-\rho_{k_m}} Cum(y(n_1)y(n_1+\rho_{k_1}),\ldots,y(n_m)y(n_m+\rho_{k_m}))$$
$$\triangleq\sum_{n_1=1}^{N-\rho_{k_1}}\cdots\sum_{n_m=1}^{N-\rho_{k_m}} \Lambda_y(n_1,\ldots,n_m)$$

Since the process $x(n)$ is bounded with probability 1, it follows that

$$|\Lambda_y|\leq\kappa\left|Cum(e(n_1)e(n_1+\rho_{k_1})+e(n_1)u(n_1+\rho_{k_1})+ u(n_1)e(n_1+\rho_{k_1})+u(n_1)u(n_1+\rho_{k_1}),\ldots,)\right|\triangleq\kappa\left|\Theta_{e,u}(n_1,\ldots,n_m)\right|$$

where κ is a constant depending on $\{A_k\}$. Using eq. (7), it can be proved that

$$\sum_{n_2=1}^{N-\rho_{k_2}}\cdots\sum_{n_m=1}^{N-\rho_{k_m}}|\Theta_{e,u}(n_1,\ldots,n_m)|<+\infty$$

Consequently, $\frac{1}{N-\rho_{k_1}}\sum_{n_1=1}^{N-\rho_{k_1}}\cdots\sum_{n_m=1}^{N-\rho_{k_m}}\Lambda_y(n_1,\ldots,n_m)=O(1)$. The same procedure can be used for the terms involving $\widehat{\mu}_y^2$. It follows that

$$\tfrac{1}{N-\rho_{k_1}}\cdots\tfrac{1}{N-\rho_{k_m}}Cum\left(\sum_{n=1}^{N-\rho_{k_1}} y(n)y(n+\rho_{k_1})-\widehat{\mu}_y^2,\ldots\right.$$
$$\left.\ldots,\sum_{n=1}^{N-\rho_{k_m}} y(n)y(n+\rho_{k_m})-\widehat{\mu}_y^2\right)=O(N^{m-1})$$

and $\Delta_N=O(N^{1-m/2})$. Therefore, $\Delta_N\underset{N\to\infty}{\longrightarrow}0$ for $m>2$. ∎

6.2. COMPUTATION OF THE COVARIANCE MATRIX Σ_ρ^y

For $\mu_u=E[u(n)]=0$, the covariance matrix Σ_ρ^y is given by:

$$\begin{aligned}\Sigma_\rho^y(\tau_1,\tau_2)=&\sum_{-\infty}^{+\infty}\{C_4^e(\tau_1,\rho,\rho+\tau_2)+C_2^e(\rho)C_2^e(\rho+\tau_2)\\&+C_2^e(\rho-\tau_1)C_2^e(\rho+\tau_2)]M_4^x(\tau_1,\rho,\rho+\tau_2)\\&+C_4^u(\rho+\tau_1,\rho,\tau_2)+C_2^u(\rho)C_2^u(\rho+\tau_1-\tau_2)\\&+C_2^u(\rho+\tau_1)C_2^u(\rho-\tau_2)+M_2^{ex}(\rho)M_2^u(\rho+\tau_2-\tau_1)\\&+M_2^{ex}(\rho-\tau_1)M_2^u(\rho+\tau_2)+M_2^{ex}(\rho+\tau_2)M_2^u(\rho-\tau_1)\\&+M_2^{ex}(\rho+\tau_2-\tau_1)M_2^u(\rho)\}\end{aligned}\qquad(12)$$

with

$$\begin{aligned}M_4^x(\tau_1,\tau_2,\tau_3)=&C_4^x(\tau_1,\tau_2,\tau_3)+C_2^x(\tau_1)C_2^x(\tau_2-\tau_3)\\&+C_2^x(\tau_2)C_2^x(\tau_1-\tau_3)+C_2^x(\tau_3)C_2^x(\tau_1-\tau_2)\end{aligned}$$

and

$$\begin{aligned}C_4^x(\tau_1,\tau_2,\tau_3)=&-\sum_{k=1}^{p}\tfrac{A_k^4}{8}\left[\cos(\omega_k(\tau_1-\tau_2-\tau_3))\right.\\&\left.+\cos(\omega_k(\tau_2-\tau_1-\tau_3))+\cos(\omega_k(\tau_3-\tau_1-\tau_2))\right]\end{aligned}$$

Proof: The complete proof is long and tedious, but does not present particular difficulty. We will only give the main guidelines. Using the fact that third-order cumulants of the process $x(n)$ are zero, it can be first proved that

$$\begin{aligned}&cov\left(\widehat{C}_2^y(\tau_1),\widehat{C}_2^y(\tau_2)\right)=cov\left(\widehat{M}_2^y(\tau_1),\widehat{M}_2^y(\tau_2)\right)=\\&cov\left(\widehat{M}_2^{ex}(\tau_1),\widehat{M}_2^{ex}(\tau_2)\right)+cov\left(\widehat{C}_2^u(\tau_1),\widehat{C}_2^u(\tau_2)\right)\\&+cov\left(\widehat{\zeta}(\tau_1),\widehat{\zeta}(\tau_2)\right)\end{aligned}\qquad(13)$$

where

$$\widehat{\zeta}(\tau)=\frac{1}{N-\tau}\sum_{n=1}^{N-\tau}(e(n)x(n)u(n+\tau)+e(n+\tau)x(n+\tau)u(n))$$

The second term of (13) is given by Bartlett's formula. However, this formula can not be used for the two others terms, which do not verify the formula's assumptions. These terms can be computed based on the two following lemmas:

lemma 1: Assume that $\sum_{n=-\infty}^{+\infty}|\lambda_n|<\infty$. Then, $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-k}\sum_{m=0}^{N-l}\lambda_{n-m}=\sum_{n=-\infty}^{+\infty}\lambda_n$.

lemma 2: If $\omega\neq 0$, $\frac{1}{N}\sum_{n=1}^{N}\cos(\omega n+\phi)=\frac{1}{N}\sin(N\omega/2)\cos\left((N+1)\omega/2+\phi\right)/\sin(\omega/2)$.

It can be shown from lemma 2 that $\lim_{N\to\infty}\frac{N}{(N-\tau_1)(N-\tau_2)}\sum_{n=1}^{N-\tau_1}\sum_{m=0}^{N-\tau_2}C_2^x(m-n+\tau)=0$ for any τ. This last result, as well as eq. (7), and lemmas 1 and 2, allow us to obtain (12) ∎.

If the mean μ_u is non-zero, the sample mean can be substracted from the measurements $y(n)$ to get a zero-mean process, so that formula (12) holds true.

7. REFERENCES

[1] O. Besson and P. Stoica, "Sinusoidal Signals with Random Amplitude: Least-Squares Estimators and their Statistical Analysis," IEEE Trans. on Signal Processing, Vol. 43, No. 11, pp. 2733-2744, November 1995.

[2] D.R. Brillinger, Time Series: Data Analysis and Theory, Holden-day Inc., San Francisco, 1981.

[3] P. J. Brockwell and R.A. Davis, Time Series: Theory and Methods, Springer Verlag, 2nd Edition, 1990

[4] M. Ghogho, "Frequency Estimation of Multiplicative ARMA Nois Data," Proc. of ICASSP'95, pp. 1613-1616, Detroit, May 1995.

[5] B. Porat, Digital Processing of Random Signals: Theory and Methods, Prentice Hall, NJ, 1994.

[6] P. Stoica, T. Söderström and F.-N. Ti, "Asymptotic properties of the high-order Yule-Walker estimates of sinusoidal frequencies", IEEE Trans. Acoustic Speech Signal Processing, Vol. ASSP-37, pp. 1721-1734, 1989.

[7] G. Zhou and G. Giannakis, "Harmonics in Multiplicative and Additive noise: parameter estimation using cyclic statistics," IEEE Trans. on Signal Processing, Vol. 43, pp. 2217-2221, Sep. 1995.

[8] G. Zhou and G. Giannakis, "Harmonics in Gaussian Multiplicative and Additive Noise: Cramér-Rao Bounds," IEEE Trans. on Signal Processing, Vol. 43, pp. 1217-1231, May 1995.