d-MUSIC, A REAL TIME ALGORITHM FOR ESTIMATING THE DOA OF COHERENT SOURCES USING A SINGLE ARRAY SNAPSHOT

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ABSTRACT

d-MUSIC estimates the DOA of two closely spaced sources using a single array snapshot. To overcome the coherent signal problem d-MUSIC utilizes additional information, specifically the derivative of an array snapshot. The combined vector set produces a full rank signal space projector. The algorithm nearly attains the Cramér-Rao bound for typical air traffic control problems. As it does not require a subspace decomposition (e.g., eigenstructure) and all operations are highly vectorized it can be readily implemented in real-time. The algorithm is tested using vertical linear array data with a low flying helicopter. With a spacing of 16% to 35% of a beamwidth between the direct and surface reflected rays, the d-MUSIC rms error is 9.6% of a beamwidth for the 4 data collections while MUSIC resolved the two rays for 2 of the 4 cases with a rms error of 18.1%.

1. INTRODUCTION

One of the challenging tasks of both air traffic control and air defense is estimating the altitude of aircraft at low elevation angles. The severe multipath contamination typically present in low angle tracking will often lead to a rank deficient problem..

Complicating the coherent signal problem is the need to resolve the sources using only a few array snapshots. As the system may rotate to provide azimuth coverage it may only be able to extract a few snapshots in the direction of the target for each dwell. Alternatively, the system may step through a sequence of transmit frequencies permitting only a few snapshots per frequency burst. Even if more than one snapshot is available the system will often need to perform some form of coherent time-series analysis (e.g., Doppler processing using a FFT) to provide gain for detection and to isolate the moving target from the clutter/interference background. In that event it may only be feasible to use a single array snapshot, specifically that of the frequency bin containing the target.

These are serious limitations. Another issue is the need to minimize costs by deriving a computationally simple algorithm that can be easily implemented in real-time.

An algorithm which attempts to meet this requirement is derivative-MUSIC or simply d-MUSIC. It bypasses the need to estimate the signal space of the 2 sources using a matrix factorization technique by constructing a full rank signal projector using the input snapshot vector and its derivative following a forward-backward operation to balance the complex amplitude of the two rays. The orthogonal vector set is used to construct an estimate for the signal space projector and the source bearings can be estimated using a root-MUSIC procedure.

Two versions of the algorithm are derived here. Both are for an uniform linear array of sensors. The primary algorithm is for the case when the centroid of the direct and surface reflected rays is known. This is typical of air traffic control problems where the aircraft altitude and range is much greater than the array height (i.e., the flat-earth approximation [1]). The second algorithm is for the case when the cluster centroid is unknown. This is typical of low flying aircraft at short ranges.

A data set is available to test the second version of the algorithm. This data was graciously provided to Raytheon by Defence Research Establishment Ottawa (Canadian DND) under a research contract. The data was collected in October of 1995 at Osborne Head, Nova Scotia using the Experimental Array Radar System (EARS). In the subset of the data to be analyzed here a helicopter hovered at an altitude of 30.5 and 61 meters over the sea at a range of 8 km and data was collected at both 8.9 and 9.4 GHz (4 collections in all). The lower 6 channels of the EARS vertical linear array of horns are available for this study.

Much of the material to be discussed here is presented in Howell's thesis [2]. Drawing upon this work the paper is organized as follows. In Section 2 both versions of the d-MUSIC algorithm are derived. In section 3 the theoretical performance analysis of [2] is summarized and the d-MUSIC variance is compared to the Cramér-Rao bound (CRB) for the case of a known cluster centroid. The error analysis and CRB derivation is based on the well known approach of Stoica and Nehorai in their analysis of the MUSIC algorithm [3]. In section 4 the Osborne Head data is analyzed using both the d-MUSIC and MUSIC algorithms.

2 d-MUSIC

The specular multipath signal model can be represented as

$$\mathbf{x}(t) = s_{\mathrm{T}}(t)\,\mathbf{a}(\theta_{\mathrm{T}}) + \rho s_{\mathrm{R}}(t)\,\mathbf{a}(\theta_{\mathrm{R}}) + \mathbf{n}(t) \tag{1}$$

where the T and R subscripts denote the direct path and surface reflected rays, respectively. θ denotes elevation angle, s the signal time series, ρ the surface reflection coefficient, $\mathbf{n}$ the combined noise/clutter/interference vector, $\mathbf{x}$ the array snapshot vector and $\mathbf{a}(\theta)$ is the linear array steering vector defined as

$$\mathbf{a}(\theta) = e^{-j\alpha(N-1)/2}\left[1 \;\; e^{j\alpha} \;\; e^{j2\alpha} \cdots e^{j(N-1)\alpha}\right]^T \tag{2}$$

where $(\cdot)^T$ denotes transpose, $\alpha = 2\pi d\sin(\theta)/\lambda$, d is the sensor spacing, λ is wavelength and N is the number of sensors.

Say that only array snapshot is available. There may in fact be more but it is assumed that some form of coherent processing (e.g., an FFT) is applied to the data to separate the target echo from the clutter/interference background as part of the detection process. Hence, the only vector that may be available is the Doppler frequency bin corresponding to the detection.

The single snapshot is modeled as

$$\mathbf{v} = k_1\,\mathbf{a}(\theta_T) + k_2\,\mathbf{a}(\theta_R) + \mathbf{w} \tag{3}$$

where k_1 and k_2 are complex constants and $\mathbf{w}$ is a noise + clutter + interference vector. If $\mathbf{w}$ is due to zero mean spatially and temporally white Gaussian noise alone, its variance will be represented as σ^2/K in recognition of the fact that some form of coherent processing involving the effective addition of K snapshots was employed.

Without any loss of generality the center of the array is chosen as the phase reference point. Clearly $\mathbf{a}(\theta)$ is a centro-Hermitian vector, i.e., $\mathbf{J}\,\mathbf{a}(\theta) = \mathbf{a}^*(\theta) = \mathbf{a}(-\theta)$ where $*$ denotes the complex conjugate and $\mathbf{J}$ is a reverse permutation matrix (ones running along the anti-diagonal and zero everywhere else).

An important property of exponential vectors such as $\mathbf{a}(\theta)$ is that its derivatives is simply an amplitude weighted version of $\mathbf{a}(\theta)$. For example, the first derivative of $\mathbf{a}(\theta)$ is

$$\dot{\mathbf{a}}(\theta) = d\,\mathbf{a}(\theta)/d\alpha = \mathbf{D}\,\mathbf{a}(\theta) \tag{4}$$

where the linear array derivative operator $\mathbf{D}$ is defined as

$$\mathbf{D} = j\,\mathrm{diag}\left\{[0\,1 \cdots N-1]^T - (N-1)/2\right\} \tag{5}$$

and diag$\{\cdot\}$ converts a column vector into a diagonal matrix. Note that the first derivative is an anti-symmetric vector. As $\mathbf{a}(\theta)$ is symmetric it follows that $\mathbf{a}^H(\theta)\dot{\mathbf{a}}(\theta) = \mathbf{0}$.

Derivative vectors can be particularly useful for resolving two closely spaced sources. Let $\delta = \alpha_T - \alpha_R$. If the sources are closely spaced then δ is small and we may use the following first-order Taylor series expansions to relate the two steering vectors

$$\begin{aligned}\mathbf{a}(\theta_T) &\approx \mathbf{a}(\theta_R) + \delta\dot{\mathbf{a}}(\theta_R)\\ \mathbf{a}(\theta_R) &\approx \mathbf{a}(\theta_T) - \delta\dot{\mathbf{a}}(\theta_T)\end{aligned} \tag{6}$$

Subtracting the two Taylor series yields

$$\mathbf{a}(\theta_T) - \mathbf{a}(\theta_R) \approx \delta\mathbf{D}\{\mathbf{a}(\theta_T) + \mathbf{a}(\theta_R)\}/2 \tag{7}$$

This is a profound result as it permits us to relate the signal space of the steering vectors to that of their derivatives. In strict terms, the space of $[\mathbf{a}(\theta_T)\ \mathbf{a}(\theta_R)]$ and $[\dot{\mathbf{a}}(\theta_T)\ \dot{\mathbf{a}}(\theta_R)]$ do not overlap, but for the special case of closely spaced sources the two spaces do have an approximate intersection point.

Say that $k_1 = k_2$ in (3) then $\mathbf{v}$ will be proportional to $\mathbf{a}(\theta_T) + \mathbf{a}(\theta_R)$ and $\mathbf{Dv}$ will be proportional to $\mathbf{a}(\theta_T) - \mathbf{a}(\theta_R)$. Two independent and orthogonal vectors that span the same signal space.

Note that the MUSIC algorithm attempts to construct the signal space projector $\mathbf{P_S}$ defined for 2 sources as

$$\mathbf{P_S} = \frac{\mathbf{p}_1\mathbf{p}_1^H}{\mathbf{p}_1^H\mathbf{p}_1} + \frac{\mathbf{p}_2\mathbf{p}_2^H}{\mathbf{p}_2^H\mathbf{p}_2} \tag{8}$$

where $\mathbf{p}_1 = \mathbf{a}(\theta_T) + \mathbf{a}(\theta_R)$ and $\mathbf{p}_2 = \mathbf{a}(\theta_T) - \mathbf{a}(\theta_R)$.

Hence, if $k_1 = k_2$ so that $\mathbf{v}$ is proportional to $\mathbf{a}(\theta_T)+\mathbf{a}(\theta_R)$ we could immediately construct an estimate for $\mathbf{P_S}$ by substituting $\mathbf{p}_1 = \mathbf{v}$ and $\mathbf{p}_2 = \mathbf{Dv}$. Even if $k_1 \neq k_2$ we may exploit other information to equalize the gain of the two steering vectors, specifically the centro-Hermitian property.

Let $\phi = (\theta_T+\theta_R)/2$, the geometric center of the cluster. The vector rotation operator is defined as $\mathbf{S_R}(\phi) = \mathrm{diag}\{\mathbf{a}^*(\phi)\}$. The product $\mathbf{S_R}(\phi)\mathbf{a}(\theta)$ effectively rotates the vector $\mathbf{a}(\theta)$ to point in the direction $\sin^{-1}\{\sin\theta - \sin\phi\}$.

Say that the cluster centroid ϕ is known *a priori* which is typical of many air traffic control problems where the aircraft range and altitude is sufficiently large relative to the antenna height such that the flat earth approximation is valid (i.e., $\theta_R = -\theta_T$). Applying the rotation property we may derive

$$\begin{aligned}\mathbf{S_R}(\phi)\mathbf{v} &= \mathbf{S_R}(\phi)(k_1\mathbf{a}(\theta_T) + k_2\mathbf{a}(\theta_R) + \mathbf{w})\\ &= k_1\,\mathbf{a}(\gamma) + k_2\,\mathbf{a}(-\gamma) + \mathbf{S_R}(\phi)\mathbf{w}\end{aligned} \tag{9}$$

where $\gamma = (\theta_T - \theta_R)/2$ is half the source spacing.

Using the centro-Hermitian property we can apply the following forward-backward operation to equalize the amplitude of the two steering vectors

$$\begin{aligned}\mathbf{u} &= (\mathbf{I} + \mathbf{J})\mathbf{S_R}(\phi)\mathbf{v}\\ &= \{k_1 + k_2\}\{\mathbf{a}(\gamma) + \mathbf{a}(-\gamma)\} + (\mathbf{I} + \mathbf{J})\mathbf{S_R}(\phi)\mathbf{w}\end{aligned} \tag{10}$$

where $\mathbf{I}$ is the identity matrix. Though the two steering vectors now have equal gain factors the cost of this procedure is convert spatially white noise into spatially correlated noise with covariance matrix $(\mathbf{I} + \mathbf{J})\,\sigma^2/K$.

As the two steering vectors now have equal gain factors we can quickly construct the d-MUSIC estimate for the signal space projector using $\mathbf{u}$ and $\dot{\mathbf{u}} = \mathbf{Du}$

$$\hat{\mathbf{P}}_\mathbf{S} = \frac{\mathbf{u}\mathbf{u}^H}{\mathbf{u}^H\mathbf{u}} + \frac{\dot{\mathbf{u}}\dot{\mathbf{u}}^H}{\dot{\mathbf{u}}^H\dot{\mathbf{u}}} \tag{11}$$

The source bearings may be quickly computed using the efficient root-MUSIC algorithm.

The above represents the primary version of the d-MUSIC algorithm, where the cluster centroid ϕ is known. Even if ϕ is not known it is still possible to employ this algorithm in a grid search scheme to locate the two source bearings.

This second version of d-MUSIC involves a grid search over a fixed grid of ϕ values to find the solution point that best fits the input vector **v**. For each value of ϕ the d-MUSIC algorithm will produce an estimate (θ_1,θ_2) for the source bearings. Using (θ_1,θ_2) we may construct a null space projector $\mathbf{P}_N = \mathbf{I} - \mathbf{P}_S$ and compute $\mathbf{v}^H \mathbf{P}_N \mathbf{v}$. The grid point with solution (θ_1,θ_2) which results in a minimum for $\mathbf{v}^H \mathbf{P}_N \mathbf{v}$ represents the true solution point.

The first algorithm for the case of known ϕ is conceptually simple. No complex matrix factorization technique such as an eigenvector or singular value decomposition is required and all other operations are highly vectorized. As such this algorithm can be easily implemented in a real-time setting.

The second algorithm requires more computation, but again no complex matrix factorization is required and all operations are highly vectorized. It too can be implemented in a real-time.

This represents a first description of the d-MUSIC concept. The fundamental principle which guides d-MUSIC is the creation of additional signal space vectors using derivatives to add to or complete the span of the signal space. Many other versions of the same basic algorithm can be constructed. The building blocks for a multi-source d-MUSIC algorithm as well as a planar array d-MUSIC algorithm is presented in [2].

The d-MUSIC algorithm is unique in that it is highly insensitive to the problem of signal correlation. As it uses only snapshot vector it is largely irrelevant if the sources are correlated or not. In fact, the results should improve for known ϕ if the sources coherently add as the SNR of **v** will increase (this conjecture is borne out in the performance analysis). There is one important case where d-MUSIC will fail, specifically $\mathbf{v} = \mathbf{a}(\theta_T) - \mathbf{a}(\theta_R)$, the vector d-MUSIC attempts to construct. This is a moot point however as the probability of detection is low for this case.

3. PERFORMANCE ANALYSIS

A theoretical error analysis of the primary d-MUSIC algorithm (known ϕ) is presented in [2]. This analysis closely follows the approach developed by Stoica and Nehorai [3] who analyzed the MUSIC algorithm and derived the Cramér-Rao lower bound for the general direction finding problem.

When ϕ is known the problem reduces to estimating one parameter, the source spacing. The Cramér-Rao bound (CRB) for the source spacing is really a special case of the general CRB developed in [3]. The modifications required to adapt the CRB of [3] to derive the source spacing CRB is listed in [2].

The theoretical model for the d-MUSIC error variance is a complicated expression involving nearly a hundred terms. The large number of terms is mainly due the fact that the d-MUSIC noise is spatially correlated and the noise of the signal space vector and its derivative is highly correlated. It is beyond the scope of this paper to present the full analysis along with the CRB derivation. Instead the reader is referred to [2] and a summary of the main findings will be presented here.

In this example a 10 sensor linear array with a spacing of $\lambda/2$ will be used. K =100 and the signal covariance matrix **S** for the equal power sources is modeled as

$$\mathbf{S} = \begin{bmatrix} 1 & e^{j\chi} \\ e^{-j\chi} & 1 \end{bmatrix} \tag{12}$$

where χ represents the phase difference between the direct and surface reflected rays. As it is inexpensive to employ forward-backward averaging of the form $(\mathbf{R} + \mathbf{J}\mathbf{R}^*\mathbf{J})/2$ in MUSIC the MUSIC variance of [3] along with the 2 parameter CRB will both assume forward-backward averaging. The result of this averaging is to replace the off-diagonal terms of **S** with cos χ.

The source spacing is referenced to the Rayleigh resolution beamwidth of the array defined as the spacing from the peak of the broadside beam to the first null.

To illustrate the main result of the performance analysis consider the case of $\sigma^2 = 1$ which corresponds to a detection SNR of about 30 dB for coherent sources. Figure 1 represents the d-MUSIC result for 4 values of χ and Figure 2 is the MUSIC result.

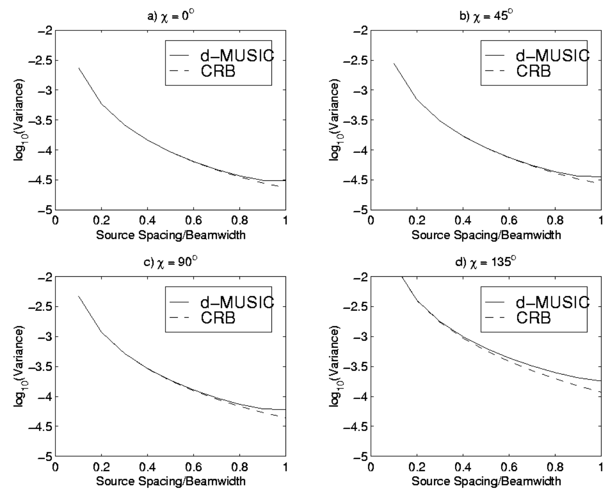

Figure 1. d-MUSIC variance for $\sigma^2 = 1$

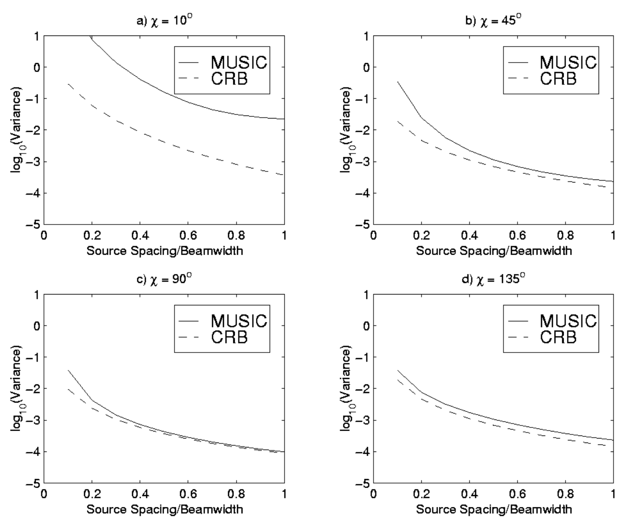

Figure 2. MUSIC variance for $\sigma^2 = 1$

Note that the d-MUSIC algorithm nearly attains the CRB with the best results arising for positive coherence ($\chi = 0$). Overall, the d-MUSIC algorithm is largely insensitive to the problem of signal correlation.

In contrast the MUSIC algorithm really only attains the CRB when the sources are uncorrelated ($\chi = 90°$). Note that the single parameter CRB of Figure 1 is at a lower level than the 2 parameter CRB of Figure 2, as expected.

Due to the complexity of the problem a theoretical performance analysis of the grid search version of d-MUSIC to locate the cluster centroid will be difficult. A large number of Monte Carlo simulation were used in [2] to investigate this algorithm. In summary the second algorithm behaved much like the main algorithm but with degraded accuracy. In comparison to MUSIC it attained a substantially lower error variance. The following experimental data will be used to support these findings.

5. EXPERIMENT RESULTS

The Osborne Head data is analyzed using the grid search version of d-MUSIC and compared to the MUSIC algorithm. A subset of the data is presented here and the full results are listed in [2].

In this experiment the lower 6 channels of the EARS vertical linear array were available for analysis. The sensor spacing is 12.5 cm with a array height of 27.8 m. The Sea King helicopter however at a height of 30.5 m and 61 m at a range of 8 km. The target SNR at detection is about 40 dB, the signal-to-sea clutter ratio is better than 30 dB. At 8.9 GHz the Rayleigh beamwidth (peak to first null) is 2.57° and the direct and surface reflected angle spacing is 16% and 33% of a beamwidth. At 9.4 GHz the beamwidth is 2.44° and the source spacings is 17% and 35% of a beamwidth. The 1900 snapshots are divided into non overlapping blocks of 100 snapshots each and a modified version of the technique of [4] is used to calibrate the array (see [2] for details).

The MUSIC algorithm failed to resolve the 2 sources for both 9.4 GHz measurements. However, grid-search d-MUSIC successfully resolved the 2 ray paths for all 4 cases. The d-MUSIC bias and rms error for these 4 cases averages to 2.3% and 9.6% of a beamwidth, respectively. For the two 8.9 GHz cases in which MUSIC resolved the sources the MUSIC bias and rms error is 6.8% and 18.1% respectively. Dropping the number of snapshots to 25 has only a marginal impact on d-MUSIC with the rms error increasing to 10.7%. In contrast the MUSIC rms error increased to 46% with the 6 dB decrease in SNR.

Figures 3 and 4 depicts the 8.9 GHz d-MUSIC and MUSIC estimates for the source spacing (has an unique relationship to target altitude [1]). The dashed line represents the true spacing.

6. SUMMARY

The d-MUSIC algorithm is attractive for the low angle direction finding problem in a multipath environment, especially for aircraft tracking scenarios where the center of the signal plus multipath cluster is known (typical of distant aircraft). It is relatively insensitive to the problem of signal correlation and is computationally simple. As it uses only one array snapshot vector it can be easily integrated into most radar applications.

Future work will focus on further testing of the algorithm using experiment data and to derive a theoretical performance analysis for the grid-search version of the technique.

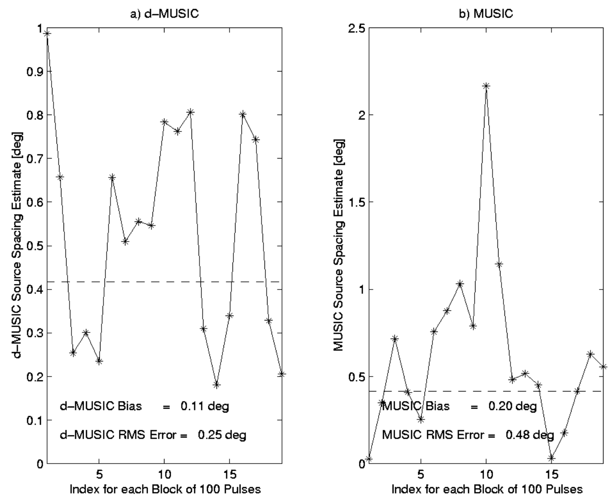

Figure 3. 8.9 GHz results for 30.5 m helicopter altitude.

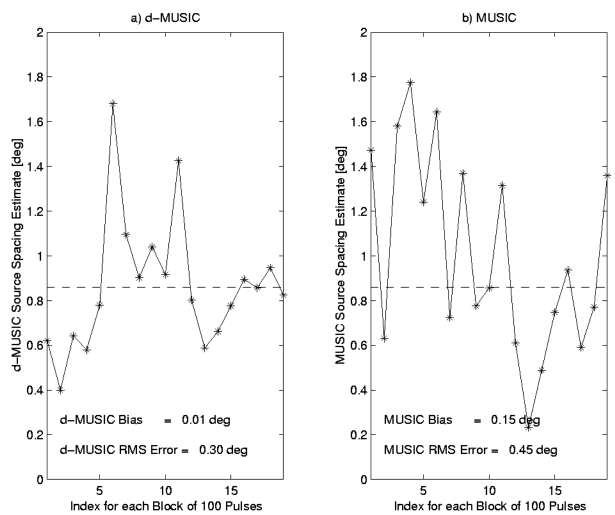

Figure 4. 8.9 GHz results for 61 m helicopter altitude.

7. REFERENCES

[1] Blake L,. *Radar Range-Performance Analysis,* Lexington Books, Lexington, MA, 1980.

[2] Howell R., *d-MUSIC, a Real-Time Algorithm for Estimating the Direction-of-Arrival of Coherent Sources using a Single Array Snapshot.* Ph.D. dissertation, Dept. Elec & Comp Eng, Univ of Victoria, Victoria, BC, Canada, 1999.

[3] Stoica P. and A. Nehorai, "MUSIC, maximum likelihood, and Cramér-Rao Bound". *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, no. 5, pp. 720-741, 1989.

[4] Wylie, M.P,. Roy, S. and H. Messer, "Joint DOA estimation and phase calibration of linear equispaced (LES) arrays," *IEEE Trans. Signal Processing*, vol. 42, no. 12, pp. 3449 - 3459, 1994.