RATIONAL SIGNAL SUBSPACE APPROXIMATION WITH APPLICATIONS TO DOA ESTIMATION

Mohammed A. Hasan[†] and Jawad A. K. Hasan[‡]

[†]Dept. of Electrical & Computer Engineering, University of Minnesota, Duluth, MN 55812

[‡]Dept. of Electrical Engineering, University of Baghdad, Baghdad, Iraq

E.mail: mhasan@d.umn.edu

ABSTRACT

In this paper, we have developed an approach for approximating the signal and noise subspaces which avoid the costly eigendecomposition or SVD. These subspaces were approximated using rational and power-like methods applied to the sample covariance matrix. It is shown that MUSIC and Minimum Norm frequency estimators can be derived using these approximated subspaces. These approximate estimators are shown to be robust against noise and overestimation of number of sources. A substantial computational saving would be gained compared with those associated with the eigendecomposition-based methods. Simulations results show that these approximated estimators have comparable performance at low signal-to-noise ratio (SNR) to their standard counterparts and are robust against overestimating the number of impinging signals.

1. INTRODUCTION

Estimation of sinusoidal frequencies and direction of arrival (DOA) embedded in white noise is a problem of interest in many fields of signal processing such as geophysics, radar, and sonar. Several high resolution techniques emerged during the years for solving this problem. For useful articles and books the reader is referred to [2], [5]-[9] and the references therein. Several subspace approximation techniques were proposed in the literature [2], [6]-[9]. In this paper, we proposed an invariant subspace decomposition based on rational approximation.

The DOA can be described as follows. Consider a linear array of p sensors and q multiple narrowband signals impinging on the array with DOA angles $\theta_1, \theta_2, \cdots, \theta_q$. The received signal at the array can be expressed as

$$\mathbf{y}(k) = A(\theta)\mathbf{s}(k) + \mathbf{v}(k), \qquad (1)$$

where $\mathbf{s}(k) \in C^q$ (C is the field of complex numbers) is a vector of complex signals of q wavefronts $\mathbf{s}(k) = [\, s_1(k) \quad s_2(k) \quad \cdots \quad s_q(k)\,]^T$, $\mathbf{v}(k)$ is a $p \times 1$ vector of additive noise in sensors with $\mathbf{v}(k) = [v_1(k) \quad v_2(k) \quad \cdots \quad v_p(k)]^T$ and A is $p \times q$ matrix $A(\theta) = [\,\mathbf{a}(\theta_1) \quad \mathbf{a}(\theta_2) \quad \cdots \quad \mathbf{a}(\theta_q)\,]$ with $\mathbf{a}(\theta)$ being the steering vector of the array toward the direction θ. It is also assumed that the signals and additive noise are stationary and zero-mean ergodic complex-valued random processes such that $E[v_i(k)v_j^*(k)] = \sigma_v^2 \delta_i(i - j)$ for $i, j = 1, \cdots, q$. Here E, and * denote the expectation, and conjugate transpose operators, respectively. Thus the spatial $p \times p$ covariance matrix of the array output is given by $R_\mathbf{y} = E[\mathbf{y}(k)\mathbf{y}^*(k)] = A(\theta)R_\mathbf{S}A(\theta)^* + \sigma_v^2 I_p$ with $R_\mathbf{S} = E[\mathbf{s}(k)\mathbf{s}^*(k)]$ is $q \times q$ matrix of amplitudes of $\mathbf{s}$ and I_p is the $p \times p$ identity matrix. If the θ's are all distinct, the unknown matrix $A \in C^{p \times q}$ is of rank q. The main objective is to estimate q and A from the noisy data $\{y(k)\}_{i=1}^N$. In the next theorem we present a rational approximation of the eigenspace of the sample covariance matrix, $\hat{R}_\mathbf{y}$.

Theorem 1. *Let $\hat{R}_\mathbf{y} = \sum_{i=1}^p \lambda_i \mathbf{u}_i \mathbf{u}_i^*$, where λ_i and $\mathbf{u}_i$ be the ith eigenvalue and corresponding eigenvector. Assume that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_q > \lambda_{q+1} = \cdots = \lambda_p = \sigma_v^2$ and $\mathbf{u}_i^*\mathbf{u}_j = \delta_{i-j}$. Let $\{b_i\}_{i=1}^r$ be a set of real number such that $0 < b_1 < b_2 < \cdots < b_r$, and set $U_0 = \sum_{j:\lambda_j < b_1} \mathbf{u}_j \mathbf{u}_j^*$, $U_i = \sum_{j:b_i < \lambda_j < b_{i+1}} \mathbf{u}_j \mathbf{u}_j^*$ for $i = 1, \cdots, r$. Then*

$$\lim_{m\to\infty} \Pi_{i=1}^r (b_i^m I_p - \hat{R}_\mathbf{y}^m)(b_i^m I_p + \hat{R}_\mathbf{y}^m)^{-1} = \sum_{i=0}^r (-1)^i U_i.$$

$$(2)$$

Proof: See [6].

In Theorem 1 if $r = 1$ and b_1 is a threshold that separates the signal and noise eigenvalues, then a rational approximation of the signal and noise subspaces can be obtained as in the following result.

Corollary 2. *Let $\hat{R}_\mathbf{y} = \sum_{i=1}^p \lambda_i \mathbf{u}_i \mathbf{u}_i^*$ such that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_q > \lambda_{q+1} = \cdots = \lambda_p = \sigma_v^2$ and $\mathbf{u}_i^*\mathbf{u}_j = \delta_{i-j}$, where λ_i and $\mathbf{u}_i$ are the ith eigenvalue and corresponding eigenvector. Let $b > 0$ be such that $\lambda_q > b > \lambda_{q+1}$ and let $U_s = \sum_{i=1}^q \mathbf{u}_i \mathbf{u}_i^*$ and $U_n = \sum_{i=p+1}^q \mathbf{u}_i \mathbf{u}_i^*$, then*

(i) $(b^m I_p - \hat{R}_y^m)(b^m I_p + \hat{R}_y^m)^{-1}$ converges to $U := U_n - U_s$ (as $m \to \infty$), and therefore $U_s = \frac{I_p - U}{2}$ and $U_n = \frac{I_p + U}{2}$.

(ii) $\hat{R}_y^m(b^m I_p + \hat{R}_y^m)^{-1}$ converges to U_s (as $m \to \infty$).

(iii) $b^m(b^m I_p + \hat{R}_y^m)^{-1}$ converges to U_n (as $m \to \infty$).

Proof: follows from Theorem 1 by setting $r = 1$.

2. HIGH RESOLUTION ESTIMATORS

Given the projections U_s and U_n onto the signal and noise subspaces, the exact MUSIC and minimum norm estimators are given by

$$P_{MUSIC}(\theta) = \frac{1}{|\mathbf{a}^*(\theta)U_n \mathbf{a}^*(\theta)|^2}. \qquad (3a)$$

and

$$P_{MN}(\theta) = \frac{1}{|\mathbf{a}^*(\theta)U_n \mathbf{e}_1|^2}, \qquad (3b)$$

where $\mathbf{e}_1$ is the first column of the $p \times p$ identity matrix.

Let b be a number that separates the signal and noise eigenvalues, i.e., $\lambda_j < b < \lambda_i$ for $i = 1, \cdots q$ and $j = q + 1, \cdots, p$. Then an approximated MUSIC and Minimum-Norm estimators based on (ii) of Corollary 2 can be written as

$$P_{b,MUSIC}^{(m)}(\theta) = \frac{1}{|p - \mathbf{a}^*(\theta)\hat{R}_y^m \{b^m I_p + \hat{R}_y^m\}^{-1}\mathbf{a}(\theta)|}$$

and

$$P_{MN}^{(m)}(\theta) = \frac{1}{|\mathbf{a}^*(\theta)(I_p - \hat{R}_y^m(b^m I_p + \hat{R}_y^m)^{-1})\mathbf{e}_1|}.$$

A rough estimate of a threshold b is $\frac{Tr(\hat{R}_y)}{p}$. This idea along with part (i) of Corollary 2 are incorporated in the following algorithms. Here we used $Tr(A)$ to mean the trace of A.

Algorithm (Rational-MUSIC and Rational-Min-Norm)

(i) Compute $\hat{R}_y(l)) = [r_y(j + l, k + l)]$, where $r_y(j, k)] = E\{y(m + j)y^*(m + k)\}$.

(ii) Choose $m \geq 1, l \geq p$ and compute

$$F_m^{(l)} = ((\frac{Tr(\hat{R}_y(l))}{p})^m I_p - \hat{R}_y(l)^m)$$

$$((\frac{Tr(\hat{R}_y(l))}{p})^m I_p + \hat{R}_y^m(l))^{-1},$$

and let $\bar{F}_m^{(l)} = \frac{F_m^{(l)} + F_m^{(l)*}}{2}$.

(iii) Compute the approximated noise subspace $\hat{U}_n = \frac{I_p + \bar{F}_m}{2}$.

(iv) Compute $P_{b,MUSIC}^{(m)}(\theta)$ and $P_{MN}^{(m)}(\theta)$ using

$$P_{b,MUSIC}^{(m)}(\theta) = \frac{1}{|\mathbf{a}^*(\theta)\hat{U}_n a(\theta)|},$$

$$P_{MN}^{(m)}(\theta) = \frac{1}{|\mathbf{a}^*(\theta)\hat{U}_n \mathbf{e}_1|}.$$

and locate the peaks. The frequencies are estimated as the angular positions of the peaks.

3. OPERATION COUNT

The methods presented in the previous sections are multiplication rich in that for a given m, $\hat{R}_y^m$ is required and followed by one matrix inversion operation. Matrix multiplication can be obtained very efficiently applying the Strassen algorithm [4]. In this algorithm, if $A \in R^{p \times p}$ and $B \in R^{p \times p}$ with p is a power of 2, then $C = AB$ can be obtained with $s \approx p^{2.807}$ multiplications. Thus asymptotically, the number of multiplications in the Strassen algorithm is $O(p^{2.807})$ compared with $O(p^3)$ in the conventional method. It should be mentioned that in [1], Bailey implemented a Strassen approach that required only 60% of the time needed by the conventional multiplication.

The number of flops in computing $(b^m I_p - \hat{R}_y^m)^{-1}(b^m I_p + \hat{R}_y^m)$ consists of approximately the number of flops in computing $\hat{R}_y^m$ and the matrix inverse. Assuming that $m = 2^r$, both of these processes cost about $rp^{2.807} + \frac{2p^3}{3}$.

The number of flops required to compute the SVD of $\hat{R}_y$ by the Golub-Reinsch algorithm is $21p^3$ [3]. For example, if we choose r to be 4 which correspond to $m = 16$, a value that is very high in most applications, the number of flops required in the rational MUSIC is $4p^{2.807} + \frac{2p^3}{3}$ which is still much less than $21p^3$ using the Golub-Reinsch Algorithm [3].

Efficient matrix inversion can be computed using the LU decomposition. Once the LU factorization of A is known, the inverse matrix A^{-1} can be computed in $\frac{p(p-1)(2p-1)}{3}$ flops [3]-[4]. Thus the total number of flops involved in computing $(b^m I_p - \hat{R}_x^m)^{-1}(b^m I_p + \hat{R}_x^m)$ is about $\frac{4p^3}{3} + 2rp^3 = (2r + 1.333)p^3$. This number is still far less than the flop count for computing the SVD, which is about $21p^3$, for $r < 10$. Note that $r = 9$ corresponds to $m = 512$ which is extremely large for most applications. Thus for all practical purposes these algorithms which are based on Corollary 2 are less costly than the truncated SVD-based methods.

4. SIMULATION RESULTS

In this section, the frequency estimators described earlier were examined on several data sets generated by the equation

$$y(n) = a_1 e^{J(2\pi f_1 n + \phi_1)} + a_2 e^{J(2\pi f_2 n + \phi_2)} + v(n), \quad (4)$$

where $a_1 = 1.0$, $a_2 = 1.0$, $f_1 = 0.5$, $f_2 = 0.52$ and $n = 1, 2, \cdots, N = 25$. The ϕ_i are independent random variables uniformly distributed over the interval $[-\pi, \pi]$. The noise $v(k)$ is assumed to be white and uncorrelated with the signal. Note that $f_2 - f_1 < \frac{1}{N}$. The SNR for either sinusoids is defined as $10 \log_{10}(\frac{\sigma_x^2}{\sigma_v^2})$, where $x(n) = a_1 e^{J(2\pi f_1 n + \phi_1)} + a_2 e^{J(2\pi f_2 n + \phi_2)}$ and σ_x^2, σ_v^2 are the variances of $x(n)$ and $v(n)$, respectively. The covariance matrix is constructed using forward-backward method to increase robustness. The size of the covariance matrix is chosen to be $p = 10$ which in the absence of noise has effective rank two. We performed experiments to compare the proposed methods versus the truncated SVD-based method. The SVD routine on MATLAB is used for the computation of the signal subspace eigenvectors and eigenvalues required to implement a SVD-based method for comparison. For each experiment (with data length and SNR fixed), we performed 100 independent trials to estimate the frequencies. We use the following performance criterion (RMSE)

$$RMSE = \sqrt{\frac{1}{N_e} \sum_{i=1}^{N_e} (\hat{f}_i - f_{true})^2}$$

to compare the results. Here N_e is the number of independent realizations, and $\hat{f}_i$ is the estimate provided from the ith realization. The simulations results of applying the Rational-MUSIC are summarized as follow. A set of 100 random experiments for different (SNR=20, 15, 10 dB) using $\hat{R}_y^m$ with $m = 3$ were used. The threshold b in these simulations was estimated by $b = \frac{Tr(\hat{R}_y)}{L}$. The peak spectrum was computed using 1000 frequency bins covering a normalized frequency range of 0 to 1. The mean and RMSE are taken only over realizations where two peaks have occured. The mean values and the RMSE of the estimated frequencies are given in Table 1. As can be noticed, the performance in these cases are almost identical to those obtained from using the exact decomposition. We repeated the experiment using the standard-MUSIC estimator. The results of testing this algorithm for different SNR were averaged over 100 trials and the mean and RMSE of each frequency was presented in Table 2. Figure 1 shows the peaks resulted from applying Rational-MUSIC for 30 independent trials with SNR=15 dB and using $R_y(10)^m$ for $m = 3$. The corresponding results obtained using the standard MUSIC are shown in Figure 2. As can be noticed from Tables

SNR	$\hat{f}_1$	$\hat{f}_2$	$RMSE_{f_1}$	$RMSE_{f_2}$
20 dB	0.500556	0.522322	0.00563	0.012522
15 dB	0.500729	0.521735	0.00652	0.014531
10 dB	0.500961	0.524952	0.00813	0.019204

Table 1: Rational-MUSIC:Mean and standard deviation of frequencies for data of two complex sinusoids at frequencies 0.50 and 0.52 in white noise with SNR=20, 15, 10 dB, dimension of data vectors p=10.

SNR	$\hat{f}_1$	$\hat{f}_2$	$RMSE_{f_1}$	$RMSE_{f_2}$
20 dB	0.50022	0.52133	0.00442	0.00514
15 dB	0.50076	0.52182	0.00781	0.00846
10 dB	0.50123	0.52214	0.01307	0.01045

Table 2: Standard-MUSIC: Mean and RMSE of frequencies for data of two complex sinusoids at frequencies 0.50 and 0.52 in white noise with SNR=20, 15, 10 dB, dimensions of data vectors and signal subspace are $p = 10, q = 2$.

1-2 and Figures 1-2, both Rational-MUSIC and the standard MUSIC are seen to have virtually identical performance.

When m is small, it is observed that overestimation of q leads to better estimation of the frequencies. The robustness of the methods against overestimation of the number of sources q can be explained as follows. Overestimation of q means additional vectors are included in the basis of the signal subspace. In our approximation, these vectors are not purely noise but contain some signal component. In the standard MUSIC, if additional vector is added to the basis, a purely noise vector is included in the signal subspace causing spurious peaks. Note that when m is small the first few signal vectors contain noise components since the noise and signal vectors are not well-separated. Thus we have to consider more vectors to capture the signal subspace. However, this separation becomes more promenant as m increases.

5. CONCLUSION

Eigendecomposition-based methods such as MUSIC and Minimum Norm, estimators are popular for their high resolution property in sinusoidal and direction of arrival estimation but they are also known to be of high computational demand. In this paper, new

·fast and robust algorithms for DOA and sinusoidal frequency estimation are presented. These algorithms approximate the required subspace using rational and power-like methods applied to the sample covariance matrix. The operation count of these algorithms were shown to be much less than that associated with the eigendecomposition-based methods. Furthermore, this approach can be refined to compute invariant subspaces of any complex matrix in different regions in the complex plane.

References

[1] Bailey D., "Extra High Speed Matrix Multiplication on Cray-2," SIAM J. Sci. and Stat. Comp. 9, pp. 603-607, 1988.

[2] Ermolaev V. T. and Gershman A. B., "Fast Algorithm for Minimum-Norm Direction-of-Arrival Estimation," IEEE Trans. on Signal Processing, Vol. 42, No. 9, pp. 2389-2394, September 1994.

[3] Golub G. H. and Reinsch C., "Singular Value Decomposition and Least Squares Solutions," Numer. Math. 14, pp. 403-420, 1970.

[4] Golub G. H. and Van Loan C. G., Matrix Computations, 2nd ed., the John Hopkins University Press, Baltimore, 1989.

[5] Hasan M. A., and Hasan A. A., "Hankel Matrices of Finite Rank with Applications to Signal Processing and Polynomials", J. of Math. Anal. and Appls, 208, pp. 218-242, 1997.

[6] Hasan M. A., "Fast Algorithms for Signal Subspace Approximation with Applications to DOA Estimation", accepted for publication in Journal of Signal Processing.

[7] Karhunen J. T., and Joutsenalo J., "Sinusoidal Frequency Estimation by signal subspace Approximation," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, Vol. ASSP-40, No. 12, pp. 2961-2972, December 1992.

[8] Kay S. M. and Shaw A. K., "Frequency Estimation by Principal Component AR Spectral Estimation Method without Eigendecomposition," IEEE Trans. on Acoustics, Speech, and Signal Processing, Vol. 36, No. 1, pp. 95-101, January 1988.

[9] Tufts D. and Melissinos C. D., "Simple, Effective Computation of Principal Eigenvectors and Their Eigenvalues and Application to High-Resolution Estimation of frequencies," IEEE Trans. on Acoustics, Speech, and Signal Processing, Vol. 34, No. 5, pp. 1046-1053, October 1986.
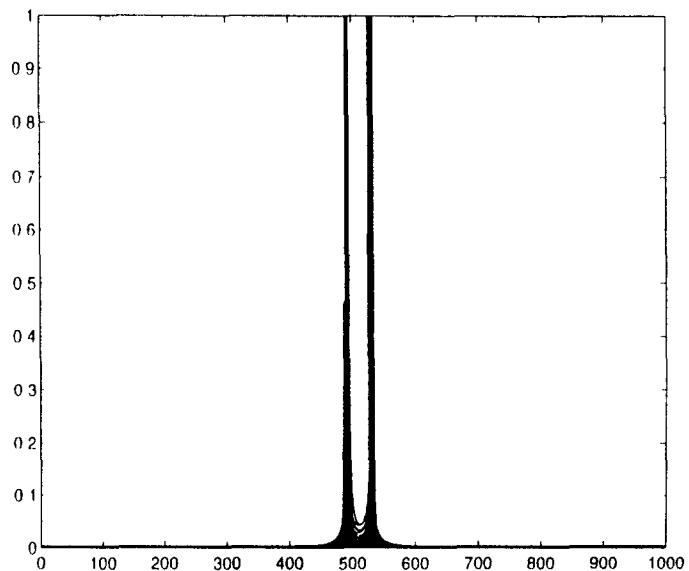

Figure 1: Spectral peaks at $f_1 = 0.5$ and $f_2 = 0.52$ with $SNR = 15dB$, resulted from applying the Rational-MUSIC, with $m = 3$ for 30 independent trials.
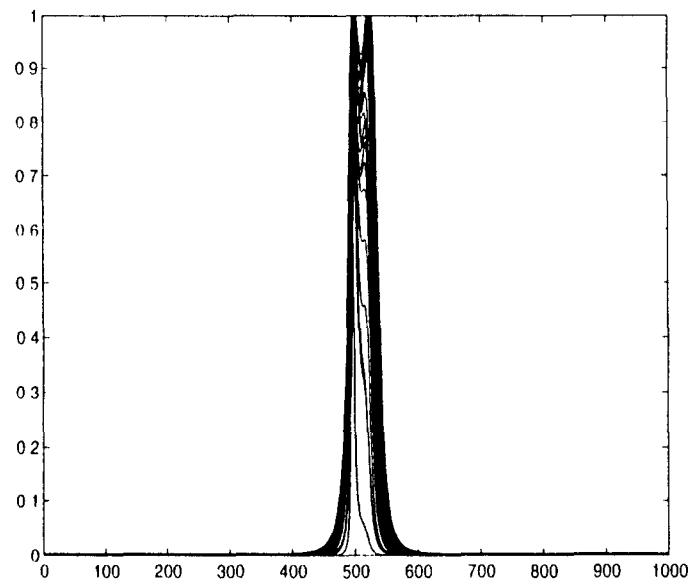

Figure 2: Spectral peaks at $f_1 = 0.5$ and $f_2 = 0.52$ with $SNR = 15dB$, resulted from applying SVD-based MUSIC on 30 independent trials with $q = 2$ and $p = 10$.