A New Suboptimal Approach to the Filtering Problem for Bilinear Stochastic Differential Systems

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Abstract

The aim of this paper is to present a new approach to the filtering problem for the class of bilinear stochastic multivariable systems, consisting in searching for suboptimal state-estimates instead of the conditional statistics. As a first result, a finite-dimensional optimal linear filter for the considered class of systems is defined. Then, the more general problem of designing polynomial finite-dimensional filters is considered. The equations of a finite-dimensional filter are given, producing a state-estimate which is optimal in a class of polynomial transformations of the measurements with arbitrarily fixed degree. Numerical simulations show the effectiveness of the proposed filter.

1 Introduction

Let us consider the class of nonlinear stochastic systems defined on some probability space, namely $(\Omega, \mathcal{F}, P)$, described by the following Ito equations:

$$dX(t) = A(t)X(t)dt + B^1(X(t), dW(t)), \tag{1.1}$$

$$dY(t) = C(t)\big(X(t)\big)dt + B^2(X(t), dW(t)), \tag{1.2}$$

where $X(t) \in \mathbb{R}^n$; $Y(t) \in \mathbb{R}^q$; $W(t) \in \mathbb{R}^p$, is a standard Wiener process with respect to some increasing family of σ-algebras, namely $\{\mathcal{F}_t\}$; $A(t), C(t)$ are matrices of proper dimensions; B^1, B^2 are bilinear forms. System (1.1), (1.2) is commonly referred in the literature as *bilinear stochastic system* (BLSS) (see for instance [4], [5], [6] and references therein).

The problem we are faced with consists in searching for finite-dimensional filters for the BLSS (1.1), (1.2). Indeed, for such system even the *linear* optimal finite-dimensional filtering problem is still an interesting one.

As well known, the optimal filter for system (1.1), (1.2), is an infinite-dimensional one. Nevertheless, from an application point of wiew, it becomes crucial to look for finite-dimensional approximations of the optimal filter, that is, finite-dimensional *suboptimal* filters showing a *better performance* with respect to the linear one. We point out that, for such system even the *linear* optimal finite-dimensional filtering problem is still an interesting one. In this paper we will derive, as an auxiliary result, the *optimal linear filter* for a BLSS in the form of (1.1), (1.2), that will result indeed to be finite-dimensional. This suboptimal approach has been recently developed for discrete-time systems in [7], [8] where a general *polynomial filter* of any arbitrarily fixed degree is defined for linear non-Gaussian systems [7] and bilinear systems [8]. The polynomial filter is able to produce recursively, the optimal state-estimate in a class of polynomials of all the currently available measurements including the linear transformations. For this reason, in a non-Gaussian setting, it represents an improvement of the classical Kalman filtering. Indeed, many numerical simulations have shown that the improvement in performance may be very large especially when noises distributions are very far from Gaussianity. In this paper we will propose this suboptimal approach for the filtering problem of continuous time BLSS's. This will allow us to define a finite-dimensional filter giving the optimal state-estimate in a suitably defined class of polynomial transformations of the measurements. The present article is a reduced version of the omonymous paper [1].

Let $T = [0\ t_M]$, and consider system (1.1), (1.2) over T, endowed with the initial conditions $X(0) = \bar{X}\ Y(0) = 0$

(these, will be understated from now on). Here, $\bar{X} \in \mathbb{R}^n$ represents an $\mathcal{F}_0$-measurable random variable, independent of W, such that: $E(\|\bar{X}\|^{2\nu}) < +\infty$, for some integer $\nu \geq 1$. We suppose that all the moments of $\bar{X}$, namely $m_{\bar{X}}^{(i)} \doteq E(\bar{X}^{[i]})$ are known for $i = 1, ..., 2\nu$. The bilinear forms in the system expression can be rewritten as

$$B^1(X(t), dW(t)) = \sum_{k=1}^{p} \big(B_k X(t) + F_k\big) dW_k(t), \qquad (1.3)$$

$$B^2(X(t), dW(t)) = \sum_{k=1}^{p} \big(D_k X(t) + G_k\big) dW_k(t), \qquad (1.4)$$

where $A(t) \in \mathbb{R}^{n\times n}$, $C(t) \in \mathbb{R}^{q\times n}$, $H(t) \in \mathbb{R}^{n\times m}$, $B_k \in \mathbb{R}^{n\times n}$, $F_k \in \mathbb{R}^n$, $D_k \in \mathbb{R}^{q\times n}$, $G_k \in \mathbb{R}^q$, for $k = 1, ..., p$, $W_k(t)$ denotes the k-th component of the standard Wiener process $W(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ is a deterministic input. In the following we shall denote with I_α, $\alpha = 0, 1, ...,$ the $\alpha \times \alpha$ identity matrix; we assume $I_0 = 1$. The following assumption will be done for the bilinear function (1.4):

Assumption 1.1. *There exists a $\bar{k}$, $1 \leq \bar{k} \leq p$, such that the matrix $D_{\bar{k}} D_{\bar{k}}^T$ is nonsingular.*

The problem we are faced with, consists in finding a finite-dimensional filter giving a state-estimate, $\widehat{X}$, such that: $\widehat{X}(t) = \Pi\big(X(t)/\mathcal{P}_t^{(\nu)}(Y)\big)$, where Π denotes the projection, and $\mathcal{P}_t^{(\nu)}(Y)$ denotes the space of the ν-th degree polynomials of process Y (see [1] for a detailed definition). We will show that there exists an augmented linear system such that the optimal linear filtering problem for it, is equivalent to the original polynomial filtering problem for system (1.1), (1.2).

2 Optimal linear filtering of BLSSs

Before treating the more general polynomial case, in this section we limit ourselves in considering the optimal linear filtering problem for the BLSS (1.1), (1.2). The reason for considering in advance this particular case is twofold. First of all, as we will see later, the polynomial case reduces to the linear one, once a suitable augmented system has been constructed. Moreover, the optimal (finite-dimensional) *linear* filtering problem for a BLSS is interesting by itself, in that it was up to now unsolved in the general case. Let $M \in \mathbb{R}^{\alpha\times\alpha}$ a symmetric positive semidefinite matrix, such that $\text{rank}(M) = \rho \leq \alpha$. As well known, there exists a full rank matrix $N \in \mathbb{R}^{\alpha\times\rho}$ such that $NN^T = M$. We will use the following notation: $M^{(\frac{1}{2})} \doteq N$, that is a "rectangular square root" of the matrix M. Note that, by definition, the matrix ${M^{(1/2)}}^T M^{(1/2)}$ is nonsingular. Let us denote $m_x(t) = E(X(t))$, $\Psi_X(t) = \text{cov}(X(t), X(t))$, the state-mean and covariance respectively. Moreover, let us denote $\bar{m}_x = E(\bar{X})$ and $\bar{\Psi}_X = \text{cov}(\bar{X}, \bar{X})$.

Theorem 2.1. *Let us consider the system* (1.1), (1.2). *Suppose that the matrix $\Psi_X(t)$ is nonsingular for any $t \in T$. Let us consider, for $k = 1, ..., p$, the integers $\rho_k \leq n$, $\sigma_k \leq q$ such that:*

$$\begin{aligned} \rho_k &\doteq \text{rank}\Big\{B_k \cdot \Psi_X(t) \cdot {B_k}^T\Big\}, \\ \sigma_k &\doteq \text{rank}\Big\{D_k \cdot \Psi_X(t) \cdot {D_k}^T\Big\}, \end{aligned} \qquad \forall t \in T.$$

Then there exists the following representation:

$$dX(t) = A(t)X(t)dt + H(t)u(t) + \sum_{k=1}^{2p} \tilde{B}_k(t) d\tilde{W}_{k,1}(t),$$

$$dY(t) = C(t)X(t)dt + \sum_{k=1}^{2p} \tilde{D}_k(t) d\tilde{W}_{k,2}(t),$$

where, for $k = 1, ..., p$: $\tilde{B}_k(t) \in \mathbb{R}^{n\times\rho_k}$ and $\tilde{D}_k(t) \in \mathbb{R}^{n\times\sigma_k}$ are given by

$$\tilde{B}_k(t) \doteq \Big(B_k \cdot \Psi_X(t) \cdot B_k^T\Big)^{(\frac{1}{2})}, \ \tilde{D}_k(t) \doteq \Big(D_k \cdot \Psi_X(t) \cdot {D_k}^T\Big)^{(\frac{1}{2})},$$

for $k = p+1, ..., 2p$:

$$\begin{aligned} \tilde{B}_k(t) &\doteq B_{k-p} E\big(X(t)\big) + F_{k-p}, \\ \tilde{D}_k(t) &\doteq D_{k-p} E\big(X(t)\big) + G_{k-p}. \end{aligned}$$

For $i = 1, 2$, the set $\{\tilde{W}_{k,i},\ k = 1, ..., 2p\}$ is a set of $2p$ mutually uncorrelated standard WSW processes. In particular, for $k = 1, ..., p$, $\tilde{W}_{k,1}(t) \in \mathbb{R}^{\rho_k}$, $\tilde{W}_{k,2}(t) \in \mathbb{R}^{\sigma_k}$; for $k = p+1, ..., 2p$: $\tilde{W}_{k,1}(t) = \tilde{W}_{k,2}(t) = W_{k-p}(t)$.

Proof. See [1] •

A sufficient condition which guarantees the nonsingularity of $\Psi_X(t)$ can be found in the time-invariant case ([1]). In the following theorem the optimal linear filter for a stationaty BLSS is defined.

Theorem 2.2. *Let be given the time-invariant BLSS as defined in equations* (1.1), (1.2), *where all matrices are constant with time. Let us suppose that* $\text{rank}(D_k) = q$ *or* $\text{rank}(G_k) = q$ *for some k. Then, the optimal linear estimate of the state process X, namely $\hat{X}$, and the error covariance $P(t) = \text{cov}\big(X(t) - \hat{X}(t)\big)$, satisfy the* *following system of equations:*

$$
\begin{aligned}
\frac{dm_X(t)}{dt} &= Am_X(t) + Hu(t), \qquad m(0) = \bar{m},\\
\frac{d\Psi_X(t)}{dt} &= A\Psi_X(t) + \Psi_X(t)A^T + \sum_{k=1}^{p} B_k\Psi_X(t)B_k^T\\
&\quad + \sum_{k=1}^{p}(B_k m_X(t) + F_k)(B_k m_X(t) + F_k)^T,\\
\Psi_X(0) &= \bar{\Psi}_X,\\
\tilde{B}_k(t) &= \begin{cases} \left(B_k \cdot \Psi_X(t) \cdot B_k^T\right)^{(\frac{1}{2})}, & 1 \le k \le p\\ B_{k-p}m_X(t) + F_{k-p}, & p+1 \le k \le 2p\end{cases}\\
\tilde{D}_k(t) &= \begin{cases} \left(D_k \cdot \Psi_X(t) \cdot D_k^T\right)^{(\frac{1}{2})}, & 1 \le k \le p\\ D_{k-p}m_X(t) + G_{k-p}, & p+1 \le k \le 2p\end{cases}\\
R(t) &= \sum_{i=1}^{2p} \tilde{D}_i(t)\tilde{D}_i(t)^T\\
\frac{dP(t)}{dt} &= AP(t) + P(t)A^T + R(t) + \Lambda_{2p}(t)R(t)^{-1}\Lambda_{2p}(t)^T,\\
\Lambda_k(t) &= \sum_{i=1}^{k} \tilde{B}_i(t)\tilde{D}_i(t)^T + P(t)C^T,\\
d\hat{X}(t) &= A\hat{X}(t)dt + \Lambda_{2p+1}(t)R(t)^{-1}(dY(t) - C\hat{X}(t)dt),\\
P(0) &= \bar{\Psi}_X, \quad \hat{X}(0) = \bar{m},
\end{aligned}
$$

Proof. See [1] •

3 Optimal Polynomial Filtering of BLSSs

In order to write down the polynomial filter equations, the following important tool is needed (vector Ito formula).

Theorem 3.1. *Let $(X_t, \mathcal{F}_t)$ be a vector continuous semimartingale in $\mathbb{R}^n$ described by the Ito's stochastic differential:*

$$dX_t = d\beta_t + dM_t,$$

where $(\beta_t, \mathcal{F}_t)$ is an a.s. continuous bounded variation process and $(M_t, \mathcal{F}_t)$ is a square integrable martingale. Let $F : \mathbb{R}^n \to \mathbb{R}^p$, be a continuous function endowed with the first and second derivatives. Then the process $Z_t = F(X_t)$ is a square integrable semimartingale, whose differential is given by

$$dZ_t = \left(\partial_x \otimes F(x)\right)_{x=X_t} dX_t + \frac{1}{2}\left(\partial_x^{[2]} \otimes F(x)\right)_{x=X_t} (dM_t)^{[2]},$$

with $(dM_t)^{[2]}$ denoting the associate quadratic variation process whose arguments are

$$(dM_t)^{[2]} = \begin{bmatrix} d\langle M_1, M_1\rangle_t\\ d\langle M_1, M_2\rangle_t\\ \vdots\\ d\langle M_n, M_n\rangle_t \end{bmatrix},$$

and ∂_x denotes the differential operator d/dx.

Proof. See [1] •

Using the vector Ito formula we can state the following Theorem, which defines the stochastic differential for the power process of the solution of a bilinear stochastic differential equation (SDE).

Theorem 3.2. *Let $\phi(t) \in \mathbb{R}^d$ the process defined by the following SDE:*

$$d\phi(t) = (\Gamma(t)\phi(t) + \gamma(t))dt + \sum_{k=1}^{p}(\Theta_k\phi(t) + \chi_k)dW_k(t),$$

where, $\Gamma(t), \Theta_k \in \mathbb{R}^{d\times d}$, $\gamma(t), \chi_k \in \mathbb{R}^d$. It results, for $i \ge 2$:

$$
\begin{aligned}
d\phi^{[i]}(t) &= \left(\mathcal{M}_i^0(t)\phi^{[i]}(t) + \mathcal{M}_i^1(t)\phi^{[i-1]}(t) + \mathcal{M}_i^2\phi^{[i-2]}(t)\right)dt\\
&\quad + \sum_{k=1}^{p}\left(\mathcal{G}_{k,i}^0\phi^{[i]}(t) + \mathcal{G}_{k,i}^1\phi^{[i-1]}(t)\right)dW_k(t),
\end{aligned}
$$

where $\mathcal{G}_{\cdot,\cdot}^{\cdot}$ and $\mathcal{M}_{\cdot}^{\cdot}$ are suitably defined matrices.

Proof. See [1] •

By means of a repeated application of Theorem 3.2, we can show that the process (X, Y) of the BLSS and its powers up to a certain degree, represents a solution of a suitably defined bilinear SDE. This will be next transformed into a linear system with WSW diffusions, generating the powers of the observation Y up to the required degree (the augmented system).

Let $x \in \mathbb{R}^d$ and h a positive integer. We recall that, the following relations hold, linking together the *reduced h-th Kroneker power of x* [7], [9], namely $x_{[h]}$ and the (ordinary) h-th Kronecker power $x^{[h]}$: $x^{[h]} = T_d^h x_{[h]}$; $x_{[h]} = \tilde{T}_d^h x^{[h]}$, where T_d^h and $\tilde{T}_d^h$ are suitably dimensioned transformation matrices [7]. Let us define the process Z as: $Z(t) \doteq \left[\, Y(t)^T \quad X(t)^T \,\right]^T$, and let $\delta = dim(Z)$. Moreover let us define the augmented process:

$$\mathcal{Z}(t) \doteq \left[\, Z(t)^T \quad \dots \quad Z_{[\nu]}(t)^T \,\right]^T.$$

First of all, we derive a SDE for the process $\mathcal{Z}$. For, note that Z satisfies the following SDE:

$$dZ(t) = \left(\tilde{A}(t)Z(t) + \alpha(t)\right)dt + \sum_{k=1}^{p}\left(B_k Z(t) + \tilde{\beta}_k\right)dW_k(t),$$

where $\tilde{A}, \alpha, B_k$ have subsequent definition. Next, an application of Theorem 3.2 to the process Z, results in a kind of SDE as the following:

$$
\begin{aligned}
dZ^{[i]}(t) &= \left(L_i^0(t)Z^{[i]}(t) + L_i^1(t)Z^{[i-1]}(t) + L_i^2 Z^{[i-2]}(t)\right)dt\\
&\quad + \sum_{k=1}^{p}\left(V_{k,i}^0 Z^{[i]}(t) + V_{k,i}^1 Z^{[i-1]}(t)\right)dW_k(t),
\end{aligned}
$$

Using the latter, and observing that, $Z^{[i]} = T_\delta^i Z_{[i]}$, $Z_{[i]} = \tilde{T}_\delta^i Z^{[i]}$, we can state the following proposition.

Proposition 3.3. *The process $\mathcal{Z}$ satisfies the following bilinear SDE,*

$$d\mathcal{Z}(t) = \big(\mathcal{A}(t)\mathcal{Z}(t)+\mathcal{U}(t)\big)dt+\sum_{k=1}^{p}\big(\mathcal{B}_k\mathcal{Z}(t)+\mathcal{V}_k\big)dW_k(t),$$

where

$$\mathcal{A}(t) = \begin{bmatrix} \tilde{A}(t) & 0 & \dots & 0 \\ L_2^1(t) & \tilde{T}_\delta^2 L_2^0(t)T_\delta^2 & \ddots & \\ & \ddots & \ddots & \\ \dots & \tilde{T}_\delta^\nu L_\nu^2 T_\delta^{\nu-2} & \tilde{T}_\delta^\nu L_\nu^1(t)T_\delta^{\nu-1} & \tilde{T}_\delta^\nu L_\nu^0(t)T_\delta^\nu \end{bmatrix},$$

$$\mathcal{B}_k = \begin{bmatrix} \tilde{B}_k & 0 & \dots & 0 \\ V_{k,2}^1 & \tilde{T}_\delta^2 V_{k,2}^0 T_\delta^2 & & \vdots \\ \vdots & & \ddots & \ddots \\ 0 & \dots & \tilde{T}_\delta^\nu V_{k,\nu}^1 T_\delta^{\nu-1} & \tilde{T}_\delta^\nu V_{k,\nu}^0 T_\delta^\nu \end{bmatrix},$$

and $\mathcal{U}(t) = \begin{bmatrix}\alpha(t)^T & L_2^{2\,T} & 0 & \dots & 0\end{bmatrix}^T$, $\mathcal{V}_k = \begin{bmatrix}\beta_k{}^T & 0 & \dots & 0\end{bmatrix}^T$.

Now, we can use Theorem 2.1 in order to rewrite the bilinear SDE of Proposition 3.3 in the form of a *linear* SDE with WSW diffusion term. The underlying hypothesis is that the covariance matrix of the process $\mathcal{Z}$, namely $\Phi_Z(t)$, is uniformly nonsingular over T.

Proposition 3.4. *Let ρ_k, $k = 1,...,p$, be the ranks of the matrices $\mathcal{B}_k$. Then the process $\mathcal{Z}$ satisfies the following SDE,*

$$d\mathcal{Z}(t) = \big(\mathcal{A}(t)\mathcal{Z}(t)+\mathcal{U}(t)\big)dt+\sum_{k=1}^{2p}\tilde{\mathcal{B}}_k(t)d\tilde{W}_k(t), \quad (3.1)$$

where $\tilde{W}_k$, $k = 1,...,2p$ are independent standard WSW processes, $\tilde{W}_k \in \mathbb{R}^{\rho_k}$, for $k = 1,...,p$, $\tilde{W}_k = W_k \in \mathbb{R}$, for $k = p+1,...,2p$, and

$$\tilde{\mathcal{B}}_k(t) \doteq \begin{cases} \big(\mathcal{B}_k\Psi_Z(t)\mathcal{B}_k^T\big)^{(\frac{1}{2})}, & 1 \le k \le p \\ \mathcal{B}_{k-p}m_z(t)+\mathcal{V}_{k-p}, & p+1 \le k \le 2p \end{cases}$$

with $m_z = E(\mathcal{Z})$.

In order to write down the equations of the *augmented system* we need to split out the vector SDE of Proposition 3.4 into two SDE's: one for the *observed* components of $\mathcal{Z}$ and the other one for the remaining entries. From the definition of Z, we see that the components of the vector $\mathcal{Z}$ are of the form: $X_1^{i_1}\cdots X_n^{i_n}\cdots Y_1^{j_1}\cdots Y_q^{j_q}$, where X_l, Y_l denote the lth component of vectors X, Y respectively, and $0 \le i_l, j_r \le \nu$ for $l = 1,...,n$, $r = 1,...,q$, $\sum_{l=1}^n i_l \le \nu$, $\sum_{r=1}^q j_r \le \nu$. The observed components are those with $i_1 = ... = i_n = 0$. Denote by $\mathcal{Y}$ the vector of all such components: $\mathcal{Y} \doteq \begin{bmatrix}Y^T & \dots & Y_{[\nu]}{}^T\end{bmatrix}^T$. Moreover, let us denote by $\mathcal{E}_\mathcal{Y}$ the $(0,1)$-matrix such that $\mathcal{Y} = \mathcal{E}_\mathcal{Y}\mathcal{Z}$. Let us denote with $\mathcal{X}$ the aggregate vector of all the components in $\mathcal{Z}$ which are not components of $\mathcal{Y}$. Then it results well defined the $(0,1)$-matrix $\mathcal{E}_\mathcal{X}$ such that $\mathcal{X} = \mathcal{E}_\mathcal{X}\mathcal{Z}$. Then, the aggregate matrix $\mathcal{I}$: $\mathcal{I} \doteq \begin{bmatrix}\mathcal{E}_\mathcal{Y}{}^T & \mathcal{E}_\mathcal{X}{}^T\end{bmatrix}^T$, results to be invertible. Let us consider the matrices $\mathcal{I}_1, \mathcal{I}_2$ such that $\mathcal{Z} = \mathcal{I}_1\mathcal{Y} + \mathcal{I}_2\mathcal{X}$. Note that the matrices $\mathcal{I}_1, \mathcal{I}_2$ are obtained by means of a suitable partition of the matrix $\mathcal{I}^{-1} = [\mathcal{I}_1\ \mathcal{I}_2]$. We can now state the following proposition.

Proposition 3.5. *The processes $\mathcal{X}$, $\mathcal{Y}$ satisfy the following pair of SDE's (augmented system):*

$$\begin{aligned} d\mathcal{X}(t) &= \big(\mathcal{A}_1(t)\mathcal{Y}(t) + \mathcal{A}_2(t)\mathcal{X}(t) + \mathcal{U}_1(t)\big)dt \\ &\quad + \sum_{k=1}^{2p}\mathcal{B}_k^1(t)d\tilde{W}_k(t), \\ d\mathcal{Y}(t) &= \big(\mathcal{C}_1(t)\mathcal{Y}(t) + \mathcal{C}_2(t)\mathcal{X}(t) + \mathcal{U}_2(t)\big)dt \\ &\quad + \sum_{k=1}^{2p}\mathcal{D}_k^1(t)d\tilde{W}_k(t), \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_1(t) &= \mathcal{E}_\mathcal{X}\mathcal{A}(t)\mathcal{I}_1, & \mathcal{A}_2(t) &= \mathcal{E}_\mathcal{X}\mathcal{A}(t)\mathcal{I}_2, \\ \mathcal{U}_1(t) &= \mathcal{E}_\mathcal{X}\mathcal{U}(t), & \mathcal{U}_2(t) &= \mathcal{E}_\mathcal{Y}\mathcal{U}(t), \\ \mathcal{C}_1(t) &= \mathcal{E}_\mathcal{Y}\mathcal{A}(t)\mathcal{I}_1, & \mathcal{C}_2(t) &= \mathcal{E}_\mathcal{Y}\mathcal{A}(t)\mathcal{I}_2, \\ \mathcal{B}_k^1(t) &= \mathcal{E}_\mathcal{X}\tilde{\mathcal{B}}_k(t), & \mathcal{D}_k^1(t) &= \mathcal{E}_\mathcal{Y}\tilde{\mathcal{B}}_k(t), \end{aligned}$$

$\mathcal{A}$, $\tilde{\mathcal{B}}_k$, $\mathcal{U}$, are the matrix coefficients of eq. (3.1)*, and $\{\tilde{W}_k,\ k = 1,...,2p\}$ is a set of mutually uncorrelated standard WSW processes.*

Proposition 3.5 states that the augmented observation process $\mathcal{Y}$ can be generated as the output process of an augmented system. This implies that the problem of finding the ν-th degree polynomial filter for the original system (1.1), (1.2) is now reduced to an *optimal linear filtering* problem for a *linear* system. In [2] the optimal linear filter is defined for the class of linear stochastic systems whose noise terms are represented by WSW processes. The augmented system comes within this class of systems, and we can use here the same approach as in [2] in order to obtain the optimal linear filter with respect to the augmented observation process $\mathcal{Y}$. In order to do this, we need to show the uniform nonsingularity in T of the output-noise covariance of the output augmented equation, namely $\mathcal{R}(t) \doteq \sum_{k=1}^{2p}\mathcal{D}_k^1(t)\mathcal{D}_k^1(t)^T$. Indeed, the uniform nonsingularity of the output-covariance is required, in order to apply the Kalman-Bucy scheme [KB].

Theorem 3.6. *The noise covariance matrix function* *of the augmented measurement equation is uniformly nonsingular over T.*

Proof. See [1] •

Now, we can prove the main Theorem, defining the ν-th degree polynomial filter for system (1.1), (1.2). We remind readers that, ρ_k is the dimension of the WSW process $\tilde{W}_k$ when $k = 1, ..., p$, and for $k = p+1, ..., 2p$, $\tilde{W}_k = W_k \in \mathbb{R}$. Let us denote with γ the dimension of the augmented process $\mathcal{Z}$. Moreover, we shall denote with $\mathrm{cov}(\chi, \eta)$ the cross-covariance between two random variables χ, η. Finally, we shall denote with $M^\dagger$ the Moore-Penrose pseudoinverse of the square matrix M.

Theorem 3.7. *The ν-th order polynomial filter for system* (1.1), (1.2) *is described by the following system of equations:*

$$\frac{dm_z(t)}{dt} = \mathcal{A}(t)m_z(t) + \mathcal{U}(t),$$
$$\bar{\mathcal{B}}_k(t) = \mathcal{B}_k m_z(t) + \mathcal{V}_k \quad 1 \le k \le p,$$
$$\frac{d\Psi_Z(t)}{dt} = \mathcal{A}(t)\Psi_Z(t) + \Psi_Z(t)\mathcal{A}(t)^T + \sum_{k=1}^{p} \mathcal{B}_k \Psi_Z(t)\mathcal{B}_k^T + \sum_{k=1}^{p} \bar{\mathcal{B}}_k(t)\bar{\mathcal{B}}_k(t)^T,$$
$$\tilde{\mathcal{B}}_k(t) = \left(\mathcal{B}_k \Psi_Z(t)\mathcal{B}_k^T\right)^{(\frac{1}{2})} \quad 1 \le k \le p,$$
$$\mathcal{J}(t) = \sum_{k=1}^{p} \mathcal{E}_{\mathcal{X}}\left(\tilde{\mathcal{B}}_k(t)\tilde{\mathcal{B}}_k(t)^T + \bar{\mathcal{B}}_k(t)\bar{\mathcal{B}}_k(t)^T\right)\mathcal{E}_{\mathcal{Y}}^T,$$
$$\mathcal{R}(t) = \sum_{k=1}^{p} \mathcal{E}_{\mathcal{Y}}\left(\tilde{\mathcal{B}}_k(t)\tilde{\mathcal{B}}_k(t)^T + \bar{\mathcal{B}}_k(t)\bar{\mathcal{B}}_k(t)^T\right)\mathcal{E}_{\mathcal{Y}}^T,$$
$$\mathcal{Q}(t) = \sum_{k=1}^{p} \mathcal{E}_{\mathcal{X}}\left(\tilde{\mathcal{B}}_k(t)\tilde{\mathcal{B}}_k(t)^T + \bar{\mathcal{B}}_k(t)\bar{\mathcal{B}}_k(t)^T\right)\mathcal{E}_{\mathcal{X}}^T,$$
$$\frac{d\mathcal{P}(t)}{dt} = \mathcal{A}_2(t)\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A}_2(t)^T + \mathcal{Q}(t) - \Theta(t)\mathcal{R}(t)^{-1}\Theta(t)^T,$$
$$d\hat{\mathcal{X}}(t) = \left(\mathcal{A}_1(t)\mathcal{Y}(t) + \mathcal{A}_2(t)\hat{\mathcal{X}}(t) + \mathcal{U}_1(t)\right)dt + \Theta(t)\mathcal{R}(t)^{-1}d\tilde{\mathcal{Y}}(t),$$
$$d\tilde{\mathcal{Y}}(t) = \left(d\mathcal{Y}(t) - \mathcal{C}_1(t)\mathcal{Y}(t) + \mathcal{C}_2(t)\hat{\mathcal{X}}(t) + \mathcal{U}_2(t)\right)dt$$
$$\Theta(t) = \mathcal{J}(t) + \mathcal{P}(t)\mathcal{C}_2(t)^T, \quad \hat{X}(t) = \mathcal{T}_\nu \hat{\mathcal{X}}(t),$$

where $\mathcal{T}_\nu$ is the operator extracting the first n entries of a vector. The initial conditions are:

$$m_z(0) = E(\mathcal{X}(0)), \quad \Psi_Z(0) = \mathrm{cov}(\mathcal{X}(0), \mathcal{X}(0)),$$
$$\hat{\mathcal{X}}(0) = E(\mathcal{X}(0)) + \Psi_{\mathcal{X}\mathcal{Y}}(0)\Psi_{\mathcal{Y}}(0)^\dagger\left(\mathcal{Y}(0) - E(\mathcal{Y}(0))\right),$$
$$\mathcal{P}(0) = \Psi_Z(0) - \Psi_{\mathcal{X}\mathcal{Y}}(0)\Psi_{\mathcal{Y}}(0)^\dagger\Psi_{\mathcal{X}\mathcal{Y}}(0)^T,$$

where
$\Psi_{\mathcal{X}\mathcal{Y}}(t) \doteq \mathrm{cov}(\mathcal{X}(t), \mathcal{Y}(t))$, $\Psi_{\mathcal{Y}}(t) \doteq \mathrm{cov}(\mathcal{Y}(t), \mathcal{Y}(t))$

Proof. See [1] •

4 Simulation example

In order to test the algorithm defined in Theorem 3.7, the filtering problem for the following second order system has been considered:

$$dX(t) = AX(t)dt + BX(t)dW(t) + UdN(t), \tag{4.1}$$
$$dY(t) = CX(t)dt + DX(t)dV(t), \tag{4.2}$$

where

$$A = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \quad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$
$$C = [1 \quad 1], \qquad D = [g \quad 0],$$

$X(0) = Y(0) = 0$, and W, N, V are mutually independent scalar Wiener processes. The well known extended Kalman filter (EKF) was up to now the classical tool for the filtering of a nonlinear system in the form of (4.1), (4.2). However, nothing is known about the working conditions the performances of the EFK. In the present case, for instance, the EKF does not work at all. Indeed the state-expectation is zero and hence the state process is expected to cross the zero. Since the term $D\hat{X}(t)\hat{X}(t)^T D^T$ (where $\hat{X}$ denotes the EKF estimation) needs to be inverted in the EKF equations, we should expect a failure of the alghoritm. This really happens in the simulations we carried out for several values of the parameters $a_1, a_2, b_1, b_2, u_1, u_2, g$. We have always observed a sudden and strong deviation to infinity. The linear, quadratic, and cubic filters have been built up. We show below our simulation results for the linear and cubic filters, with the following values of the parameters: $a_1 = -0.01$, $a_2 = -0.5$, $u_1 = 30$, $u_2 = 2$, $b_1 = 0.1$, $b_2 = 0.1$, $g = 0.1$. We do not show graphs for the quadratic filter simulation because, in our case, the quadratic filter does not show any valuable improvement with respect to the linear case. Differently to the EKF, for the polynomial filters we are able to compute the *a priori* state-estimate error variances, that are entries of the matrix $\mathcal{P}(t)$, that is: $\mathcal{P}_{1,1}(t)$, $\mathcal{P}_{2,2}(t)$ for the first and second state components respectively. In our example these values are growing with time. The reason of this is that system (4.1), (4.2), is unstable with the chosen values. Nevertheless, as expected the time-evolution of $\mathcal{P}_{1,1}(t)$ for the cubic filter (namely $P_C(t)$) is ever less than the $\mathcal{P}_{1,1}(t)$ for the linear filter (namely $P_L(t)$). We observed that the evolution of the ratio $\rho(t) = P_L(t)/P_C(t)$ stabilizes over the value $\bar{\rho} = 1.30$. Hence, the improvement in the a priori performance of the cubic filter with respect to the linear one can be considered almost of 30%.

The time-evolutions of filtered paths, for the linear and cubic filters, compared with the corresponding true 1-th component state path, are reported in figs. 1,2. As we can see, even a visual comparision between the signal time-evolutions show a valuable improvement in the estimation quality of the cubic filter with respect to the linear one. Several Monte Carlo runs have been carried out. For each one of these runs, the ratio, namely ρ_S, between the sampled error variances of the linear and cubic filters has been computed. We have chosen the paths with $\rho_S = 1.35$.

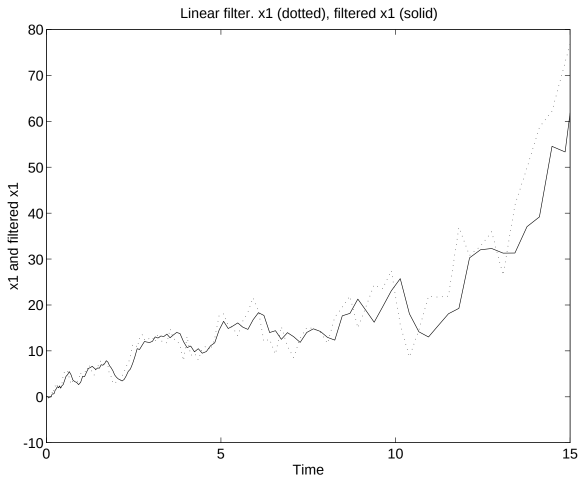

Fig. 1.

The simulation of the EKF confirms also in this case its unsatisfactory behaviour. Indeed, after almost 0.01 time units the EKF estimate starts up and quickly goes to infinity.

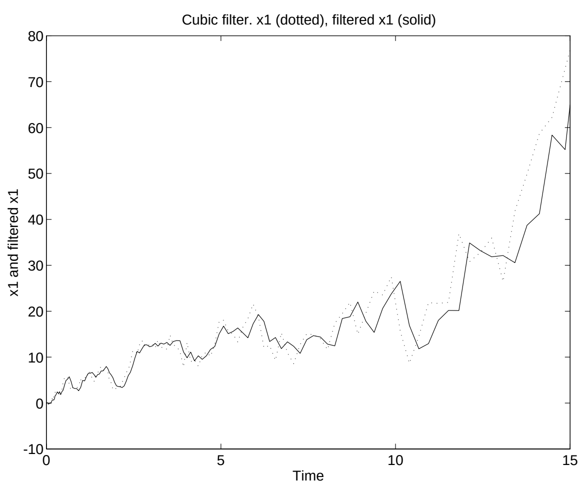

Fig. 2.

5 Conclusions

Theorem 3.7 defines a finite-dimensional filter for the BLSS (1.1), (1.2) which is optimal in a class of polynomial estimates. The considered class includes the linear estimates, and moreover it defines a not decreasing sequence of spaces for increasing polynomial degree. This implies that the polynomial filter had to improve the estimation performance for increasing polynomial degree. We underline that the proposed filter is finite-dimensional. Of course, it is always possible to approximate the optimal filter with an arbitrary approximation degree. However, the more accurate the approximation level is chosen, the heavier the computational burden of the algorithm is. The computational effort is prohibitive even for small approximation degrees. Moreover, it has no sense, within this approach, to use a large approximation degree in order to make really implementable the filtering algorithm. Counterwise, our suboptimal approach allows to get meaningful estimates also for small polynomial degrees, which does not present difficult implementation problems. In the numerical simulation a second-order BLSS has been filtered using the polynomial filters up to the third degree. The EFK has been also simulated, however its performance is resulted to be unsatisfactory at all. The simulation results show that the estimation quality really improves as polynomial degree grows, and for the cubic filter we obtained an improvement valuable over 30% with respect to the linear filter.

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