ASYMPTOTIC STABILITY AND ENERGY DECAY RATES FOR SOLUTIONS OF THE WAVE EQUATION WITH MEMORY

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Abstract

We study the asymptotic stability and give the energy decay rates for solutions of the wave equation with boundary dissipation of the memory type.

1 Introduction

A basic linear model for the evolution of sound in a compressible fluid is the system of partial differential equations

$$\rho v_t(t,x) + grad\, p(x,t) = 0 \tag{1.1}$$

$$\alpha p_t(t,x) + div\, v(t,x) = 0, \quad t>0, \ x \in \mathbf{R}^n, \tag{1.2}$$

where p denotes acoustic pressure and v the velocity field; cf., e.g., Leis [9]. In the sequel the equilibrium density ρ and the compressibility α will be assumed to be constant and then without loss of generality to be equal to 1. Eliminating v from this system one obtains a wave equation for the pressure p

$$p_{tt}(t,x) = \Delta p(t,x), \quad t>0, \ x \in \mathbf{R}^n. \tag{1.3}$$

When the fluid is enclosed in a region $\Omega \subset \mathbf{R}^n$, (1.3) has to be supplemented by conditions at $\partial\Omega$, the boundary of Ω. The following three dissipative boundary conditions are discussed in the mathematical literature on time domain models for acoustics.

Firstly, equating the acoustic impedance $\xi(x) \in \mathbf{C}$ of the boundary surface at x with the ratio between the fluid's pressure and its velocity normal to the surface results in

$$\frac{\partial p}{\partial \nu}(t,x) + \xi(x) p_t(t,x) = 0 \quad t>0, \ x \in \partial\Omega \tag{1.4}$$

where $\partial \frac{\partial}{\partial \nu}$ denotes the derivative in the direction of the outer normal of $\partial\Omega$. The well-posedness of the equation with boundary conditions (1.4) and the asymptotic behaviour of its solutions has been investigated in [3, 4, 15].

Secondly, adding a friction term $\beta(x)p_t(t,x)$, $\beta > 0$, to the classic Robin condition, yields, for $t>0$ and $x \in \partial\Omega$,

$$\frac{\partial p}{\partial \nu}(t,x) + \beta(x)p_t(t,x) + \alpha(x)p(x,t) = 0. \tag{1.5}$$

This condition has been studied, e.g., in [8].

Thirdly, modelling the boundary surface as independent oscillations and equating the velocity δ_t of the impenetrable surface with the normal velocity of the fluid at boundary points, leads to

$$n(x)\delta_{tt}(t,x) + d(x)\delta_t(t,x) + k(x)\delta(t,x) = -p(t,x)$$

and

$$\frac{\partial p}{\partial \nu}(t,x) + \delta_{tt}(t,x) = 0, \quad t>0, \ x \in \partial\Omega. \tag{1.6}$$

In [2], where this boundary model is formulated for the velocity potential, spectral properties of the generator of the solution semigroup are given.

Looking for more general results, we find in [11], equation (6.3.11), that the pressure of the combination of a wave $F(T_i)$, $T_i = t-(x_1 \sin\theta - x_2\cos\theta)$, that is incident at angle θ onto the surface $x_2 = 0$, with the reflected wave in direction $T_r - t = -(x_1\cos\theta + x_2\cos\theta)$, is of the form

$$p = F(T_i) + F(T_r) + \int_{-\infty}^{+\infty} F(\tau) W(T_r - \tau)\, d\tau. \tag{1.7}$$

Here W represents the modification of the reflected wave that is caused by the motion of the surface. This means that a general linear reflection process is to be modelled by convolution of the acoustic wave with a function that characterizes the boundary material. In order to cast (1.4)-(1.6) into a common form, we write

$$\frac{\partial p}{\partial \nu}(t,x) + dk * p_t(t,x) = 0, \quad t > 0, \; x \in \partial\Omega \tag{1.8}$$

where $dk * p_t(t,x) = \int_0^t dk(\tau,x) p_t(t-\tau,x)$.

Another approach that also leads to convolution boundary conditions of the form (1.8) is the modelling of the boundary as surface of a viscoelastic material.

In this paper, we are concerned more precisely in the asymptotic behavior of solutions to the following problem

$$u'' - \Delta u = 0 \quad \text{in } \Omega \times \mathbf{R}_+, \tag{1.9}$$

$$u = 0 \quad \text{on } \Gamma_0 \times \mathbf{R}_+, \tag{1.10}$$

$$\frac{\partial u}{\partial \nu} + \int_0^t k(t-s,x) u'(s)\, ds + a(x) g(u') = 0, \tag{1.11}$$

on $\Gamma_1 \times \mathbf{R}_+$, and

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad x \in \Omega, \tag{1.12}$$

where $\Omega \subset \mathbf{R}^n$ is an open bounded domain with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2, $a : \Gamma_1 \to \mathbf{R}_+ \in L^\infty(\Gamma_1)$ is such that $a(x) \geq a_0 > 0$, $k : \mathbf{R}_+ \times \Gamma_1 \to \mathbf{R}_+ \in C^2(\mathbf{R}_+, L^\infty(\Gamma_1))$ and $g : \mathbf{R} \to \mathbf{R}$ is a continuous nondecreasing function such that $g(0) = 0$ and $|g(x)| \leq 1 + C|x|, \; C > 0$.

Problems related to (1.9)-(1.12) were studied by many authors, e.g., Muñoz-Rivera [12], Tadayuki-Rinko [17], Prüss [14], Dix-Torrejon [5], Guesmia [6], Propst-Prüss [13], Renardy et al. [16] and the references therein.

Our paper is organized as follows. In Section 2, we state our main results and in Section 3 we give an idea of the proofs.

2 Main Results

The following hypotheses are made on Ω and on the functions k and g:

$$\Gamma_0 \neq \emptyset, \quad \inf_{\Gamma_1 \times \mathbf{R}_+} k \neq 0, \tag{2.1}$$

$$m \cdot \nu \geq \delta > 0 \quad \text{on } \Gamma_1, \quad m \cdot \nu \leq 0 \quad \text{on } \Gamma_0, \tag{2.2}$$

where $m(x) = x - x^0 (x^0 \in \mathbf{R}^n)$,

$$k \geq 0 \quad \text{on } \Gamma_1 \times \mathbf{R}_+, \tag{2.3}$$

$$k' \leq 0 \quad \text{on } \Gamma_1 \times \mathbf{R}_+, \tag{2.4}$$

$$\exists \alpha > 0 \quad k'' \geq -\alpha k' \quad \text{on } \Gamma_1 \times \mathbf{R}_+, \tag{2.5}$$

$$C_1 |x|^p \leq |g(x)| \leq C_2 |x|^{1/p} \quad \text{if } |x| \leq 1, \tag{2.6}$$

$$C_3 |x| \leq |g(x)| \leq C_4 |x| \quad \text{if } |x| \geq 1, \tag{2.7}$$

where $p \geq 1$ and $C_i \, (1 \leq i \leq 4)$ are four positive constants.

Remarks 2.1. (a) An example of function k satisfying (2.3)-(2.5) is $k(t,x) = f(x)\, e^{-\alpha t} + g(x)$ on $\Gamma_1 \times \mathbf{R}_+$, where $f, g \in L^\infty(\Gamma_1, \mathbf{R}_+)$.

(b) The condition (2.1) implies that the formula $\int_\Omega |\nabla u|^2\, dx + \int_{\Gamma_1} k\, |u|^2\, d\Gamma$ defines a norm on $H^1(\Omega)$ equivalent to the usual one.

(c) The condition (2.2) implies that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, and Ω is star-shaped with respect to x^0. It could be weakened as it was done in [8].

For the sake of completeness, we give a brief outline of the well-posedness of problem (1.9)-(1.12). This problem could be rewritten in the following form

$$\langle u'', v \rangle + \int_0^t d\beta\,(t-\tau, u'(\tau), v) + \alpha(u,v) + \int_{\Gamma_1} a(x) g(u') v \, d\Gamma = 0 \tag{2.8}$$

$$u(0) = u_0; \quad u'(0) = u_1 \tag{2.9}$$

where, for $t > 0$ and

$$u, v \in H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega);\; u = 0 \;\text{ on }\; \Gamma_0 \right\},$$

$$\beta(t,u,v) = \int_\Gamma k(t,x) u(x) v(x)\, d\Gamma,$$

$$\beta(0,u,v) = 0,$$

$$\alpha(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad u, v \in H^1_{\Gamma_0}(\Omega).$$

The problem (2.8)-(2.9) could be reformulated as an evolutionary integral equation of variational type, and thanks to the monotocity and to the growth condition $|g(x)| \leq 1 + C|x|$ assumed on the function g, we can have access to the results and methods developed in Section 6 of Prüss [14] to obtain the following

Theorem 2.1. *For all given initial data $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, problem (1.9)-(1.12) admits a unique global weak solution*

$$u \in C\left(\mathbf{R}_+, H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left(\mathbf{R}_+, L^2(\Omega)\right). \tag{2.10}$$

Furthermore, if $(u_0, u_1) \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega)$ and g is globally Lipschitz continuous, then the solution has the following regularity

$$u \in C\left(\mathbf{R}_+, H^2(\Omega)\right) \cap C^1\left(\mathbf{R}_+, H^1_{\Gamma_0}(\Omega)\right) \cap C^2\left(\mathbf{R}_+, L^2(\Omega)\right). \tag{2.11}$$

We define the energy of the solution given by (2.10) by the following formula

$$E(t) := \frac{1}{2}\int_{\Omega}\left(\left|u'(t,x)\right|^2 + \left|\nabla u(t,x)\right|^2\right)dx$$
$$+\frac{1}{2}\int_{\Gamma_1} k(t,x)\left|u(t,x)-u_0(x)\right|^2 d\Gamma \qquad (2.12)$$
$$-\frac{1}{2}\int_{\Gamma_1}\int_0^t k'(t-s,x)\left|u(t,x)-u(s,x)\right|^2 ds\, d\Gamma.$$

Our main results are the following:

Theorem 2.2. *Assume that hypotheses* (2.1)-(2.7) *hold and that*

$$\alpha \inf_{\Gamma_1} k(0) > -2\inf_{\Gamma_1} k'(0). \qquad (2.13)$$

Then there exist $C, \omega > 0$ *such that*

$$E(t) \le CE(0)e^{-\omega t} \quad \text{if } p = 1,\ t \ge 0; \qquad (2.14)$$

$$E(t) \le \frac{CE(0)}{(1+t)^{2/(p-1)}} \quad \text{if } p > 1,\ t \ge 0, \qquad (2.15)$$

for every weak solution to (1.9)-(1.12) *and initial data* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Remark 2.2 Theorem 2.2 has a serious drawback: it never can be applied for bounded functions g (because of $C_3 > 0$ in (2.7)). The purpose of the following theorem is to obtain a variant of Theorem 2.2 for bounded feedback functions.

Theorem 2.3. *Assume* (2.1)-(2.5) *and assume that the function* g *is bounded, globally Lipschitz continuous and that the inequalities* (2.6) *are satisfied with some positive constants* C_1, C_2 *and with a number* p *satisfying*

$$p \ge n-1. \qquad (2.16)$$

Then for every strong solution u *to* (1.9)-(1.12) *we have*

$$E(t) \le \frac{CE(0)}{(1+t)^{2/(p-1)}}, \quad C > 0,\ t \ge 0. \qquad (2.17)$$

To end this section, we recall the following useful lemma:

Lemma 2.4. [7, *Lemma* 9.1] *Let* $E : \mathbf{R}_+ \to \mathbf{R}_+$ *be a non-increasing function and assume that there exist two constants* $\alpha > 0$ *and* $T > 0$ *such that*

$$\int_t^{+\infty} E^{\alpha+1}(s)\,ds \le TE(0)^{\alpha}E(t), \quad \forall t \in \mathbf{R}_+, \qquad (2.18)$$

then we have

$$E(t) \le E(0)\left(\frac{T+\alpha t}{T+\alpha T}\right)^{-1/\alpha}, \quad \forall t \ge T. \qquad (2.19)$$

3 Sketch of the Proof

Without lost of generality, we will transform the boundary condition (1.11) in another more practical considering $u_0 = 0$ on Γ_1.

A simple integration by parts yields:

$$\int_0^t k(t-s,x)u'(s,x)\,ds$$
$$= \left[k(t-s,x)u(s,x)\right]_0^t + \int_0^t k'(t-s,x)u(s,x)\,ds$$
$$= \int_0^t k'(t-s,x)u(s,x)\,ds + k(0,x)u(t,x).$$

Hence, the problem (1.9)-(1.12) is now transformed to

$$u'' - \Delta u = 0 \quad \text{in } \Omega \times \mathbf{R}_+, \qquad (3.1)$$

$$u = 0 \quad \text{on } \Gamma_0 \times \mathbf{R}_+, \qquad (3.2)$$

$$\frac{\partial u}{\partial \nu} + \int_0^t k'(t-s,x)u(s,x)\,ds$$
$$+ k(0)u + a(x)g(u') = 0 \quad \text{on } \Gamma_1 \times \mathbf{R}_+, \qquad (3.3)$$

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in } \Omega. \qquad (3.4)$$

The proofs of Theorem 2.2 and Theorem 2.3 are based on the following technical lemmas combined with Lemma 2.4 of the previous section.

Lemma 3.1. *The energy defined by* (2.12) *is non-increasing and it holds that*

$$\frac{1}{2}\int_S^T\int_{\Gamma_1}\int_0^t k''(t-s)\left(u(t)-u(s)\right)^2 ds\, d\Gamma\, dt$$
$$+\int_S^T\int_{\Gamma_1} a(x)g(u')u'\, d\Gamma\, dt$$
$$-\frac{1}{2}\int_S^T\int_{\Gamma_1} k'\left|u\right|^2 d\Gamma\, dt = E(S) - E(T) \le E(S),$$

for every $0 \le S < T < +\infty$. *In particular,* $\int g(u')u' \le c|E'(0)|$ *for a suitable constant* c.

Lemma 3.2. *Setting* $Mu := 2(m\cdot\nabla u) + (n-1)u$, *then it holds that*

$$\int_S^T E^{\frac{p-1}{2}}\int_{\Omega}\left(\left|u'\right|^2 + \left|\nabla u\right|^2\right)dx\,dt$$
$$= -\left[E^{\frac{p-1}{2}}\int_{\Omega} u'(Mu)\,dx\right]_S^T$$
$$+\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_{\Omega} u'(Mu)\,dx\,dt$$
$$+\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_0}(m\cdot\nu)\left|\nabla u\right|^2 d\Gamma\, dt$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\left(|u'|^2-|\nabla u|^2\right)d\Gamma\,dt$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(Mu)\frac{\partial u}{\partial\nu}\,d\Gamma\,dt.$$

Lemma 3.3. *It holds that*

$$\int_S^T E^{\frac{p-1}{2}}\int_\Omega\left(|u'|^2+|\nabla u|^2\right)dx\,dt$$

$$\leq C(\varepsilon)E(S)+\varepsilon\int_S^T E^{\frac{p+1}{2}}(t)\,dt+C(\varepsilon)E(S)$$

$$+(n-1)\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}u\frac{\partial u}{\partial\nu}d\Gamma\,dt$$

$$+\frac{R^2}{\delta}\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma\,dt,$$

where $0\leq S<T<+\infty$, $R=||m||_{L^\infty(\Omega)}$, C *is a positive constant,* ε *is an arbitrary small real number and* δ *comes from* (2.2).

Lemma 3.4. *It holds that*

$$-\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}u\frac{\partial u}{\partial\nu}d\Gamma\,dt$$

$$\leq CE(S)+\varepsilon\int_S^T E^{\frac{p-1}{2}}\int_\Omega|u'|^2\,dx\,dt$$

$$+\varepsilon\int_S^T E^{\frac{p+1}{2}}(t)dt+C(\varepsilon)E(S),$$

for every $\varepsilon>0$, $0\leq S<T<\infty$.

Lemma 3.5. *Define* $\gamma:=\gamma(x)=\frac{\lambda}{k(0)}$ *with*

$$\lambda>\max\left\{\frac{n-1}{2},\frac{R^2}{\delta}\,||k(0)||_{L^\infty(\Gamma_1)}\right\}.$$

Then it holds that

$$\int_S^T E^{\frac{p+1}{2}}(t)\,dt\leq CE(S)$$

$$+(1-\lambda)\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}k(0)\,|u|^2\,d\Gamma\,dt$$

$$+2\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}\gamma\left(\int_0^t k'(t-s)u(s)\,ds\right)^2 d\Gamma\,dt.$$

Lemma 3.6. *Let* $\varepsilon>0$ *be such that*

$$\varepsilon\inf_{\Gamma_1}k'(0)+1>0. \tag{3.8}$$

Then, for any $0\leq S<T<\infty$ *we have*

$$\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}\gamma\left(\int_0^t k'(t-s,x)u(s)\,ds\right)^2 d\Gamma dt$$

$$\leq CE(S)+\frac{\lambda}{\varepsilon\alpha}\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}|u|^2\,d\Gamma dt.$$

Proof of Theorem 2.2. We deduce from Lemma 3.5 that

$$\int_S^T E^{\frac{p+1}{2}}(t)dt\leq CE(S)$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}\left((1-\lambda)\,k(0)+\frac{2\lambda}{\varepsilon\alpha}\right)|u|^2\,d\Gamma dt.$$

We would like to make a choice of λ such that $(1-\lambda)\,k(0)+\frac{2\lambda}{\varepsilon\alpha}\leq 0$, and then by Lemma 2.6 we deduce the desired decay rates.

The condition $\alpha\inf_{\Gamma_1}k(0)>-2\inf_{\Gamma_1}k'(0)$ implies that

$$\exists\varepsilon'>0\ \text{ such that }\ \ \alpha\inf_{\Gamma_1}k(0)>-(2+\varepsilon')\inf_{\Gamma_1}k'(0).$$

We choose $\varepsilon>0$ such that $-\varepsilon\inf_{\Gamma_1}k'(0)=\frac{\varepsilon'+4}{2(\varepsilon'+2)}$, (we always have $\varepsilon\inf_{\Gamma_1}k'(0)+1>0$). Then, we have

$$\begin{aligned}(1-\lambda)k(0)&+\frac{2\lambda}{\varepsilon\alpha}\\ &=\frac{\lambda}{\varepsilon\alpha}\left(2-\varepsilon\alpha k(0)\right)+k(0)\\ &\leq\frac{\lambda}{\varepsilon\alpha}\left(2+\varepsilon(2+\varepsilon')\inf_{\Gamma_1}k'(0)\right)+k(0)\\ &=\frac{-\varepsilon'}{2\varepsilon\alpha}\lambda+k(0).\end{aligned}$$

Hence, if we choose

$$\lambda=\max\left\{n-1,\left(\frac{R^2}{\delta}+\frac{2\varepsilon\alpha}{\varepsilon'}\right)||k(0)||_{L^\infty(\Gamma_1)}\right\}$$

(remark that $\lambda>\max\left\{\frac{n-1}{2},\frac{R^2}{\delta}\,||k(0)||_{L^\infty(\Gamma_1)}\right\}$ still holds) we obtain $(1-\lambda)k(0)+\frac{2\lambda}{\varepsilon}\leq 0$. $\diamond$

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