# Indirect Obstacle Control Problem for Variational Inequalities

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## 1 Elliptic Systems

This paper is concerned with the optimal control of systems governed by a variational inequality coupled with a semilinear partial differential equation via the constraint of obstacle. In the stationary case, we consider an elliptic obstacle variational inequality

$$\begin{cases} Ay \geq f(x,y,u) & \text{in } \Omega, \\ y \geq \varphi & \text{in } \Omega, \\ (Ay-f)(y-\varphi) = 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

where the obstacle $\varphi$ is the solution of a controlled semilinear elliptic equation

$$\begin{cases} A\varphi = g(x,\varphi,u) & \text{in } \Omega, \\ \varphi|_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

Our goal is to minimize the following cost functional

$$J(y,\varphi,u) = \int_\Omega L(x,y(x),\varphi(x),u(x))dx. \tag{1.3}$$

The above problem is called *indirect obstacle optimal control problem*, since, if $f$ dose not explicitly depend on $u$, the control acts upon the state $y$ by means of the obstacle $\varphi$, the solution of the controlled equation (1.2). This amounts to a design of the shape of the string by choosing a suitable *curvature* of the obstacle, if the one-dimensional obstacle problem of the string is considered. Also, its parabolic counterpart may appear in the studying of Mathematical Finance (such as the pricing model of interest rate derivatives etc., cf. [15]).

The above problem is essentially the optimal control problem for a class of *nonsmooth distributed parameter systems*, which has attracted much attention in the literature (cf. [1, 2, 6, 7, 10, 13] for examples). In this paper, our main purpose is to study the existence and optimality conditions (in the form of *Pontryagin principle*).

If we define a *multifunction* $\beta$ as follows:

$$\beta(r) = \begin{cases} 0 & r > 0, \\ (\infty, 0] & r = 0, \\ \emptyset & r < 0, \end{cases}$$

then the obstacle variational inequality (1.1) can be written as

$$Ay + \beta(y-\varphi) \ni f.$$

The main feature of our problem is that the action of control (via the obstacle $\varphi$) gets into the multivalued operator $\beta$.

To begin with, we introduce the following assumptions.

($H_1$) $\Omega \subset \mathbb{R}^n$ is a bounded region with $C^{1,1}$ boundary $\partial\Omega$; $U$ is a Polish space.

($H_2$) Operator $A$ is defined by

$$Ay(x) = -\sum_{i,j=1}^n D_j(a_{ij}(x)D_i y(x))$$

with

$$a_{ij} \in C^1(\overline{\Omega}), \quad a_{ij} = a_{ji}, \quad 1 \leq i,j \leq n,$$

and for some $\lambda > 0$,

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda \sum_{i=1}^n |\xi_i|^2,$$

$$\forall x \in \Omega,\ (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n.$$

($H_3$) The functions $f, g\colon \Omega \times \mathbb{R} \times U \to \mathbb{R}$ have the following properties: $f(\cdot,y,u)$, $g(\cdot,\varphi,u)$ are measurable on $\Omega$, and $f(x,\cdot,u), g(x,\cdot,u)$ are in $C^1(\mathbb{R})$ with $f(x,\cdot,\cdot)$, $f_y(x,\cdot,\cdot)$, $g(x,\cdot,\cdot)$ and $g_\varphi(x,\cdot,\cdot)$ continuous on $\mathbb{R} \times U$. Moreover, there exists a constant $K > 0$, such that

$$-K \leq f_y,\ g_\varphi \leq 0 \qquad \text{on} \quad \Omega \times \mathbb{R} \times U,$$

and

$$|f(x,0,u)| + |g(x,0,u)| \leq K \qquad \text{on} \quad \Omega \times U.$$

($H_4$) The function $L\colon \Omega \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}$ satisfies the following: $L(\cdot,y,\varphi,u)$ is measurable on $\Omega$, $L(x,\cdot,\cdot,u)$ is in $C^1(\mathbb{R}\times\mathbb{R})$ with $L(x,\cdot,\cdot,\cdot), L_y(x,\cdot,\cdot,\cdot)$ and $L_\varphi(x,\cdot,\cdot,\cdot)$ continuous on $\mathbb{R}\times\mathbb{R}\times U$, and for any $R > 0$, there exists a constant $K_R > 0$, such that

$$|L| + |L_y| + |L_\varphi| \leq K_R$$

$$\text{on} \quad \Omega \times [-R,R] \times [-R,R] \times U.$$

We define

$$\mathcal{U} = \{u\colon \Omega \to U |\, u(\cdot) \text{ is measurable}\}$$

Any element $u \in \mathcal{U}$ is referred to as a control. Clearly, under (H$_1$)–(H$_4$), for each $u \in \mathcal{U}$, there corresponds a unique pair of state $(y, \varphi)$ and the cost functional (1.3) is well-defined. We can write $J(y, \varphi, u)$ as $J(u)$ without any ambiguity and state our control problem as follows.

**Problem (C).** Find a $\overline{u} \in \mathcal{U}$, such that

$$J(\overline{u}) = \inf_{u \in \mathcal{U}} J(u).$$

To establish the existence for Problem (C), we introduce the following set:

$$\Lambda(x, y, \varphi) = \{(\xi, \eta, \zeta) \in \mathrm{R}^3 |\, \xi \geq L(x, y, \varphi, u),$$

$$\eta = f(x, y, u), \zeta = g(x, \varphi, u), u \in U\},$$

and make the following assumption:

(H$_5$) For almost all $x \in \Omega$, the mapping $(y, \varphi) \mapsto \Lambda(x, y, \varphi)$ has the Cesari property on $\mathrm{R}^2$.

Using the so called *Cesari condition* (H$_5$) and the measurable selection theorem (cf. [10]) we have

**Theorem 1.1** Let (H$_1$)–(H$_5$) hold. Then Problem (C) admits at least one optimal control $\overline{u} \in \mathcal{U}$.

Using the spike variation technique and the Ekeland variational principle we obtain the first-order necessary conditions for optimal triples.

**Theorem 1.2** Let (H$_1$)–(H$_4$) hold and $(\overline{y}, \overline{\varphi}, \overline{u})$ be an optimal triple for Problem (C). Then there exist $\overline{z}, \overline{\psi} \in H_0^1(\Omega)$ and $\overline{\mu} \in H^{-1}(\Omega) \cap \mathcal{M}(\overline{\Omega})$, such that

$$\begin{cases} A\overline{z} - f_y(x, \overline{y}, \overline{u})\overline{z} = L_y(x, \overline{y}, \overline{\varphi}, \overline{u}) - \overline{\mu} & \text{in}\, \Omega, \\ A\overline{\psi} - g_\varphi(x, \overline{\varphi}, \overline{u})\overline{\psi} = L_\varphi(x, \overline{y}, \overline{\varphi}, \overline{u}) + \overline{\mu} & \text{in}\, \Omega, \\ \overline{z}|_{\partial\Omega} = 0, \qquad \overline{\psi}|_{\partial\Omega} = 0, \end{cases}$$

$$\textit{supp}\, \overline{\mu} \subset \{x \in \Omega |\, \overline{y}(x) = \overline{\varphi}(x)\} \tag{1.4}$$

and

$$H(x, \overline{y}(x), \overline{\varphi}(x), \overline{u}(x), \overline{z}(x), \overline{\psi}(x))$$
$$= \min_{u \in U} H(x, \overline{y}(x), \overline{\varphi}(x), u, \overline{z}(x), \overline{\psi}(x))$$
$$\text{a.e.} x \in \Omega,$$

where $\mathcal{M}(\overline{\Omega})$ is the set of all regular signed measures on $\overline{\Omega}$, and

$$H(x, y, \varphi, u, z, \psi)$$
$$= zf(x, y, u) + \psi g(x, \varphi, u) + L(x, y, \varphi, u)$$

for any $(x, y, \varphi, u, z, \psi)$

$$\in \Omega \times \mathrm{R} \times \mathrm{R} \times U \times \mathrm{R} \times \mathrm{R}.$$

The condition (1.4) is understood as the following: For any $\eta \in C(\overline{\Omega})$ with $\textit{supp}\, \eta \subset \Omega^+$

$$< \overline{\mu},\ \eta >_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0,$$

where

$$\Omega^+ = \{x \in \Omega |\, \overline{y}(x) > \overline{\varphi}(x)\}.$$

Note that our control domain is merely a separable metric space and dose not necessarily have any algebraic structure. Neither the convexity of control domain nor the smoothness of control is imposed. The regularity of the obstacle $\varphi$, which is required in the state analysis of our problem, relies on the governing equation (1.2). This makes the control manner more realizable.

# 2 Minimax Control Problem

For the same state system (1.1)–(1.2), we may also pose the following cost functional:

$$I(u) = \operatorname*{esssup}_{x \in \Omega} L(x, y(x), \varphi(x), u(x)) \tag{2.1}$$

where $(y, \varphi, u)$ is a triple of state and control satisfying (1.1)–(1.2).

One of the motivation of the above cost functional is the following: Consider the deformation of a membrane constrained by an obstacle. We would like to design the shape of the membrane by choosing a suitable obstacle so that the largest deviation of the perpendicular displacement $y$ from the desired one, say $y_d$, is minimized. In this case, we could take $L(x, y, \varphi, u) = |y - y_d(x)|^2$. Since the problem is to minimize a "maximum", it is usually referred to as a *minimax control problem.*

Minimax control problems sometimes seem to arise more naturally in applications than the standard problem involving integral cost, especially when one is attempting to minimize the maximum deviation from what is desired. Nevertheless, in our knowledge, such problems were less studied (especially for infinite-dimensional systems). As we know, the minimax control problem for ordinary differential equations was studied by several authors (cf. [3]). The first infinite-dimensional version of Pontryagin principle for minimax problem was presented in [10]. The usual integral cost problem is smooth (in some sense), whereas the $L^\infty$ norm (as the cost functional) is *nonsmooth.* This leads to more complicated necessary conditions for minimax control problems. That is one of the reasons why the problems with integral cost are more often studied than that with $L^\infty$ cost.

Let us retain all assumptions (H$_1$)–(H$_5$) given above. Since the state equations are the same, we may quote, from the previous section, all the analyses of state system without modifications. As before, let $\mathcal{U}$ be the set of all controls, our minimax control problem can be stated as follows.

**Problem (M).** Find a $\overline{u} \in \mathcal{U}$, such that

$$I(\overline{u}) = \inf_{u \in \mathcal{U}} I(u)$$

where $I(u)$ is defined by (2.1).

For the sake of convenience, let us make some reductions. First, by scaling, we may assume that $m(\Omega) = 1$. Next, from $W^{2,p}$-estimate of state and Sobolev's embedding, it follows that $y$ and $\varphi$ are uniformly bounded, independent of $u \in \mathcal{U}$. Thus, by ($H_4$), we may assume without loss of generality, that

$$0 < a \leq L(x, y, \varphi, u) \leq b < 1$$

$$\forall (x, y, \varphi, u) \in \Omega \times \mathrm{R} \times \mathrm{R} \times U$$

for some constants $a$ and $b$. We will keep the two reductions in this section.

The existence for Problem (M) can be established similarly to last section.

**Theorem 2.1** Let ($H_1$)–($H_5$) hold. Then Problem (M) admits at least one optimal control $\overline{u} \in \mathcal{U}$.

The major novelty of our problem lies in the simultaneous presence of the nonsmooth state equation (variational inequality) and the nonsmooth cost functional (the sup norm). We need to introduce a new regularization and to overcome some new difficulties arising in the discussion on convergency of the approximation.

Before going further, let us assume

($H_6$) $L(x, y, \varphi, u)$ is continuous on $\Omega \times \mathrm{R} \times \mathrm{R} \times U$ and there exists a nondecreasing continuous function $\omega: [0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$, such that

$$|L(\tilde{x}, \tilde{y}, \tilde{\varphi}, u) - L(x, y, \varphi, u)|$$

$$\leq \omega(|\tilde{x} - x| + |\tilde{y} - y| + |\tilde{\varphi} - \varphi|)$$

$$\forall (x, y, \varphi, u),\ (\tilde{x}, \tilde{y}, \tilde{\varphi}, u) \in \Omega \times \mathrm{R} \times \mathrm{R} \times U.$$

Under ($H_6$), we can prove a convergence theorem which is crucial in deriving the optimality conditions. Then, the Pontryagin principle for Problem (M) is available.

**Theorem 2.2** Let ($H_1$)–($H_4$) and ($H_6$) hold and $(\overline{y}, \overline{\varphi}, \overline{u})$ be an optimal triple for Problem (M). Then there exist $\overline{z}, \overline{\psi} \in W^{1,p'}(\Omega)$ with $p' = \frac{p}{p-1} \in (1, \frac{n}{n-1})$ and $\overline{\lambda}, \overline{\mu}, \overline{\nu} \in L^\infty(\Omega)^*$, such that

$$\begin{cases} A\overline{z} - f_y(x, \overline{y}, \overline{u})\overline{z} = \overline{\lambda} L_y(x, \overline{y}, \overline{\varphi}, \overline{u}) + \overline{\mu} & \text{in}\, \Omega, \\ A\overline{\psi} - g_\varphi(x, \overline{\varphi}, \overline{u})\overline{\psi} = \overline{\lambda} L_\varphi(x, \overline{y}, \overline{\varphi}, \overline{u}) + \overline{\nu} & \text{in}\, \Omega, \\ \overline{z}|_{\partial\Omega} = 0, \qquad \overline{\psi}|_{\partial\Omega} = 0 \end{cases} \tag{2.2}$$

and

$$\overline{z}(x) f(x, \overline{y}(x), \overline{u}(x)) + \overline{\psi}(x) g(x, \overline{\varphi}(x), \overline{u}(x))$$

$$= \min_{u \in U_0(x)} [\overline{z}(x) f(x, \overline{y}(x), u) + \overline{\psi}(x) g(x, \overline{\varphi}(x), u)]$$

$$\text{a.e.} x \in \Omega_0 \tag{2.3}$$

where

$$\begin{cases} \Omega_0 = \{x \in \Omega |\, L(x, \overline{y}(x), \overline{\varphi}(x), \overline{u}(x)) < \overline{J}\}, \\ U_0(x) = \{u \in U |\, L(x, \overline{y}(x), \overline{\varphi}(x), u) < \overline{J}\} \\ \qquad x \in \Omega, \end{cases}$$

$$\overline{\lambda}(\Omega) \equiv < \overline{\lambda},\, \chi_\Omega > \geq a > 0. \tag{2.4}$$

Moreover, in the case $m(\Omega_0) > 0$, for any $0 < \sigma < m(\Omega_0)$, there exists a measurable set $S_\sigma \subset \Omega_0$ with $m(S_\sigma) \geq \sigma$, such that

$$\overline{\lambda}(S_\sigma) = 0.$$

We note that in general, the above $\overline{\lambda}$ is only a finitely additive measure and is not necessarily in $\mathcal{M}(\overline{\Omega})$. If $\overline{\lambda}$ happens to be in $\mathcal{M}(\overline{\Omega})$, then there exists a measurable set $S \subset \Omega_0$ with $m(\Omega_0 \setminus S) = 0$, such that

$$\overline{\lambda}(S) = 0.$$

This means that the support of $\overline{\lambda}$ is disjoint with $\Omega_0$.

**Remark 1** If $L$ is independent of $u$, $\Omega_0$ and $U_0(x)$ can be replaced by $\Omega$ and $U$, respectively.

**Remark 2** If $(\overline{z}, \overline{\psi}) \neq 0$, then (2.3) gives a necessary condition for the optimal control $\overline{u}$. Whereas, if $(\overline{z}, \overline{\psi}) = 0$, then (2.3) is trivial. In this case, (2.2) tells us that

$$\begin{cases} \overline{\lambda} L_y(x, \overline{y}, \overline{\varphi}, \overline{u}) + \overline{\mu} = 0, \\ \overline{\lambda} L_\varphi(x, \overline{y}, \overline{\varphi}, \overline{u}) + \overline{\nu} = 0. \end{cases} \tag{2.5}$$

This gives (implicitly, if $L$ is independent of $u$) a necessary condition for $\overline{u}$. Due to (2.4), (2.5) is nontrivial.

Also, if $m(\Omega_0) = 0$, (2.3) tells us nothing. But, in this case, we must have

$$L(x, \overline{y}(x), \overline{\varphi}(x), \overline{u}(x)) = \overline{J} \qquad \text{a.e.}\, x \in \Omega.$$

This has already given us some information about the optimal triple $(\overline{y}, \overline{\varphi}, \overline{u})$.

# 3 Parabolic Systems with State Constraint

In this section, we are concerned with the following controlled evolutionary obstacle variational inequality

$$\begin{cases} y \in W_2^{2,1}(Q) \cap L^2(0, T; H_0^1(\Omega)), & \text{in}\, \Omega, \\ y|_{t=0} = y_0, & \text{in}\, \Omega, \\ y_t - \Delta y \geq f(x, t, y, u) & \text{in}\, Q, \\ y \geq \varphi & \text{in}\, Q, \\ (y_t - \Delta y - f)(y - \varphi) = 0 & \text{in}\, Q \end{cases} \tag{3.1}$$

where the obstacle $\varphi$ is time-dependent and solves a semilinear parabolic equation with distributed control:

$$\begin{cases} \varphi_t - \Delta\varphi = g(x,t,\varphi,u) & \text{in } Q, \\ \varphi|_\Sigma = 0, \varphi|_{t=0} = \varphi_0. \end{cases} \quad (3.2)$$

Our cost functional is taken to be

$$F(u) = \int_Q L(x,t,y(x,t),\varphi(x,t),u(x,t))dxdt \quad (3.3)$$

where $(y,\varphi,u)$ is a triple satisfying (3.1)–(3.2).

With respect to the control domain and the data involved, we make the following assumptions.

($H_7$) $\Omega \subset \mathrm{R}^n$ is a bounded region with $C^{1,1}$ boundary $\partial\Omega$; $U$ is a Polish space and

$$\mathcal{U} = \{u: Q \to U \,|\, u(\cdot,\cdot) \text{ is measurable}\}.$$

($H_8$) For some $\alpha \in (0,1)$ and any $p > 1$,

$$y_0, \varphi_0 \in C_0^\alpha(\Omega) \cap W^{2-1/p,p}(\Omega).$$

Moreover,

$$y_0 \geq \varphi_0 \qquad \text{a.e.in } \Omega.$$

($H_9$) The functions $f, g: \Omega \times [0,T] \times \mathrm{R} \times U \to \mathrm{R}$ have the following properties: $f(\cdot,\cdot,y,u)$, $g(\cdot,\cdot,\varphi,u)$ are measurable on $\Omega \times [0,T]$, and $f(x,t,\cdot,u)$, $g(x,t,\cdot,u)$ are in $C^1(\mathrm{R})$ with $f(x,t,\cdot,\cdot)$, $f_y(x,t,\cdot,\cdot)$, $g(x,t,\cdot,\cdot)$ and $g_\varphi(x,t,\cdot,\cdot)$ continuous on $\mathrm{R}\times U$. Moreover, there exists a constant $K > 0$, such that

$$|f_y| + |g_\varphi| \leq K \qquad \text{on} \quad \Omega \times [0,T] \times \mathrm{R} \times U$$

and

$$|f(x,t,0,u)| + |g(x,t,0,u)| \leq K$$
$$\text{on} \quad \Omega \times [0,T] \times U.$$

($H_{10}$) The function $L: \Omega \times [0,T] \times \mathrm{R} \times \mathrm{R} \times U \to \mathrm{R}$ satisfies the following: $L(\cdot,\cdot,y,\varphi,u)$ is measurable on $\Omega \times [0,T]$, $L(x,t,\cdot,\cdot,u)$ is in $C^1(\mathrm{R}\times\mathrm{R})$ with $L(x,t,\cdot,\cdot,\cdot)$, $L_y(x,t,\cdot,\cdot,\cdot)$ and $L_\varphi(x,t,\cdot,\cdot,\cdot)$ continuous on $\mathrm{R}\times\mathrm{R}\times U$, and for any $R > 0$, there exists a constant $K_R > 0$, such that

$$|L| + |L_y| + |L_\varphi| \leq K_R$$
$$\text{on} \quad \Omega \times [0,T] \times [-R,R] \times [-R,R] \times U.$$

Let

$$W = \{y \in L^2(0,T;H_0^1(\Omega)) |\, y_t \in L^2(0,T;H^{-1}(\Omega))\}.$$

By [14] (Lemma 3.2, Ch.II) we know that if $y \in W$, then $y$ is almost every where equal to a function that is continuous from $[0,T]$ into $L^2(\Omega)$. Hence, our initial condition is meaningful for any $y \in W$.

Any element $u \in \mathcal{U}$ is referred to as a control. It is known that (cf.[5, 12]), under ($H_7$)–($H_{10}$), for each $u \in \mathcal{U}$, there corresponds a unique pair of state $(y,\varphi) \in [W \cap C(\overline{Q})]^2$ and the cost functional (3.3) is well defined. Keeping assumptions ($H_7$)–($H_{10}$), we may talk about the state constraint of form

$$G(y) \in S. \quad (3.4)$$

For the state constraint (3.4), we assume that

($H_{11}$) $Z$ is a Banach space with the dual $Z^*$ being strictly convex. $S \subset Z$ is convex and closed, and is of finite codimension in $Z$ (see [10] for the definition of finite codimentional subset). The map $G: C_0(\overline{Q}) \to Z$ is continuously Fréchet differentiable, where

$$C_0(\overline{Q}) = \{\eta \in C(\overline{Q}) |\, \eta|_\Sigma = 0\}.$$

Any triple $(y,\varphi,u)$ satisfying (3.1)–(3.2) and the constraint (3.4) is called an admissible triple and the corresponding $u$ is called an admissible control. We denote by $\mathcal{U}_{ad}$ the set of all admissible controls. In what follows, we assume that $\mathcal{U}_{ad} \neq \emptyset$. Then, we may state our optimal control problem as follows.

**Problem (P).** Find an admissible control $\overline{u} \in \mathcal{U}_{ad}$, such that

$$F(\overline{u}) = \inf_{u \in \mathcal{U}_{ad}} F(u).$$

In analogy to the previous sections, we introduce the following set:

$$\Lambda(x,t,y,\varphi) = \{(\xi,\eta,\zeta) \in \mathrm{R}^3 |\, \xi \geq L(x,t,y,\varphi,u),$$
$$\eta = f(x,t,y,u), \zeta = g(x,t,\varphi,u), u \in U\}$$

and make the following assumption:

($H_{12}$) For almost all $(x,t) \in Q$, the mapping $(y,\varphi) \mapsto \Lambda(x,t,y,\varphi)$ has the Cesari property on $\mathrm{R}^2$.

Then, we can prove the existence result for Problem (P).

**Theorem 3.1** Let ($H_7$)–($H_{10}$) and ($H_{12}$) hold. Then Problem (P) admits at least one optimal control $\overline{u} \in \mathcal{U}_{ad}$.

We are mainly interested in deriving optimality conditions in the form of Pontryagin principle, which, for the optimal control problems of parabolic variational inequalities with nonconvex control domain, have never been established before.

There are many contributions devoted to the derivation of Pontryagin principle for evolutionary systems. See, for examples, [9, 10, 11], in which an abstract evolution equation setting was commonly used. Since our constraint (3.4) is quite general and, in many cases, it requires pointwise behavior of the state, we use the framework of partial differential equation instead of the abstract framework. The Pontryagin maximum principle for semilinear parabolic equations with pointwise state constraints has been proved in [8] (and recently in [4] for boundary control problems) without using the abstract evolution equations. However, both of them have *not* contained the case where the nonlinear term $f$ is multivalued.

Due to the state constraint, we need the *stability* of the optimal cost with respect to small perturbation of the state constraint ( ($H_{13}$) below, cf. [10]) in order the Ekeland variational principle applies, and need the finite codimensionality of the state constraint set to ensure the nontriviality of the Lagrange multipliers (see (3.5)).

($H_{13}$) For any $(y_k, \varphi_k, u_k)$ satisfying (3.1)–(3.2) and

$$d_S(G(y_k)) \to 0 \qquad (k \to \infty),$$

it holds that

$$\varliminf_{k\to\infty} J(u_k) \geq \inf_{u\in\mathcal{U}_{ad}} J(u)$$

where the *distance function* (to the convex and closed subset $S \subset Z$)

$$d_S(z) = \inf_{\eta\in S} \|z-\eta\|_Z \qquad z \in Z$$

is involved.

Now, we are in a position to give the Pontryagin principle for Problem (P).

**Theorem 3.2** Let ($H_7$)–($H_{11}$) and ($H_{13}$) hold and let the following *compatibility condition*, for the set $S$, the map $G$ and the initial state $y_0$, hold:

$$supp\, G'(\eta)^*\partial d_S(G(\eta)) \subset Q \cup (\Omega \times \{T\})$$

$$\forall \eta \in C_0(\overline{Q}) \text{ with } G(\eta) \in S, \quad \eta|_{t=0} = y_0(x).$$

Let $(\overline{y}, \overline{\varphi}, \overline{u})$ be an optimal triple for Problem (P). Then there exist $\overline{z}, \overline{\psi} \in L^q(0,T; W_0^{1,q}(\Omega))$ $(1 < q < \frac{n+2}{n+1})$, $\overline{\lambda} \in [0,1]$, $\overline{\mu} \in \mathcal{M}_0(\overline{Q})$ and $\overline{\nu} \in \partial d_S(G(\overline{y})) \subset Z^*$, such that

$$\overline{\lambda} + \|\overline{\nu}\|_{Z^*} > 0, \tag{3.5}$$

$$\begin{cases} -\overline{z}_t - \Delta\overline{z} - f_y(x,t,\overline{y},\overline{u})\overline{z} \\ = \overline{\lambda}L_y(x,t,\overline{y},\overline{\varphi},\overline{u}) - \overline{\mu} + G'(\overline{y})^*\overline{\nu}|_Q \quad \text{in } Q, \\ -\overline{\psi}_t - \Delta\overline{\psi} - g_\varphi(x,t,\overline{\varphi},\overline{u})\overline{\psi} \\ = \overline{\lambda}L_\varphi(x,t,\overline{y},\overline{\varphi},\overline{u}) + \overline{\mu} \quad \text{in } Q, \\ \overline{z}|_\Sigma = 0, \\ \overline{\psi}|_\Sigma = 0, \\ \overline{z}|_{t=T} = G'(\overline{y})^*\overline{\nu}|_{\Omega\times\{T\}}, \\ \overline{\psi}|_{t=T} = 0, \end{cases} \tag{3.6}$$

$$< \overline{\nu},\ \eta - G(\overline{y}) > \leq 0 \quad \forall \eta \in S, \tag{3.7}$$

$$supp\,\overline{\mu} \subset \{(x,t) \in Q|\, \overline{y}(x,t) = \overline{\varphi}(x,t)\} \tag{3.8}$$

and

$$H(x,t,\overline{y}(x,t),\overline{\varphi}(x,t),\overline{u}(x,t),\overline{\lambda},\overline{z}(x,t),\overline{\psi}(x,t))$$
$$= \min_{u\in U} H(x,t,\overline{y}(x,t),\overline{\varphi}(x,t),u,\overline{\lambda},\overline{z}(x,t),\overline{\psi}(x,t))$$
$$\text{a.e.}(x,t) \in Q \tag{3.9}$$

where

$$H(x,t,y,\varphi,u,\lambda,z,\psi)$$
$$= zf(x,t,y,u) + \psi g(x,t,\varphi,u) + \lambda L(x,t,y,\varphi,u)$$

for any $(x,t,y,\varphi,u,\lambda,z,\psi)$

$$\in \Omega \times [0,T] \times \mathrm{R} \times \mathrm{R} \times U \times \mathrm{R} \times \mathrm{R} \times \mathrm{R}.$$

Moreover, if

$$\mathcal{N}(G'(\overline{y})^*) = \{0\},$$

then,

$$(\overline{\lambda}, \overline{z}, \overline{\psi}) \neq 0.$$

In the above, $\mathcal{M}_0(\overline{Q}) = C_0(\overline{Q})^*$ is the set of all Radon measures on $\overline{Q}$ with the support contained in $Q\cup(\Omega\times\{0,T\})$; (3.6), (3.7) and (3.9) are referred to as the adjoint equation (along the given optimal triple), the transversality condition and Pontryagin's condition, respectively. The condition (3.8) is understood as the following: For any $\eta \in C_0(\overline{Q})$ with $supp\,\eta \subset Q^+$,

$$< \overline{\mu},\ \eta >_{\mathcal{M}_0(\overline{Q}),C_0(\overline{Q})} = 0$$

where $Q^+ = \{(x,t) \in Q|\, \overline{y}(x,t) > \overline{\varphi}(x,t)\}$.

**Acknowledgement** The author is grateful to Professor Xunjing Li and Professor Jiongmin Yong for their bringing him into the research area of optimal control theory.

# References

[1] D.R.Adams, S.M.Lenhart and J.Yong, Optimal control of the obstacle for an elliptic variational inequality, *Appl. Math. Optim.*, **38**(1998), 121–140.

[2] V.Barbu, Optimal Control of Variational Inequalities, Pitman, London, 1984.

[3] E.N.Barron, The Pontryagin maximum principle for minimax problems of optimal control, *Nonlinear Anal.*, **15**(1990), 1155–1165.

[4] E.Casas, Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations, *SIAM J. Control & Optim.*, **35**(1997), 1297–1327.

[5] P.Charrier and G.M.Troianiello, On strong solutions to parabolic unilateral problems with obstacle dependent on time, *J. Math. Anal. Appl.*, **65**(1978), 110–125.

[6] A.Friedman, Optimal control for variational inequalities, *SIAM J. Control* & *Optim.*, **24**(1986), 439–451.

[7] A.Friedman, Optimal control for parabolic variational inequalities, *SIAM J. Control* & *Optim.*, **25**(1987), 482–497.

[8] B.Hu and J.Yong, Pontryagin maximum principle for semilinear and quasilinear parabolic equations with pointwise state constraints, *SIAM J. Control* & *Optim.*, **33**(1995), 1857–1880.

[9] J.L.Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York, 1971.

[10] X.Li and J.Yong, Optimal Control Theory for Infinite Dimensional Systems, Birkhäuser, Boston, 1995.

[11] P.Neittaanmäki and D.Tiba, Optimal Control of Nonlinear Parabolic Systems, Marcel Dekker Inc., New York, 1994.

[12] J.F.Rodrigues, On a class of parabolic unilateral problems, *Nonlinear Anal.*, **10**(1986), 1357–1366.

[13] D.Tiba, Optimal Control of Nonsmooth Distributed Parameter Systems, Lecture Notes in Math., Vol.1459, Springer-Verlag, Berlin, 1990.

[14] R.Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1988.

[15] P.Wilmott, J.Dewynne and S.Howison, Option Pricing: Mathematical Models and Computation, Oxford Financial Press, Oxford, 1993.