

Shape design of channel flows for steady, incompressible flows

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Abstract

A shape design problem for stationary, viscous, incompressible, two-dimensional channel flows is considered. The shape of part of the boundary is determined so that the viscous drag is minimized. The adjoint equation method is used to derive an optimality system and the shape gradient of the design functional.

1 Introduction

Shape design problems associated with viscous, incompressible flows have wide application. The complete problem is concerned with questions of existence, regularity of solutions, and differentiability properties. Previous studies devoted to optimal shape design problems for the Stokes and Navier-Stokes equations can be found in [3, 4, 5, 7, 9, 13, 14, 15, 16, 18]. Some of these deal with the existence and regularity of solutions but generally lack a coherent first-order necessary condition and often the regularity assumed cannot be used in numerical algorithms. Other papers deal with reformulations of the problem, mainly to simplified situations. Again, results are far from complete. The embedding domain technique explored in [17] provides an equivalent formulation of the shape design problem on a fixed domain. An explicit formula for shape variations for the Navier-Stokes equation is proposed. We not only recover this result but extend it in the framework of a more general Lagrange multiplier technique that can easily take into account other constraints.

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The main focus is to find an appropriate formulation of the optimal shape design problem that is attractive for consistent numerical computations. We consider the full Navier-Stokes case and recover the first-order necessary conditions. The resulting optimality condition is a system of equations and variational inequalities which express the problem in a compact and coherent mathematical formulation. Although we deal with a specific, two-dimensional drag minimization problem, the approach used here is discussed in general terms and can be used for many other optimal control problems involving different objective functionals, classes of shape controls, and more complicated domains. For details concerning the results of this paper, see [10].

1.1 The model shape control problem

We consider the two-dimensional, incompressible flow of a viscous fluid through the channel Ω shown in Figure 1. The velocity \vec{u} and pressure p satisfy the stationary Navier-Stokes system

$$-\nu\Delta\vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p = \vec{f} \quad \text{in } \Omega \quad (1.1)$$

$$\nabla \cdot \vec{u} = 0 \quad \text{in } \Omega \quad (1.2)$$

$$\vec{u} = \vec{g} = \begin{cases} \vec{g}_i & \text{on } \Gamma_i, \quad i = 1, 3 \\ \vec{0} & \text{on } \Gamma_2 \cup \Gamma_4 \end{cases} \quad (1.3)$$

$$\int_{\Gamma} \vec{g} \cdot \vec{n} \, ds = 0, \quad (1.4)$$

where \vec{f} is the given body force, ν is the inverse of the Reynolds number whenever appropriate nondimensionalizations are used, and \vec{g}_1 and \vec{g}_3 are given velocities at the inflow Γ_1 and outflow Γ_3 . Along the bottom Γ_4 and top Γ_2 of the channel the velocity vanishes. The set $\Gamma_\alpha = \{\vec{x} = (x, z) \in \mathbb{R}^2 \mid x \in (a, b), z = \alpha(x)\} \subset \Gamma_2$ is the shape which is to be determined. We denote the interval (a, b) by I and the domain Ω by $\Omega(\alpha)$.

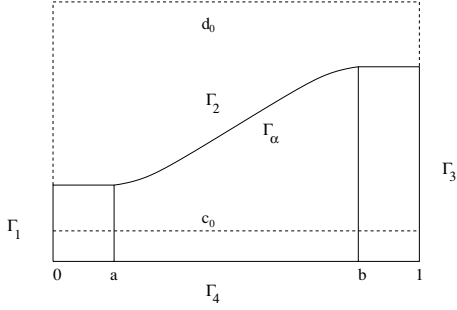


Figure 1: The flow domain $\Omega(\alpha)$ and its boundary; Γ_α denotes the part of the boundary whose shape is to be determined by toptimization.

We define a set of allowable shapes in the following way. Let c_0, d_0, c_2 , and d_2 be positive constants and z_1 and z_2 be the location of the controlled surface Γ_α at $x = a$ and $x = b$, respectively. Then, the set

$$\{\alpha(x) \in C^1(I) \mid c_i \leq \alpha^{(i)} \leq d_i \text{ for } i = 0, 2, \\ \alpha(a) = z_1, \alpha(b) = z_2, \text{ and } \alpha^{(1)}(a) = \alpha^{(1)}(b) = 0\}$$

may be a suitable set of allowable shapes, where $\alpha^{(i)}$ denotes the i -th derivative of α .

In order to enforce the regularity of the boundary, we take α to be the solution of the Poisson equation

$$\frac{d^2\alpha}{dx^2} = q \text{ on } I, \quad \alpha(a) = z_1, \quad \alpha(b) = z_2, \quad (1.5)$$

where $c_2 \leq q \leq d_2$ for all $x \in I$ is an unknown function. The other boundary conditions on α , i.e., $\alpha'(a) = \alpha'(b) = 0$, impose constraints on allowable functions q . In fact, if α and q are related by (1.5), then $\alpha'(a) = \alpha'(b) = 0$ if and only if

$$\int_a^b q(x) dx = 0, \quad \int_a^b \int_a^x q(\xi) d\xi dx = z_2 - z_1. \quad (1.6)$$

Since q is bounded, we have that $\alpha \in C^1(I)$ and Γ is piecewise $C^{1,1}$ with convex corners. We note that the lower bound for α , i.e., $\alpha(x) \geq c_0 \forall x \in I$, is necessary to avoid the intersection of Γ_2 and Γ_4 (see Figure 1) and that the second derivative bounds are necessary to assure that the curvature on Γ_α can be computed.

We focus on the minimization of the cost functional

$$\mathcal{J}(\vec{u}, q, \alpha) = 2\nu \int_{\Omega(\alpha)} D(\vec{u}) : D(\vec{u}) d\vec{x} + \frac{\beta}{2} \int_I q^2 dx, \quad (1.7)$$

where $D(\vec{u}) = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$ is the deformation tensor for the flow \vec{u} and β is a nonnegative constant. For $\beta = 0$, the functional (1.7) represents the rate of energy dissipation due to deformation and physically, except for an additive constant, can be identified with the viscous drag of the flow.

Formally speaking, the design problem we consider is to find \vec{u} , α , and q such that the functional (1.7) is minimized subject to the Navier-Stokes system (1.1)–(1.4) and the relations (1.5) and (1.6) being satisfied.

1.2 Notations

Depending on the context, C and K denote generic constants whose values also depend on context. We denote by $H^s(\mathcal{O})$, $s \in \mathbb{R}$, the standard Sobolev space of order s with respect to the set \mathcal{O} , which is either the flow domain Ω , or its boundary Γ , or part of its boundary. Whenever m is a nonnegative integer, the inner product over $H^m(\mathcal{O})$ is denoted by $(f, g)_m$ and (f, g) denotes the inner product over $H^0(\mathcal{O}) = L^2(\mathcal{O})$. Hence, we associate with $H^m(\mathcal{O})$ its natural norm $\|f\|_{m, \mathcal{O}} = \sqrt{(f, f)_m}$. Whenever possible, we will neglect the domain label in the norm. For vector-valued functions and spaces, we use boldface notation. For example, $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$ denotes the space of \mathbb{R}^n -valued functions such that each component belongs to $H^s(\Omega)$. Of special interest is the space

$$\mathbf{H}^1(\Omega) = \left\{ v_j \in L^2(\Omega) \mid \frac{\partial v_j}{\partial x_k} \in L^2(\Omega) \text{ for } j, k = 1, 2 \right\}$$

equipped with the norm $\|\vec{v}\|_1 = (\sum_{k=1}^2 \|v_k\|_1^2)^{1/2}$. We define the space

$$\mathbf{V}(\Omega) = \{ \vec{u} \in \mathbf{H}^1(\Omega) \mid \nabla \cdot \vec{u} = 0 \}$$

and the space of infinite differentiable solenoidal functions by

$$\mathcal{V}(\Omega) = \{ \vec{u} \in \mathbf{C}_0^\infty(\bar{\Omega}) \mid \nabla \cdot \vec{u} = 0 \}.$$

For $\Gamma_s \subset \Gamma$ with nonzero measure, we also consider the subspace

$$\mathbf{H}_{\Gamma_s}^1(\Omega) = \{ \vec{v} \in \mathbf{H}^1(\Omega) \mid \vec{v} = \vec{0} \text{ on } \Gamma_s \}.$$

Also, we write $\mathbf{H}_0^1(\Omega) = \mathbf{H}_\Gamma^1(\Omega)$. For any $\vec{v} \in \mathbf{H}^1(\Omega)$, we write $\|\nabla\vec{v}\|$ for the seminorm. Let $(\mathbf{H}_{\Gamma_s}^1)^*$ denote the dual space of $\mathbf{H}_{\Gamma_s}^1$. Note that $(\mathbf{H}_{\Gamma_s}^1)^*$ is a subspace of $\mathbf{H}^{-1}(\Omega)$, where the latter is the dual space of $\mathbf{H}_0^1(\Omega)$. The duality pairing between $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$.

Let \vec{g} be an element of $\mathbf{H}^{1/2}(\Gamma)$. It is well known that $\mathbf{H}^{1/2}(\Gamma)$ is a Hilbert space with norm

$$\|\vec{g}\|_{1/2, \Gamma} = \inf_{\vec{v} \in \mathbf{H}^1(\Omega); \gamma_\Gamma \vec{v} = \vec{g}} \|\vec{v}\|_1,$$

where γ_Γ denotes the trace mapping $\gamma_\Gamma : \mathbf{H}^1(\Omega) \rightarrow \mathbf{H}^{1/2}(\Gamma)$. We let $(\mathbf{H}^{1/2}(\Gamma))^*$ denote the dual space of $\mathbf{H}^{1/2}(\Gamma)$ and $\langle \cdot, \cdot \rangle_\Gamma$ denote the duality pairing between $(\mathbf{H}^{1/2}(\Gamma))^*$ and $\mathbf{H}^{1/2}(\Gamma)$. From the definition of the dual norm, we have

$$\|\vec{s}\|_{-1/2, \Gamma} = \sup_{\vec{g} \in \mathbf{H}^{1/2}(\Gamma); \vec{g} \neq \vec{0}} \frac{\langle \vec{s}, \vec{g} \rangle_\Gamma}{\|\vec{g}\|_{1/2}}.$$

Let Γ_s be a smooth subset of Γ . Then, the trace mapping $\gamma_{\Gamma_s} : \mathbf{H}^1(\Omega) \rightarrow \mathbf{H}^{1/2}(\Gamma_s)$ is well defined and $\mathbf{H}^{1/2}(\Gamma_s) = \gamma_{\Gamma_s}(\mathbf{H}^1(\Omega))$.

Since the pressure is only determined up to an additive constant by the Navier-Stokes system with velocity boundary conditions, we define the space of square integrable function having zero mean over Ω as

$$L_0^2(\Omega) = \{ p \in L^2(\Omega) \mid \int_{\Omega} p \, d\vec{x} = 0 \}.$$

In order to define a weak form of the Navier-Stokes equations, we introduce the continuous bilinear and trilinear forms

$$a(\vec{u}, \vec{v}) = 2\nu \int_{\Omega} D(\vec{u}) : D(\vec{v}) \, d\vec{x} \quad (1.8)$$

$$b(\vec{v}, q) = - \int_{\Omega} q \, \nabla \cdot \vec{v} \, d\vec{x} \quad (1.9)$$

$$c(\vec{w}; \vec{u}, \vec{v}) = \int_{\Omega} \vec{w} \cdot \nabla \vec{u} \cdot \vec{v} \, d\vec{x}. \quad (1.10)$$

for all $\vec{u}, \vec{v}, \vec{w} \in \mathbf{H}^1(\Omega)$ and $q \in L_0^2(\Omega)$.

For details concerning the function spaces we have introduced, one may consult [1, 19] and for details about the bilinear and trilinear forms and their properties, one may consult [6, 19].

1.3 The associated boundary value problem

We consider the formulation of the direct problem for the Navier-Stokes system (1.1)–(1.3) for which the boundary and all the data functions are known. Let $\Gamma(\alpha)$ be the boundary which includes the segment Γ_{α} (see Figure 1) defined for a given $\alpha \in H^2(I)$. Given α , we can compute q by using (1.5).

A weak formulation of the Navier-Stokes system is given as follows: *given $\vec{f} \in \mathbf{H}^{-1}(\Omega(\alpha))$ and $\vec{g} \in \mathbf{H}^{1/2}(\Gamma(\alpha))$, find $(\vec{u}, p) \in \mathbf{H}^1(\Omega(\alpha)) \times L_0^2(\Omega(\alpha))$ satisfying*

$$\begin{cases} a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{v}, p) = \langle \vec{f}, \vec{v} \rangle \\ b(\vec{u}, q) = 0 \\ \langle \vec{u}, \vec{s} \rangle_{\Gamma(\alpha)} = \langle \vec{g}, \vec{s} \rangle_{\Gamma(\alpha)} \end{cases} \quad (1.11)$$

$\forall (\vec{v}, q, \vec{s}) \in \mathbf{H}_0^1(\Omega(\alpha)) \times L_0^2(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma(\alpha))$. Existence, uniqueness, and regularity results for solutions of the system (1.11) are contained in the following theorem; see, e.g., [6, 16, 19].

Theorem 1 *Let $\Omega(\alpha)$ be an open, bounded set of \mathbb{R}^2 with Lipschitz-continuous boundary $\Gamma(\alpha)$. Let $\vec{f} \in \mathbf{H}^{-1}(\Omega(\alpha))$ and $\vec{g} \in \mathbf{H}^{1/2}(\Gamma(\alpha))$ and let \vec{g} satisfy the compatibility condition (1.4). Then, there exists at least one solution $(\vec{u}, p) \in \mathbf{H}^1(\Omega(\alpha)) \times L^2(\Omega(\alpha))$ of (1.11); the set of velocity fields that are solutions of (1.11) is*

closed in $\mathbf{H}^1(\Omega(\alpha))$ and is compact in $\mathbf{L}^2(\Omega(\alpha))$; and if $\nu > \nu_0(\Omega(\alpha), \vec{f}, \vec{g})$ for some positive ν_0 whose value is determined by the given data, then the set of solutions of (1.11) consists of a single element. Now, let $\Gamma(\alpha)$ be piecewise $C^{1,1}$ with convex corners, $\vec{g} \in \mathbf{H}^{3/2}(\Gamma(\alpha))$, and $\vec{f} \in \mathbf{L}^2(\Omega(\alpha))$. Let (\vec{u}, p) denote a solution of (1.11). Then, $(\vec{u}, p) \in \mathbf{H}^2(\Omega(\alpha)) \times H^1(\Omega(\alpha)) \cap L_0^2(\Omega(\alpha))$ and the set of solutions of (1.11) is closed in $\mathbf{H}^2(\Omega(\alpha))$ and compact in $\mathbf{H}^1(\Omega(\alpha))$.

2 The shape design problem

We now formulate the model shape design problem. We define the closed convex set

$$\begin{aligned} \mathcal{Q}_{ad} = \{ \alpha \in H^2(I) \mid 0 < c_0 \leq \alpha \leq d_0 \\ \alpha(a) = z_1, \alpha(b) = z_2, \text{ and } \alpha'(a) = \alpha'(b) = 0 \} \end{aligned}$$

and introduce the variable q belonging to the set

$$\mathcal{B}_{ad} = \{ q \in L^2(I) \mid c_2 \leq q \leq d_2 \text{ almost everywhere} \}$$

defined by

$$\int_I qv \, dx = \int_I \frac{d^2\alpha}{dx^2} v \, dx \quad \forall v \in L^2(I). \quad (2.1)$$

The constants c_2 and d_2 are such that the set \mathcal{Q}_{ad} is not empty. From the Sobolev imbedding theorem, we have that $H^2(I) \subset C^1(\bar{I}) \subset C^{0,1}(\bar{I})$ and therefore, if $\alpha \in \mathcal{Q}_{ad}$ and $q \in \mathcal{B}_{ad}$, then $\alpha \in C^{1,1}$ at least.

The shape design problem can then be stated in the following way: *given $\vec{f} \in \mathbf{L}^2(\Omega(\alpha))$ and $\vec{g} \in \mathbf{H}^{3/2}(\Gamma(\alpha))$ satisfying the compatibility condition (1.4), find (\vec{u}, p, q, α) such that*

$$\mathcal{J}(\vec{u}, q, \alpha) \leq \mathcal{J}(\tilde{u}, \tilde{q}, \tilde{\alpha}) \quad (2.2)$$

for all $(\tilde{u}, \tilde{p}, \tilde{q}, \tilde{\alpha}) \in \mathbf{H}^2(\Omega(\alpha)) \times \mathbf{H}^1(\Omega(\alpha)) \cap L_0^2(\Omega(\alpha)) \times \mathcal{B}_{ad} \times \mathcal{Q}_{ad}$ satisfying (1.11) and (2.1).

The admissible set of states and controls is given by

$$\begin{aligned} \mathcal{A}_{ad} = \{ (\vec{u}, p, q, \alpha) \in \mathbf{H}^2(\Omega(\alpha)) \cap \mathbf{V}(\Omega(\alpha)) \\ \times H^1(\Omega(\alpha)) \cap L_0^2(\Omega(\alpha)) \times \mathcal{B}_{ad} \times \mathcal{Q}_{ad} \\ \text{such that } \mathcal{J}(\vec{u}, q, \alpha) < \infty \\ \text{and } (\vec{u}, p, q, \alpha) \text{ satisfies (1.11) and (2.1)} \}. \end{aligned}$$

The existence of optimal solutions for shape design problem is given by the following result.

Theorem 2 *There exists at least one optimal solution $(\vec{u}, p, q, \alpha) \in \mathcal{A}_{ad}$ of the optimal shape design problem (2.2).*

3 The Lagrange multiplier method

3.1 Preliminaries

We introduce auxiliary variables that allow us to transform the inequality constraints into equalities and then invoke well-known derivations for equality constrained minimization problems; see, e.g., [2] or [20]. We begin by replacing

$$c_0 \leq \alpha \leq \bar{d}_0 \quad \text{and} \quad c_2 \leq q \leq d_2 \quad \forall x \in I \quad (3.1)$$

by

$$|\alpha - \alpha_0|^2 - \alpha_m^2 + s_0^2 = 0 \quad \forall x \in I \quad (3.2)$$

$$|q - q_0|^2 - q_m^2 + s_2^2 = 0 \quad \forall x \in I \quad (3.3)$$

for some $s_2 \in L^2(I)$ and $s_0 \in H^2(I)$, where $\alpha_0 = (c_0 + d_0)/2$, $q_0 = (c_2 + d_2)/2$, $\alpha_m = (d_0 - c_0)/2$, and $q_m = (d_2 - c_2)/2$. Clearly, if (3.2)–(3.3) are satisfied, then so are (3.1). Also, note that if (\vec{u}, p, q, α) is a solution of the shape design problem, then there exist s_0, s_2 such that α, q and s_0, s_2 satisfy (3.2)–(3.3).

We let $\Gamma(\alpha)$ be piecewise $C^{1,1}$ in agreement with the proposed model problem and $\vec{g} \in \mathbf{H}^{3/2}(\Gamma(\alpha))$ where $\int_{\Gamma(\alpha)} \vec{g} \cdot \vec{n} ds = 0$ with $\vec{g} = \vec{0}$ on $\Gamma_2 \cap \Gamma_4$, $\vec{g} = \vec{g}_1 \in \mathbf{H}^{3/2}(\Gamma_1)$ on Γ_1 and $\vec{g} = \vec{g}_3 \in \mathbf{H}^{3/2}(\Gamma_3)$ on Γ_3 .

Let $\mathbf{B}_1 = (\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)) \times (L_0^2(\Omega) \cap H^1(\Omega)) \times \mathcal{B}_{ad} \times \mathcal{Q}_{ad} \times H^2(I) \times L^2(I)$, $\mathbf{B}_2 = \mathbf{H}^{-1}(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{1/2}(I) \times \mathbf{H}^{1/2}(\Gamma(\alpha) - \Gamma_\alpha) \times L^2(I) \times H^2(I) \times L^1(I)$ and $\mathbf{B}_3 = \mathbf{H}^{-1}(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{1/2}(I) \times \mathbf{H}^{1/2}(\Gamma(\alpha) - \Gamma_\alpha) \times L^2(I) \times W^{2,1}(I) \times L^1(I)$. We equip \mathbf{B}_1 , \mathbf{B}_2 and \mathbf{B}_3 with the usual graph norms for the product spaces involved. We define the nonlinear mapping $M : \mathbf{B}_1 \rightarrow \mathbf{B}_3$ by $M(\vec{U}) = \vec{b}$ for $\vec{U} = (\vec{u}, p, q, \alpha, s_0, s_2) \in \mathbf{B}_1$ and $\vec{b} = (\bar{l}_1, l_2, \bar{l}_3, \bar{l}_4, l_5, l_6, l_7) \in \mathbf{B}_3$ if and only if

$$\left\{ \begin{array}{l} \nu a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{v}, \vec{p}) \\ - \int_{\Omega} \vec{f} \cdot \vec{v} d\vec{x} = \int_{\Omega} \bar{l}_1 \cdot \vec{v} d\vec{x} \quad \forall \vec{v} \in \mathbf{H}_0^1(\Omega) \\ b(\vec{u}, z) = \int_{\Omega} l_2 z d\vec{x} \quad \forall z \in L_0^2(\Omega) \\ \int_{\Gamma_\alpha} \vec{u} \cdot \vec{s} ds = \int_I \bar{l}_3 \cdot \vec{s} ds \quad \forall \vec{s} \in \mathbf{H}^{-1/2}(I) \\ \int_{\Gamma(\alpha) - \Gamma_\alpha} (\vec{u} - \vec{g}) \cdot \vec{s} ds = \int_{\Gamma(\alpha) - \Gamma_\alpha} \bar{l}_4 \cdot \vec{s} ds \\ \quad \forall \vec{s} \in \mathbf{H}^{-1/2}(\Gamma(\alpha) - \Gamma_\alpha) \\ \int_I v q dx + \int_I \frac{d\alpha}{dx} \frac{dv}{dx} dx = \int_I l_5 v dx \\ \quad \forall v \in H_0^1(I) \\ (\alpha - \alpha_0)^2 - \alpha_m^2 + s_0^2 = l_6 \quad \forall x \in I \\ (q - q_0)^2 - q_m^2 + s_2^2 = l_7 \quad \forall x \in I \end{array} \right. \quad (3.4)$$

with $\alpha(a) = z_1$, $\alpha(b) = z_2$, and $\alpha'(a) = \alpha'(b) = 0$. The set of constraint equations in the optimal shape design problem can be expressed as $M(\vec{u}, p, q, \alpha, s_0, s_2) = (\vec{0}, 0, \vec{0}, \vec{0}, 0, 0, 0)$.

Given $(\vec{u}_1, p_1, q_1, \alpha_1) \in \mathcal{A}_{ad}$, we define another nonlinear mapping $Q : \mathbf{B}_1 \rightarrow \mathbb{R} \times \mathbf{B}_3$ by $Q(u, p, q, \alpha, s_0, s_2) = \vec{b} = (a, \bar{l}_1, l_2, \bar{l}_3, \bar{l}_4, l_5, l_6, l_7)$ if and only if

$$\begin{pmatrix} \mathcal{J}(\vec{u}, q, \alpha) - \mathcal{J}(\vec{u}_1, q_1, \alpha_1) \\ M(\vec{u}, p, q, \alpha, s_0, s_2) \end{pmatrix} = \begin{pmatrix} a \\ \vec{b} \end{pmatrix}. \quad (3.5)$$

3.2 Differentiability

These mappings are strictly differentiable, as is shown in the following lemma. We recall the notion of strict differentiability (see [20]). Let X and Y denote Banach spaces, then the mapping $\varphi : X \rightarrow Y$ is strictly differentiable at $x \in X$ if there exists a bounded, linear mapping D from X to Y such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\|x - x_1\|_X < \delta$ and $\|x - x_2\|_X < \delta$ for $x_1, x_2 \in X$, then

$$\|\varphi(x_1) - \varphi(x_2) - D(x_1 - x_2)\|_Y \leq \epsilon \|x_1 - x_2\|_X.$$

The strict derivative D at the point $x \in X$, if it exists, will often be denoted by $D = \varphi'(x)$. The value of this mapping on an element $\tilde{x} \in X$ will often be denoted by $\varphi'(x) \cdot \tilde{x}$. In the next theorem we can identify $X = \mathbf{B}_1$ and $Y = \mathbf{B}_2$.

Lemma 1 *Let the nonlinear mappings $M : \mathbf{B}_1 \rightarrow \mathbf{B}_3$ and $Q : \mathbf{B}_1 \rightarrow \mathbb{R} \times \mathbf{B}_3$ be defined by (3.4) and (3.5), respectively. Then, these mappings are strictly differentiable at a point $(u, p, q, \alpha, s_0, s_2) \in \mathbf{B}_1$ and its strict derivative is given by the bounded linear operator $M'(u, p, q, \alpha, s_0, s_2) : \mathbf{B}_1 \rightarrow \mathbf{B}_2$, where $M'(u, p, q, \alpha, s_0, s_2) \cdot (\tilde{u}, \tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{s}_0, \tilde{s}_2) = \vec{b}$ for $(\tilde{u}, \tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{s}_0, \tilde{s}_2) \in \mathbf{B}_1$ and $\vec{b} = (\bar{l}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4, \bar{l}_5, \bar{l}_6, \bar{l}_7) \in \mathbf{B}_2$ if and only if*

$$\left\{ \begin{array}{l} \nu a(\tilde{u}, \vec{v}) + c(\tilde{u}; \tilde{u}, \vec{v}) + c(\vec{u}; \tilde{u}, \vec{v}) + b(\vec{v}, \tilde{p}) \\ = \int_{\Omega} \bar{l}_1 \cdot \vec{v} d\vec{x} \quad \forall \vec{v} \in \mathbf{H}_0^1(\Omega) \\ b(\tilde{u}, z) = \int_{\Omega} \bar{l}_2 z d\vec{x} \quad \forall z \in L_0^2(\Omega) \\ \int_{\Gamma_\alpha} \tilde{u} \cdot \vec{s} ds + \int_{\Gamma_\alpha} (\vec{V}(\tilde{\alpha}) \cdot \vec{n}) (k + \frac{\partial}{\partial n}) \tilde{u} \cdot \vec{s} ds \\ = \int_I \bar{l}_3 \cdot \vec{s} ds \quad \forall \vec{s} \in \mathbf{H}^{-1/2}(I) \\ \int_{\Gamma(\alpha) - \Gamma_\alpha} \tilde{u} \cdot \vec{s} ds = \int_{\Gamma(\alpha) - \Gamma_\alpha} \bar{l}_4 \cdot \vec{s} ds \\ \quad \forall \vec{s} \in \mathbf{H}^{-1/2}(\Gamma(\alpha) - \Gamma_\alpha) \\ \int_I \tilde{q} v dx + \int_I \frac{d\tilde{\alpha}}{dx} \frac{dv}{dx} dx = \int_I \bar{l}_5 v dx \\ \quad \forall v \in H_0^1(I) \\ 2\tilde{\alpha}(\alpha - \alpha_0) + 2\tilde{s}_0 s_0 = \bar{l}_6 \quad \forall x \in I \\ 2\tilde{q}(q - q_0) + 2\tilde{s}_2 s_2 = \bar{l}_7 \quad \forall x \in I \\ \tilde{\alpha}(a) = \tilde{\alpha}(b) = \tilde{\alpha}'(a) = \tilde{\alpha}'(b) = 0, \end{array} \right. \quad (3.6)$$

where $\vec{V}(\tilde{\alpha}) = (0, \tilde{\alpha})$, κ denotes the curvature, and \vec{n} is the normal vector to Γ_α . Moreover, the strict derivative

of Q at a point $\vec{U} = (\vec{u}, p, q, \alpha, s_0, s_2) \in \mathbf{B}_1$ is given by the bounded linear operator $Q'(\vec{U}) : \mathbf{B}_1 \rightarrow \mathbb{R} \times \mathbf{B}_2$, where $Q'(\vec{U}) \cdot \vec{U} = (\bar{a}, \bar{b})$, for $\vec{U} = (\vec{u}, \tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{s}_0, \tilde{s}_2) \in \mathbf{B}_1$ and $(\bar{a}, \bar{b}) \in \mathbb{R} \times \mathbf{B}_2$ if and only if

$$\begin{pmatrix} \mathcal{J}'(\vec{u}, q, \alpha) \cdot \vec{U} \\ M'(\vec{U}) \cdot \vec{U} \end{pmatrix} = \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}, \quad (3.7)$$

where

$$\begin{aligned} \mathcal{J}'(\vec{u}, q, \alpha) \cdot \vec{U} &= \beta \int_I q \tilde{q} dx \\ &+ \nu \left(\int_{\Gamma_\alpha} \frac{\partial \vec{u}}{\partial n} \frac{\partial \vec{u}}{\partial n} (\vec{V}(\tilde{\alpha}) \cdot \vec{n}) ds + 2 \int_{\Omega(\alpha)} (\nabla \vec{u} \cdot \nabla \vec{u}) d\vec{x} \right). \end{aligned}$$

From (3.6), we note that the regularity of the Gateaux derivative cannot be the same as the solution of the Navier-Stokes system. In fact, the boundary conditions for the Gateaux derivative implies a different degree of regularity.

Next, we prove some further properties of the derivatives of the mappings M and Q .

Lemma 2 *Let $(\vec{u}, p, q, \alpha, s_0, s_2) \in \mathbf{B}_1$ denote a solution of the optimal control problem. Then we have*

- i) the operator $M'(\vec{u}, p, q, \alpha, s_0, s_2)$ has closed range in \mathbf{B}_2 ;
- ii) the operator $Q'(\vec{u}, p, q, \alpha, s_0, s_2)$ has closed range in $\mathbb{R} \times \mathbf{B}_2$;
- iii) the operator $Q'(\vec{u}, p, q, \alpha, s_0, s_2)$ is not onto $\mathbb{R} \times \mathbf{B}_2$.

The first-order necessary condition follows easily from the fact that the operator $Q'(\vec{u}, p, q, \alpha, s_0, s_2)$ is not onto $\mathbb{R} \times \mathbf{B}_2$; see, e.g., [8, 11, 12].

Theorem 3 *Given $(\vec{u}, p, q, \alpha) \in \mathcal{A}_{ad}$. If $\vec{U} = (\vec{u}, p, q, \alpha, s_0, s_2) \in \mathbf{B}_1$ is a solution of the optimal shape design problem, then there exists a nonzero Lagrange multiplier $(\lambda, \vec{W}) \in \mathbb{R} \times \mathbf{B}_2^*$, where $\vec{W} = (\vec{w}, r, \vec{\theta}, \vec{\eta}, \mu, \tau_0, \tau_2)$, satisfying the Euler equations*

$$\begin{aligned} \lambda \mathcal{J}'(\vec{u}, q, \alpha) \cdot \vec{U} + \langle \vec{W}, M'(\vec{U}) \cdot \vec{U} \rangle &= 0 \\ \forall \vec{U} &= (\vec{u}, \tilde{r}, \tilde{q}, \tilde{\alpha}, \tilde{s}_0, \tilde{s}_2) \in \mathbf{B}_1, \end{aligned} \quad (3.8)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between \mathbf{B}_2 and \mathbf{B}_2^* .

3.3 The optimality system

We now examine the first-order necessary condition (3.8) to derive an optimality system from which optimal states and controls may be determined.

Theorem 4 *Let $(\vec{u}, p, q, \alpha, s_0, s_2) \in \mathbf{B}_1$ denote a solution of the optimal design problem. Then, if $s_0 \neq 0$ and $s_2 \neq 0$, μ is the solution of*

$$\begin{aligned} \int_I \frac{d\mu}{dx} \frac{d\zeta}{dx} dx + \nu \int_{\Gamma_\alpha} \left(\frac{\partial \vec{u}}{\partial n} \frac{\partial \vec{u}}{\partial n} - \frac{\partial \vec{u}}{\partial n} \frac{\partial \vec{w}}{\partial n} \right) \\ \cdot (\vec{V}(\zeta) \cdot \vec{n}) ds = 0 \quad \forall \zeta \in H_0^1(I) \end{aligned} \quad (3.9)$$

$$\int_I (\mu + \beta q) v dx = 0 \quad \forall v \in L^2(I) \quad (3.10)$$

for all $\alpha \in H^2(I)$ with $\alpha(a) = z_1$, $\alpha(b) = z_2$, and $\alpha'(a) = \alpha'(b) = 0$, where $(\vec{w}, r) \in \mathbf{H}_0^1(\Omega(\alpha)) \times L_0^2(\Omega(\alpha))$ satisfies the adjoint problem

$$\begin{cases} \nu a(\vec{w}, \vec{v}) + c(\vec{v}; \vec{u}, \vec{w}) + c(\vec{u}; \vec{v}, \vec{w}) + b(\vec{v}, r) = \\ -2\nu \int_{\Omega(\alpha)} \nabla \vec{u} \cdot \nabla \vec{v} d\vec{x} \quad \forall \vec{v} \in \mathbf{H}_0^1(\Omega(\alpha)) \\ b(\vec{w}, q) = 0 \quad \forall q \in L_0^2(\Omega(\alpha)) \\ \vec{w} \in \mathbf{H}_0^1(\Omega(\alpha)). \end{cases} \quad (3.11)$$

If $s_0 = 0$ we have $\alpha = c_0$ or $\alpha = d_0$. If $s_2 = 0$ we have $q = c_2$ or $q = d_2$ which gives α through (2.1) and the appropriate boundary conditions.

As a consequence of the optimality system we have to solve

$$\begin{cases} \nu a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{v}, p) = \langle \vec{f}, \vec{v} \rangle \\ \forall \vec{v} \in \mathbf{H}_0^1(\Omega(\alpha)) \\ b(\vec{u}, r) = 0 \quad \forall r \in L_0^2(\Omega(\alpha)) \\ \langle \vec{u}, \vec{s} \rangle_{\Gamma(\alpha)} = \langle \vec{g}, \vec{s} \rangle_{\Gamma(\alpha)} \quad \forall \vec{s} \in \mathbf{H}^{1/2}(\Gamma(\alpha)) \end{cases} \quad (3.12)$$

$$\begin{cases} \nu a(\vec{w}, \vec{v}) + c(\vec{v}; \vec{u}, \vec{w}) + c(\vec{u}; \vec{v}, \vec{w}) + b(\vec{v}, r) = \\ -2\nu \int_{\Omega(\alpha)} \nabla \vec{u} \cdot \nabla \vec{v} d\vec{x} \quad \forall \vec{v} \in \mathbf{H}_0^1(\Omega(\alpha)) \\ b(\vec{w}, r) = 0 \quad \forall r \in L_0^2(\Omega(\alpha)) \\ \vec{w} = \vec{0} \quad \text{on} \quad \Gamma(\alpha) \end{cases} \quad (3.13)$$

$$\begin{aligned} \int_I \frac{d\mu}{dx} \frac{d\zeta}{dx} dx + \nu \int_{\Gamma_\alpha} \left(\frac{\partial \vec{u}}{\partial n} \frac{\partial \vec{u}}{\partial n} - \frac{\partial \vec{u}}{\partial n} \frac{\partial \vec{w}}{\partial n} \right) \\ \cdot (\vec{V}(\zeta) \cdot \vec{n}) ds = 0 \quad \forall \zeta \in H_0^1(I) \end{aligned} \quad (3.14)$$

$$- \int_I \frac{d\alpha}{dx} \frac{dv}{dx} dx = \int_I q v dx \quad \forall v \in H_0^1(I) \quad (3.15)$$

$$\int_I (\beta q + \mu)(\tilde{q} - q) \geq 0 \quad (3.16)$$

with $\alpha(a) = z_0$ and $\alpha(b) = z_1$ and for all $\tilde{\alpha} \in \mathcal{Q}_{ad}$ and for all $\tilde{q} \in \mathcal{B}_{ad}$.

3.4 The shape gradient

The numerical solution of the coupled system (3.12)–(3.16) of variational equations and inequalities is

formidable. In practice, one does not solve the system simultaneously, instead, one invokes an iteration such that at each step the method requires the sequential solution of the Navier-Stokes system (3.12) and the adjoint system in (3.13). The solution of the two Poisson equations (3.15) and (3.16) is not very expensive but care should be taken to check that the shape function α is in \mathcal{C}_{ad} . The iteration is often implemented so that at each step a guess for the boundary shape Γ_α is made, after which the state system (3.12) can be solved and the design functional evaluated. To determine the new iterate for the boundary shape, a gradient or quasi-Newton method can be used. Efficient implementations of such methods usually require knowledge of the shape gradient, i.e., the gradient of the functional with respect to changes in the boundary shape.

For any candidate optimizer (\vec{u}, p, q, α) satisfying the state system (3.12), the shape gradient is given in terms of the adjoint variables by

$$\begin{aligned} \mathcal{J}'(\vec{u}, q, \alpha)(\vec{w}, \mu, \zeta) &= \int_I \frac{d\mu}{dx} \frac{d\zeta}{dx} dx \\ &+ \nu \int_{\Gamma_\alpha} \left(\frac{\partial \vec{u}}{\partial n} \frac{\partial \vec{w}}{\partial n} - \frac{\partial \vec{u}}{\partial n} \frac{\partial \vec{w}}{\partial n} \right) (\vec{V}(\zeta) \cdot \vec{n}) ds. \end{aligned} \quad (3.17)$$

The vector $\vec{V}(\zeta)$ is simply $(0, \zeta)$, \vec{n} is the unit normal to Γ_α and the adjoint variables \vec{w} and μ are determined from (3.13) and $\mu + \beta q = 0$, respectively. The shape gradient can be computed for multiple directions ζ with a single linear adjoint system solution. On the other hand, if sensitivity equations are used, one must solve a linear sensitivity system for each distinct direction ζ .

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