Controllability and stabilization of liquid vibration in a container during transportation

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Abstract

This paper deals with the modeling and the mathematical analysis of problems involving a rectangular container. The container is controlled via a longitudinal acceleration in order to move it from one location to another, and the key problem is the suppression of sloshing during transportation. Practical control problems involving this system have been studied, from a numerical and experimental point of view. For these aspects we refer to [8], where the mathematical analysis is not deep enough for the study of controllability or stabilization problems. Here we develop a suitable theoretical framework which is similar to the one we have used in [6], since the physical system is the same, but with different input and output operators. This framework allows us to show that approximate controllability in finite time does not hold. We also study the stability of the system when the elevation of the surface is measured at the right end of the container, and a static negative acceleration feedback is used. We show that strong stability holds (but with a non-uniform decay), although the perturbation caused by the feedback on the system operator is not dissipative in the natural topology.

1 Model of the container

We consider the rectangular container on figure 1. The container can be controlled my means of a longitudinal acceleration in the x direction. It is supposed to be wide enough to consider that the motion of the fluid does not depend on the variable z. This allows to use a bidimensional model, where the domain Ω is the rectangle $[-\frac{L}{2}, \frac{L}{2}] \times [0, h]$, represented on figure 1. The boundary $\Gamma = \Gamma_s \cup \Gamma_w$ is represented by $\Gamma_s = \{(x, h) \mid -\frac{L}{2} < x < \frac{L}{2}\}$, the free surface and Γ_w, the bottom, left end and right end of the container. The fluid is supposed to be perfect, incompressible and irrotational. Let $\vec{V}(x, y, t)$ be the velocity field at time t. From the hypothesis curl $\vec{V} = 0$, there exists a potential ψ defined by

$$\vec{V}(x, y, t) = \vec{\nabla}\psi(x, y, t).$$

We now study the boundary conditions on Γ. The boundary condition on Γ_s is a dynamic condition. Let us call $\eta(x, t)$ the elevation of a point $M(x, h)$ of Γ_s with respect to its equilibrium position. From the hypothesis we have made the static

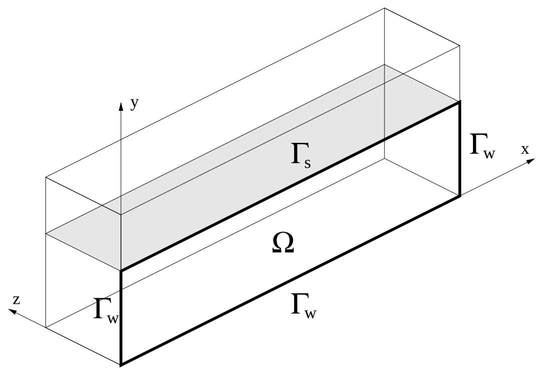

Figure 1: Definition of domain Ω

pressure $P(x, y, t)$ is related to the velocity potential ψ by the Bernoulli condition :

$$\frac{P}{\rho} + \frac{1}{2}|\vec{\nabla}\psi|^2 + \dot{\psi} + g_0\eta + \mathcal{U} = \frac{P_a}{\rho}, \text{ in } \Omega,$$

where ρ is the (constant) volumic mass, g_0 is the acceleration of gravity and P_a the atmospheric pressure (supposed to be constant). The potential $\mathcal{U}$ is such that the volumic acceleration applied $\vec{a}$ are given by $\vec{a} = \vec{\nabla}\mathcal{U}$. Since the acceleration only occurs in the x direction we have

$$\mathcal{U}(x, y, t) = (x + c)u(t),$$

where u is the amplitude of the acceleration of the container and c is some arbitrary constant.

If we express the continuity of the pressure across the free surface then the Bernoulli condition takes the following form :

$$\frac{1}{2}|\vec{\nabla}\psi|^2 + \dot{\psi} + g_0\eta + (x + c)u = 0, \text{ on } \Gamma_s.$$

Once linearized under the hypothesis of small fluid motion, this condition takes the form

$$\dot{\psi} + g_0\eta + (x + c)u = 0, \text{ on } \Gamma_s. \quad (1)$$

The kinematic condition, which expresses the fact that the vertical component of the velocity of a fluid particle $M(x, h)$ of the free surface is equal to the time derivative of $\eta(x, t)$, takes the form

$$\dot{\eta} = \partial_n\psi, \text{ on } \Gamma_s. \quad (2)$$

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The normal velocity of the fluid is zero on boundary Γ_w: if one calls $\vec{n}$ the outer normal to Γ_w, we have

$$\partial_n\psi \equiv \vec{\nabla}\,\psi.\vec{n} = 0, \text{ on } \Gamma_w, \tag{3}$$

Finally, a straightforward analysis shows that the value $c = 0$ is physically compatible with the incompressibility of the fluid, and we obtain the following equations:

$$\begin{aligned} \Delta\psi &= 0 \text{ in } \Omega \times [0,\tau], & (4)\\ \dot{\eta} - \partial_n\psi &= 0 \text{ on } \Gamma_s \times [0,\tau], & (5)\\ \dot{\psi} + g_0\eta + x\,u &= 0 \text{ on } \Gamma_s \times [0,\tau], & (6)\\ \partial_n\psi &= 0 \text{ on } \Gamma_w \times [0,\tau],, & (7) \end{aligned}$$

As we will see in the sequel, we only need to specify initial conditions on Γ_s

$$\psi(0) = \varphi_0, \eta(0) = \eta_0, \text{ on } \Gamma_s. \tag{8}$$

2 Mathematical analysis

To simplify the notations in the sequel and without loss of generality, we will take $g_0 = 1$, $L = \pi$ and $h = 1$.

Let D denote the "Dirichlet map" i.e. the continuous map D : $H^{1/2}(\Gamma_s) \to H^1(\Omega)$ defined by $\mathrm{D}\varphi = \Phi$ where

$$\begin{cases} \Delta\Phi = 0, & \text{in } \Omega,\\ \Phi = \varphi, & \text{on } \Gamma_s,\\ \partial_n\Phi = 0, & \text{on } \Gamma_w. \end{cases} \tag{9}$$

If we define $\varphi = \psi|_{\Gamma_s}$, then the original problem can be transformed in two coupled one dimensional problems on Γ_s,

$$\begin{cases} \dot{\varphi} + \eta = -x\,u, & \text{on } \Gamma_s \times [0,\tau]\\ \dot{\eta} - \mathcal{A}\varphi = 0, & \text{on } \Gamma_s \times [0,\tau],\\ \varphi(0) = \varphi_0, &\\ \eta(0) = \eta_0, & \end{cases} \tag{10}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\varphi = \partial_n \mathrm{D}\varphi|_{\Gamma_s}. \tag{11}$$

It is easy to see that this abstract formulation is related to the original equations by $\psi = \mathrm{D}\varphi$.

2.1 Regularity results

The application of elementary theorems for elliptic problems allows to claim that $\mathcal{A}$ is a linear unbounded operator in $H^{-1/2}(\Gamma_s)$ with domain $H^{1/2}(\Gamma_s)$. If we apply the results of Grisvard (see [4]), for $\varphi \in H^{3/2}(\Gamma_s)$ and the additional compatibility conditions

$$x^{-1/2}\varphi'(\frac{\pi}{2} - x) \in L^2(\Gamma_s),\ (\frac{\pi}{2} + x)^{-1/2}\varphi'(x) \in L^2(\Gamma_s),$$

we have $\mathrm{D}\varphi \in H^2(\Omega)$ (we will note this space $H_c^{3/2}(\Gamma_s)$). Thus we can consider $\mathcal{A}$ as a linear unbounded operator in $H^{1/2}(\Gamma_s)$ with domain $H_c^{3/2}(\Gamma_s)$. Results of interpolation theory (see [5]) allow to finally obtain the following result:

Theorem 2.1 *The operator $\mathcal{A}$ is a linear unbounded operator in $L^2(\Gamma_s)$ with domain $H^1(\Gamma_s)$.*

2.2 Reduction of the state-space

If we consider zero initial data and apply a control u in a time interval $[0,\tau]$, we can see formally that the functions $\varphi(t)$ and $\eta(t)$ will be odd functions, and this will be also the case if we consider odd initial data. Hence, in the following, we will consider the Hilbert space

$$H = \left\{\varphi \in L^2(\Gamma_s),\ \varphi \text{ odd}\right\} \equiv \tilde{L}^2(\Gamma_s),$$

and the domain of operator $\mathcal{A}$

$$D(\mathcal{A}) = \left\{\varphi \in H^1(\Gamma_s),\ \varphi \text{ odd}\right\} \equiv \tilde{H}^1(\Gamma_s).$$

We obtain the following results on $\mathcal{A}$ by classical operator theory and interpolation theory (see [6] for a detailed proof):

Proposition 2.1 *The operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is strictly positive, self-adjoint, and $R(\lambda I + \mathcal{A}) = H$ for $\lambda > 0$.*

This result ensures that the operator $\mathcal{A}^{1/2}$ is also well-defined and we have

$$D(\mathcal{A}^{1/2}) = [D(\mathcal{A}), H]_{1/2} = \tilde{H}^{1/2}(\Gamma_s).$$

Unfortunately we don't have an explicit representation of $\mathcal{A}^{1/2}$, but we have for φ and w in $\tilde{H}^{1/2}(\Gamma_s)$

$$\langle \mathcal{A}\varphi\,,\,w\rangle_{\tilde{H}^{-1/2},\tilde{H}^{1/2}} = \left\langle \mathcal{A}^{1/2}\varphi\,,\,\mathcal{A}^{1/2}w\right\rangle = \int_\Omega \nabla\mathrm{D}\varphi.\nabla\mathrm{D}w.$$

2.3 Spectral analysis

The eigenvalues and associated eigenfunctions of $\mathcal{A}$, i.e. the functions $w_k(x) \in D(\mathcal{A})$ and the numbers λ_k such that $\mathcal{A}\,w_k = \lambda_k\,w_k$, for $k > 0$ integer, are obtained by solving the *Steklov* problem

$$\begin{cases} \Delta W_k = 0 & \text{in } \Omega,\\ \partial_n W_k = \lambda_k W_k & \text{on } \Gamma_s,\\ \partial_n W_k = 0 & \text{on } \Gamma_w, \end{cases} \tag{12}$$

where $w_k = W_k|_{\Gamma_s}$. One easily obtains by separation of variables the following form for W_k :

$$W_k = \alpha\cosh ky \cos k(x + \tfrac{\pi}{2}),$$

where α is an arbitrary constant. Since only odd values of k correspond to eigenpairs because of the choice of $D(\mathcal{A})$, we can choose

$$w_k(x) = \sin(2k-1)x,\ x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}],\ k = 1, 2, \ldots$$

and the eigenvalues λ_k are given by

$$\lambda_k = (2k-1)\tanh(2k-1),\ k = 1, 2, \ldots$$

2.4 Formulation as a first order system

We adopt the following abstract formulation of the original system :

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & A\xi + Bu, \\ \xi(0) & = & \xi_0 \in X. \end{array} \right. \tag{13}$$

The variable ξ is related to the original variables φ and η in (10) by $\xi = (\varphi, \eta)$ and the state space X is the following :

$$X = D(\mathcal{A}^{1/2}) \times H = \tilde{H}^{1/2}(\Gamma_s) \times \tilde{L}^2(\Gamma_s). \tag{14}$$

The operators A, B are defined as follows :

$$A\xi = (-\xi_2, \mathcal{A}\xi_1),\ Bu = (-x\,u, 0), \tag{15}$$

and the domain of A is given by

$$D(A) = D(\mathcal{A}) \times D(\mathcal{A}^{1/2}) = \tilde{H}^1(\Gamma_s) \times \tilde{H}^{1/2}(\Gamma_s). \tag{16}$$

We define the inner product in X by

$$\langle \xi\,,\,\zeta \rangle_X \quad = \quad \int_\Omega \nabla \mathrm{D}\xi_1 . \nabla \mathrm{D}\zeta_1 + \int_{\Gamma_s} \xi_2 \overline{\zeta_2}.$$

We adopt the notation $\|.\|_X$ for the associated norm, which is defined by

$$\|\xi\|_X^2 \quad = \quad \langle \xi\,,\,\xi \rangle_X$$

This norm is equal, up to the constant $\frac{1}{2}$, to the natural energy, in terms of the original potential ψ,

$$E(\psi, \eta) = \frac{1}{2} \int_\Omega \|\nabla \psi\|^2 + \frac{1}{2} \int_{\Gamma_s} |\eta|^2,$$

which is equal to the sum of kinetic and potential energies of the fluid, since we have $\psi = \mathrm{D}\xi_1$.

2.5 Spectral analysis and semigroup generation

One can show that the eigenpairs $(\mu_k, \phi_k)_{k \in \mathbf{Z}^*}$ of operator A are given by $\mu_k = i\omega_k$, where $\omega_k = \sqrt{\lambda_k}$ for $k > 0$, $\omega_k = -\sqrt{\lambda_{-k}}$ for $k < 0$, and

$$\phi_k = \frac{1}{\sqrt{\pi}} (\frac{1}{\mu_k} w_k, w_k),\ \text{for } k \in \mathbf{Z}^*.$$

The family ϕ_k can be shown to be an orthonormal basis of X, with $\|\phi_k\|_X = 1$. This means that A is a Riesz-spectral operator (for a complete theory of Riesz-spectral systems see [3]) and this property immediately gives the the following result (see [3], Theorem 2.3.5) :

Proposition 2.2 *The operator A is the infinitesimal generator of a strongly continuous semigroup of contractions $T(t)$ on $X = \tilde{H}^{1/2}(\Gamma_s) \times \tilde{L}^2(\Gamma_s)$, given by the formula*

$$T(t)\xi = \sum_{k \in \mathbf{Z}^*} e^{i\omega_k t} \langle \xi\,,\,\phi_k \rangle_X\, \phi_k.$$

3 Controllability problems

Let us first recall some definitions:

Definition 3.1 *The controllability map of system (13) on $[0, \tau]$ (for a finite $\tau > 0$) is the bounded map $\mathcal{H}_\tau : L_2(0, \tau; U) \to X$ defined by*

$$\mathcal{H}_\tau u = \int_0^\tau T(\tau - s) B u(s)\, ds.$$

The concept of approximate controllability is defined in the following way :

Definition 3.2 *The system (13) is approximately controllable on $[0, \tau]$ (for a finite $\tau > 0$) if for $\varepsilon > 0$, it is possible to steer from the origin at a distance ε from all elements of X in a finite time τ, say*

$$\overline{\mathrm{Ran}\, \mathcal{H}_\tau} = X. \tag{17}$$

To show (17) one usually tries to show that H_τ^* is one to one ([3], Theorem 4.1.7) :

$$B^* T(t)^* \xi = 0 \text{ on } [0, \tau] \Rightarrow \xi = 0. \tag{18}$$

3.1 Lack of approximate controllability on $[0, \tau]$

The following lemma is crucial in the proof of our main result:

Lemma 3.1 *For any $\tau > 0$, the system $\left\{ e^{i\omega_k t} \right\}_{k \in \mathbf{Z}^*}$ is complete and linked in $L^2(0, \tau)$. Moreover, for any $\tau > 0$, there exists a subset $\mathcal{S}_\tau \subset \mathbf{Z}^*$ such that $\left\{ e^{i\omega_k t} \right\}_{k \in \mathcal{S}_\tau}$ is a Riesz basis of $L^2(0, \tau)$.*

We recall that the terminology "linked system" means that every element of the system is in the closed subspace spanned by the other elements.

Proof: The proof is rather technical and relies mainly on Kadec $\frac{1}{4}$-Theorem (see [9]) and results of Schwartz (see [7]). The detailed proof of a similar result is given in [6]. ∎

As announced above, we have the following negative result:

Theorem 3.1 *The system (13) is not approximately controllable on $[0, \tau]$, for any $\tau > 0$.*

Proof: We have for $k > 0$

$$B^* \phi_k = \langle \phi_k\,,\,(-x, 0) \rangle_X = 2i\sqrt{\lambda_k} \frac{(-1)^{k+1}}{\sqrt{\pi}(2k-1)^2}, \tag{19}$$

and there exists a constant $C > 0$ such that

$$\omega_k^3 |B^* \phi_k| \to C$$

as $k \to \infty$, since $\omega_k = \sqrt{\lambda_k}$ is equivalent to $(2k-1)^{\frac{1}{2}}$. Moreover, we have shown in Lemma 3.1 that the system $\{e^{i\omega_k t}\}_{k\in\mathcal{S}_\tau}$ is a Riesz basis of $L^2(0,\tau)$ and the set $\mathbf{Z}^*\backslash\mathcal{S}_\tau$ is not finite. Hence we can apply a result of Avdonin and Ivanov (see [1], Theorem II.6.6, page 141) to claim that there exists a nonzero sequence $\{a_k\}_{k\in\mathbf{Z}^*}$ such that $\sum_{k\in\mathbf{Z}^*}|a_k|^2|\omega_k|^6 < \infty$ and

$$\sum_{k\in\mathbf{Z}^*} a_k e^{i\omega_k t} = 0, \text{ in } L^2(0,\tau).$$

Hence, if we consider the vector ξ defined by

$$\xi = \sum_{k\in\mathbf{Z}^*} \frac{a_k}{B^*\phi_k}\phi_k,$$

we have $\xi \in X$ since the series

$$\sum_{k\in\mathbf{Z}^*} |\langle \xi\,,\,\phi_k\rangle|^2 = \sum_{k\in\mathbf{Z}^*}\left|\frac{a_k}{B^*\phi_k}\right|^2$$

is convergent. Finally, we have by construction

$$\begin{aligned} B^*T^*(t)\xi &= \sum_{k\in\mathbf{Z}^*} B^*\phi_k\,\langle \xi\,,\,\phi_k\rangle\, e^{i\omega_k t}, \\ &= \sum_{k\in\mathbf{Z}^*} a_k e^{i\omega_k t} = 0, \text{ in } L^2(0,\tau). \end{aligned}$$

This ends the proof. ■

3.2 Generic approximate controllability

One can define a generic concept of approximate controllability on $[0,\infty)$. This is the concept used by R. Curtain and H. Zwart [3]. Its definition is the following one :

Definition 3.3 *Let us call $\mathcal{R}$ the reachable subspace*

$$\mathcal{R} = \bigcup_{\tau\in\mathbf{R}^+} \operatorname{Ran}\mathcal{H}^\tau.$$

The system is approximately controllable on $[0,\infty)$ if $\overline{\mathcal{R}} = X$. ■

As for the case where the controllability time is finite, this type of approximate controllability corresponds to an observability property on the dual system, i.e. approximate controllability on $[0,\infty)$ will hold if

$$B^*T^*(t)\xi = 0, \forall t > 0 \Longrightarrow \xi = 0. \tag{20}$$

Since we have shown that A is a Riesz-Spectral operator, we can use Theorem 4.2.3 in [3] which claims that (20) will hold if

$$B^*\phi_k \neq 0,\ \forall k \in \mathbf{Z}^*,$$

where $\phi_k,\ k \in \mathbf{Z}$ are the eigenfunctions of operator A. The following result follows directly from (19):

Proposition 3.1 *The system (13) is approximately controllable on $[0,\infty)$.*

4 Stabilization

As in the previous sections, we take $g_0 = 1$, $L = \frac{\pi}{2}$, $h = 1$, and we consider the first order formulation (13).

It makes sense to measure the time derivative of the elevation of the surface $\eta = \xi_2$ at $x = \frac{\pi}{2}$. We use an approximation of this pointwise measurement by defining the observation operator C_ε such that

$$\begin{aligned} y(t) &= C_\varepsilon\xi(t) = \frac{1}{\varepsilon}\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}} \dot{\xi}_2(x,t)\,dx, \\ &= \frac{1}{\varepsilon}\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}} \mathcal{A}\xi_1(x,t)\,dx \end{aligned}$$

where $\varepsilon > 0$ is arbitrary small. This kind of observation requires enough regularity for ξ_1 on Γ_s, this point will be clarified in section 4.1.

We also modify the original input operator B : for $u \in \mathbf{R}$ we define $B_\varepsilon u$ as $B_\varepsilon u = (-f_\varepsilon(x)u, 0)$, where

$$f_\varepsilon(x) = \begin{cases} x, & \text{if} \quad 0 \le x \le \frac{\pi}{2}-\varepsilon, \\ \frac{\pi}{2}-\varepsilon+\frac{\varepsilon^2-(x-\frac{\pi}{2})^2}{2\varepsilon}, & \text{if} \quad \frac{\pi}{2}-\varepsilon \le x \le \frac{\pi}{2}. \end{cases}$$

We thus have a modified version of the original system

$$\dot{\xi}(t) = A\xi(t) + B_\varepsilon u(t),$$

together with the observation $y(t) = C_\varepsilon\xi(t)$.

In the following we will study the system that is obtained when $y(t)$ is fed back to the control with a negative sign, i.e. when $u(t) = -y(t)$. The obtained closed loop system is the following

$$\dot{\xi}(t) = A_F\xi(t), \tag{21}$$

where $A_F = A - B_\varepsilon C_\varepsilon$.

Remark 4.1 *We can see that when ξ is regular enough, we have,*

$$C_\varepsilon\xi \to C\xi = \mathcal{A}\xi_1|_{x=\frac{\pi}{2}},$$

as $\varepsilon \to 0$. Moreover B_ε is arbitrary close to the original B, since $f_\varepsilon(x) \to x$ in $L^2(\Gamma_s)$, as $\varepsilon \to 0$. But if we consider the limit of these two operators (i.e. take $\varepsilon = 0$) then the system (21) is not well-posed for regularity reasons (a similar problem is addressed in [6]). Anyway, the modification of C can be seen as a "realistic" interpretation of the idealized pointwise measurement. As far as B_ε is concerned, we can see that we have $f'_\varepsilon(\frac{\pi}{2}) = f'_\varepsilon(-\frac{\pi}{2}) = 0$, this can be seen as a (very simple) modeling of the boundary layer.

4.1 Ad hoc energy

The *ad hoc* energy we propose is based on the bilinear form

$$F(\xi,\zeta) = \sum_{k\in\mathbf{Z}^*} d_k\,\langle \xi\,,\,\phi_k\rangle_X\,\overline{\langle \zeta\,,\,\phi_k\rangle}_X, \tag{22}$$

where the bar denotes the complex conjugate, ϕ_k denotes the k^{th} eigenfunction of A, and

$$d_k = \frac{C_\varepsilon \phi_k}{B_\varepsilon^* \phi_k}.$$

Straightforward computations give for $k > 0$

$$\begin{aligned} B_\varepsilon^* \phi_k &= \langle \phi_k \,,\, (-f_\varepsilon, 0) \rangle_X \,, \\ &= 2i\sqrt{\lambda_k} \frac{(-1)^{k+1} \sin(2k-1)\varepsilon}{\varepsilon\sqrt{\pi}(2k-1)^3}, \end{aligned} \tag{23}$$

and

$$C_\varepsilon \phi_k = i\sqrt{\lambda_k} \frac{(-1)^{k+1} \sin(2k-1)\varepsilon}{\varepsilon\sqrt{\pi}(2k-1)},$$

which gives $d_k = d_{-k} = \frac{1}{2}(2k-1)^2$, for $k > 0$.

Remark 4.2 *The bilinear form $F(\xi, \zeta)$ is a re-weighted form of the classical inner product associated with the open-loop system $\dot{\xi} = A\xi$, i.e.*

$$\langle \xi \,,\, \zeta \rangle_X \quad = \quad \sum_{k \in \mathbf{Z}^*} \langle \xi \,,\, \phi_k \rangle_X \overline{\langle \zeta \,,\, \phi_k \rangle}_X.$$

The positivity of d_k allows to claim that the bilinear form $F(.,.)$ defines a scalar product and

$$F(\xi, \xi) = \sum_{k \in \mathbf{Z}^*} d_k |\, \langle \xi \,,\, \phi_k \rangle_X \,|^2, \tag{24}$$

can be used as a norm defined on the associated energy space, which is to be identified.

Proposition 4.1 *The energy space defined by the convergence of the series in (24) is equal to*

$$X_F = \tilde{H}_c^{3/2}(\Gamma_s) \times \tilde{H}^1(\Gamma_s).$$

Proof: If we use the expression of the eigenfunctions of A we have

$$F(\xi, \xi) = \frac{2}{\pi} \sum_{k>0} d_k \left(\lambda_k |\, \langle \xi_1 \,,\, w_k \rangle \,|^2 + |\, \langle \xi_2 \,,\, w_k \rangle \,|^2 \right), \tag{25}$$

and since we can easily show that

$$\lim_{k \to \infty} \frac{d_k}{\lambda_k^2} = \frac{1}{2},$$

the series (25) is equivalent to the series

$$\frac{2}{\pi} \sum_{k>0} \lambda_k^3 |\, \langle \xi_1 \,,\, w_k \rangle \,|^2 + \lambda_k^2 |\, \langle \xi_2 \,,\, w_k \rangle \,|^2.$$

Thus (25) is convergent if $\xi \in D(\mathcal{A}^{3/2}) \times D(\mathcal{A})$, and we have

$$D(\mathcal{A}^{3/2}) = \left\{ \varphi \in D(\mathcal{A}),\ \mathcal{A}\varphi \in D(\mathcal{A}^{1/2}) \right\} = \tilde{H}_c^{3/2}(\Gamma_s).$$

■

Proposition 4.2 *The domain of A_F is*

$$D(A_F) = \tilde{H}_c^2(\Gamma_s) \times \tilde{H}_c^{3/2}(\Gamma_s),$$

where the space $H_c^2(\Gamma_s)$ is defined by

$$\tilde{H}_c^2(\Gamma_s) = \left\{ \varphi \in \tilde{H}^2(\Gamma_s),\ \varphi'(-\tfrac{\pi}{2}) = \varphi'(\tfrac{\pi}{2}) = 0 \right\}.$$

Proof: The domain of A_F is by definition

$$\begin{aligned} D(A_F) &= \{ \xi \in X_F,\ A_F \xi \in X_F \}\,, \\ &= D(\mathcal{A}^2) \times D(\mathcal{A}^{3/2}), \end{aligned}$$

where $D(\mathcal{A}^2) = \{\varphi \in D(\mathcal{A}),\ \mathcal{A}\varphi \in D(\mathcal{A})\}$. We can identify $D(\mathcal{A}^2)$ by making the following analysis : we will have $\varphi \in D(\mathcal{A}^2)$ if

$$\sum_{k>0} \lambda_k^4 |\, \langle \varphi \,,\, w_k \rangle \,|^2 < +\infty. \tag{26}$$

It is well known that the operator $-\frac{d}{dx^2}$ in $H = \tilde{L}^2(\Gamma_s)$ with the boundary conditions $\varphi'(-\frac{\pi}{2}) = \varphi'(\frac{\pi}{2}) = 0$, has the eigenpairs $((2k-1)^2, w_k)$, for $k > 0$. Since λ_k^4 is equivalent to $(2k-1)^4$, we can claim that the series in (26) is convergent if φ is in the domain of this latter operator, which is exactly $\tilde{H}_c^2(\Gamma_s)$. ■

Hence, in the following, the space X_F will be endowed with the norm

$$\|\xi\|_F^2 \equiv F(\xi, \xi),$$

where $F(.,.)$ is defined by (22), and for $\xi, \zeta \in X_F$, the associated inner product will be

$$\langle \xi \,,\, \zeta \rangle_F \equiv F(\xi, \zeta),$$

and the superscript * will denote adjoint operators with respect to $\langle . \,,\, . \rangle_F$.

The main result of this section relies on the following lemma:

Lemma 4.1 *The operators B_ε and C_ε verify*

$$B_\varepsilon^* = C_\varepsilon,$$

where B_ε^* is the adjoint operator of B_ε, with respect to the inner product $\langle . \,,\, . \rangle_F$.

Proof: For $\xi \in X_F$ we have $B_\varepsilon^* \xi = \langle \xi \,,\, b_\varepsilon \rangle_F$, with $b_\varepsilon = (-f_\varepsilon(x), 0)$. But we have also

$$\begin{aligned} \langle \xi \,,\, b_\varepsilon \rangle_F &= \sum_{k \in \mathbf{Z}^*} d_k \langle \xi \,,\, \phi_k \rangle \overline{\langle b_\varepsilon \,,\, \phi_k \rangle}, \\ &= \sum_{k \in \mathbf{Z}^*} \frac{C_\varepsilon \phi_k}{\langle \phi_k \,,\, b_\varepsilon \rangle} \langle \xi \,,\, \phi_k \rangle \overline{\langle b_\varepsilon \,,\, \phi_k \rangle}, \\ &= \sum_{k \in \mathbf{Z}^*} C_\varepsilon \phi_k \langle \xi \,,\, \phi_k \rangle = C_\varepsilon \xi, \end{aligned}$$

where we have used the fact that $\langle b_\varepsilon \, , \, \phi_k \rangle = \overline{\langle \phi_k \, , \, b_\varepsilon \rangle}$. ■

The situation where $B_\varepsilon^* = C_\varepsilon$ is often denoted by "collocation" of the sensor and the actuator. It is interesting to see that the change of inner product has revealed a rather favorable situation, which was hidden in the original topology.

4.2 Strong stability

We note that the result given by Lemma 4.1 shows that C_ε is bounded for the topology of X_F, although this operator was unbounded in the "natural" topology. We will use this fact in the proof of our main result :

Theorem 4.1 *Suppose that ε is such that $B_\varepsilon^* \phi_k \neq 0$, $\forall k \in$ $\mathbf{Z}^*$. Then, the system*

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & A_F \xi, \ t > 0, \\ \xi(0) & = & \xi_0, \end{array} \right. \qquad (27)$$

with an initial data ξ_0 in X_F, is strongly stable, i.e.

$$lim_{t \to \infty} \| \xi(t) \|_F = 0,$$

moreover, the decay rate of $\| \xi(t) \|_F$ cannot be uniform.

Proof: We first recall that $A_F = A - B_\varepsilon C_\varepsilon = A - B_\varepsilon B_\varepsilon^*$, and we can easily show that $A^* = -A$ for the inner product $\langle . \, , \, . \rangle_F$. Hence we have for $\xi \in D(A_F)$

$$\begin{array}{rcl} \text{Re } \langle A_F \xi \, , \, \xi \rangle_F & = & \text{Re } \langle A \xi \, , \, \xi \rangle_F - \langle B_\varepsilon B_\varepsilon^* \xi \, , \, \xi \rangle_F \, , \\ & = & -|B_\varepsilon^* \xi|^2 \leq 0, \end{array}$$

so that $A - B_\varepsilon B_\varepsilon^*$ is dissipative. We then apply the result given in [2] : if A generates a contraction semigroup in X_F and has a compact resolvent, then $A - B_\varepsilon B_\varepsilon^*$ generates a strongly stable semigroup provided that the pair (A, B_ε) is approximately controllable in the sense of definition 4.14, i.e. $B_\varepsilon^* \phi_k \neq 0$, $\forall k \in \mathbf{Z}^*$.

The operator $A - \lambda I$ is maximal in X_F for $\lambda > 0$ (this is a consequence of proposition 2.1 together with the definition of $D(\mathcal{A}^2)$). Hence, the resolvent $(\lambda I - A)^{-1} : X_F \to D(A_F)$ is bounded. It is also compact, since the injection of $D(A_F) = D(\mathcal{A}^2) \times D(\mathcal{A}^{3/2})$ into $X_F = D(\mathcal{A}^{3/2}) \times D(\mathcal{A})$ can be easily shown to be compact. Thus A generates a contraction semigroup in X_F and has a compact resolvent in X_F.

Finally, the decay rate cannot be uniform since $B_\varepsilon B_\varepsilon^*$ is compact in X_F (bounded and one dimensional range in our case). ■

Remark 4.3 *The approach we have used to show the stability result cannot be applied if we take $\varepsilon = 0$. In this case the perturbation operator BC is not A-bounded in the natural topology. This may suggest that uniform stability could eventually hold, but in this case, even the well-posedness of the feedback system seems not to be a trivial issue.*

5 Numerical results

An interactive real-time simulation, illustrating the benefits of the feedback in terms of sloshing reduction, can be found at the URL

http://www.dma.utc.fr/˜mottelet/CDC2000

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