Equilibrium conditions for nonlocal problems.

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Abstract

This paper is concerned with the obtainment of generalized optimality conditions for a kind of nonlocal variational problems.

1 Introduction

This paper is devoted to the obtainment of generalized equilibrium conditions for the 1-dimensional scalar variational principle

$$\inf \{J(u) : u \in \mathcal{A}\}, \tag{1}$$

where

$$J(u) = \int_{I \times I} W(u'(x_1), u'(x_2))\, dx_1 dx_2, \tag{2}$$

$$\mathcal{A} = \left\{ u \in W^{1,p}(I) : u - u_0 \in W_0^{1,p}(I) \right\} \tag{3}$$

I is an open interval in $\mathbf{R}$ and $u_0 \in W^{1,p}(I)$ such that $J(u_0) < \infty$.

It is well known that the existence of solutions is strongly related to the weak lower semicontinuity of the functional (2). For the problem above this property is equivalent to a sort of inequalities of nonlocal nature: the energy density W must satisfy

$$\sum_{i,j=1}^{2n} W(\lambda_i, \lambda_j) \geq 4 \sum_{i,j=1}^{n} W\left(\frac{\lambda_{2i-1} + \lambda_{2i}}{2}, \frac{\lambda_{2j-1} + \lambda_{2j}}{2} \right) \tag{4}$$

for any $n \in \mathbf{N}$ and any choice $\lambda_1, \lambda_2, ..., \lambda_{2n} \in \mathbf{R}$ (see [14]).

Unfortunately inequalities like (4) are complicated to check and that makes the analysis of existence too difficult. Moreover, this kind of problems is different if we compare it to the relaxation of nonconvex classical variational problems where we can substitute the original problem by its convexified version. The nonlocal nature of problems like (1)-(3) seems to block this approach.

Notice we restrict our attention to the homogeneous problem, $W = W(u'(x_1), u'(x_2))$. The general case $W = W(x_1, x_2, u'(x_1), u'(x_2))$ is briefly discussed in Section 5.

Our approach to the problem is not new (cf. Young [22], McShane [12]). In order to study the original problem (1)-(3) we shall consider its relaxation in terms of the Young measures, generated by sequences of gradients of admissible functions. We shall call it the *generalized problem* (see (10)-(12)). Its analysis on problems without the weak lower semicontinuity property can be used to anticipate the oscillatory behavior of minimizing sequences. Also, the existence of solutions to the original problem (1)-(3) can be dealt once the generalized problem is solved.

The contribution of this work is to provide a tool to study the generalized problem. We mainly concentrate on the obtainment of necessary conditions of optimality for this principle. We get generalized equilibrium conditions (Theorems 4 and 5). They are established by making variations on the Young measures and are only useful when we are dealing with the homogeneous problem (i.e. when W depends only on the gradients). In that case they enable us to solve and describe the optimal structure in some examples. It also helps us to understand the appearance of microstructure and its dependence on the imposed boundary conditions.

Some basic references on Optimization and Relaxation are [6], [7], [9], [15], [18] and [?]. About Variational Calculus using Young measures [11], [13]-[16], [20] and [?] can be looked at. The analysis of principles similar to (1)-(3) can be found in [1], [8], [14] or [17].

The paper is organized as follows: in Section 2 we revise some preliminaries and tools. Section 3 is devoted to the variational analysis and the equilibrium conditions. In Section 4 we apply those conditions exploring one example, already proposed in [14]. In the final section we talk about the limitations of our method for the nonhomogeneous problem.

2 Some preliminaries and tools

Consider the optimization problem (1)-(3). We assume the density energy $W : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is smooth and

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satisfies the bounds

$$\begin{aligned} c\left(|A_1|^p+|A_2|^p-1\right) &\leq W\left(A_1,A_2\right), & (5)\\ W\left(A_1,A_2\right) &\leq C\left(|A_1|^p+|A_2|^p+1\right), & (6)\\ \left|\frac{\partial W}{\partial A_i}\left(A_1,A_2\right)\right| &\leq C\left(|A_1|^{p-1}+|A_2|^{p-1}+1\right) & (7)\\ \left|\frac{\partial^2 W}{\partial A_i^2}\left(A_1,A_2\right)\right| &\leq C\left(|A_1|^{p-2}+|A_2|^{p-2}+1\right) & (8)\end{aligned}$$

$i=1,2$, $2<p<\infty$, $0<c<C$. For simplicity we assume $I=(0,1)$ and $u_0(x)=\gamma x$, $\gamma\in\mathbf{R}$, so that (1)-(3) can be written in a simpler way:

$$\inf\left\{\int_{I\times I} W\left(u'(x_1),u'(x_2)\right)dx_1dx_2 : u\in W^{1,p}(I),\right. \\ \left.\text{with } u(0)=0 \text{ and } u(1)=\gamma\right\}. \qquad (9)$$

As we have mentioned, the lack of weak lower semicontinuity property or the difficulties to check it induces us to consider the following problem:

$$\inf\left\{\overline{J}(\nu):\nu\in\overline{\mathcal{A}}\right\}, \qquad (10)$$

where

$$\overline{J}(\nu)=\int_{I\times I}\int_{\mathbf{R}\times\mathbf{R}} W\left(\lambda_1,\lambda_2\right)d\nu_{x_1}\left(\lambda_1\right)d\nu_{x_2}\left(\lambda_2\right)dx_1dx_2, \qquad (11)$$

and $\overline{\mathcal{A}}$ is the set of Young measures $\nu=\{\nu_x\}_{x\in I}$ such that

$$\int_I\int_{\mathbf{R}}|\lambda|^p\,d\nu_x(\lambda)\,dx<\infty, \quad \int_I\int_{\mathbf{R}}\lambda d\nu_x(\lambda)\,dx=\gamma. \qquad (12)$$

Here, we follow [14] from which we stress the following results. First, Theorem 1 characterizes the Young measures generated by weakly convergent sequences of the form $\{(u'(x_1),u'(x_2))\}$ (this result can also be directly deduced using denseness results of Dirac Young measures (cf. [20], [2])). By using Theorem 1 we can easily prove the second result, Theorem 2, which guarantees that under the preceding hypotheses, (10)-(12) is a generalized version of (9).

Theorem 1 *$\Lambda_{(x_1,x_2)}$ is the Young measure generated by $\left\{\left(u_j'(x_1),u_j'(x_2)\right)\right\}$, $\{u_j(x)\}$ a bounded sequence in $W^{1,p}(I)$, if and only if*

$$\Lambda_{(x_1,x_2)}=\nu_{x_1}\otimes\nu_{x_2}$$

and

$$\int_I\int_{\mathbf{R}}|\lambda|^p\,d\nu_x(\lambda)\,dx<\infty,$$

where $\nu=\{\nu_x\}_{x\in I}$ is the Young measure generated by $\left\{u_j'(x)\right\}$.

Theorem 2 *Under (5) there exists $v\in\overline{\mathcal{A}}$ such that*

$$m=\overline{J}(v)=\inf\left\{\overline{J}(\nu):\nu\in\overline{\mathcal{A}}\right\}, \qquad (13)$$

where m is the infimum given in (1).

Regarding Theorem 1 and the fundamental theorem of Young measures ([4], [15], [18]), we have the representation

$$\lim_{j\to\infty}\int_{I\times I}\psi\left(u_j'(x_1),u_j'(x_2)\right)dx_1dx_2 \\ =\int_{I\times I}\int_{\mathbf{R}\times\mathbf{R}}\psi\left(\lambda_1,\lambda_2\right)d\nu_{x_1}\left(\lambda_1\right)d\nu_{x_2}\left(\lambda_2\right)dx_1dx_2$$

where $\nu=\left\{\nu_{(x_1,x_2)}\right\}_{(x_1,x_2)\in I\times I}$ is the Young measure generated by the sequence of pairs $\left\{\left(u_j'(x_1),u_j'(x_2)\right)\right\}$, provided ψ is continuous and $\left\{\psi\left(u_j'(x_1),u_j'(x_2)\right)\right\}$ converges weakly in $L^1(I\times I)$. Besides, Theorem 2 guarantees at least the existence of a minimizer $v\in\overline{\mathcal{A}}$ for the generalized functional $\overline{J}$. Thus, for any minimizing sequence $\{w_j\}\subset\mathcal{A}$ there is a Young measure $v\in\overline{\mathcal{A}}$ such that $m=\overline{J}(v)=\lim_{i\to\infty}J(w_j)$.

Let us now consider: ν, a homogeneous Young measure in $\overline{\mathcal{A}}$, $\{u_j(x)\}$, a sequence in $\mathcal{A}$ such that $\left\{u_j'(x)\right\}$ generates ν, and $\{\psi_j(x)\}$, a bounded sequence in $W_0^{1,p}(I)$ generating a homogeneous Young measure. Let $\mu=\{\mu_x\}_{x\in I}$ be the Young measure generated by the sequence of pairs $\left\{\left(u_j'(x),\psi_j'(x)\right)\right\}$. Notice that $\left\{\left(u_j'(x),\psi_j'(x)\right)\right\}$ does not necessarily generate a homogeneous Young measure even if each one of its components does.

Under the above circumstances, for any $x\in I$ and any $(\lambda_1,\lambda_2)\in\operatorname{supp}\mu_x$ we have

$$\mu_x\left(\lambda_1,\lambda_2\right)=\mu_x\left(\lambda_2|\,\lambda_1\right)\otimes\nu\left(\lambda_1\right), \qquad (14)$$

where $\mu_x(\cdot|\,\lambda_1)$ is a probability measure for any $\lambda_1\in\operatorname{supp}\nu$ and the map

$$\lambda_1\to\int_{\mathbf{R}}f\left(\lambda_1,\lambda_2\right)d\mu_x\left(\lambda_2|\,\lambda_1\right)$$

is $\nu-$measurable, provided f is integrable with respect to μ_x (see [10], [19]).

The decomposition (14)[1] implies

$$\int_{\mathbf{R}\times\mathbf{R}} f(\lambda_1,\lambda_2)\, d\mu_x(\lambda_1,\lambda_2) = \int_{\mathbf{R}} \left(\int_{\mathbf{R}} f(\lambda_1,\lambda_2)\, d\mu_x(\lambda_2|\lambda_1) \right) d\nu(\lambda_1).$$

We also need the following result:

Proposition 3 *[16]*
(i) If H is continuous and is assumed to verify $|H(\lambda)| \leq C\left(|\lambda|^{p-1}+1\right)$, $p>1$, $C>0$, and

$$\int_{\mathbf{R}} H(\lambda)\Upsilon(\lambda)\, d\nu(\lambda_1) = 0$$

for any $\nu-$measurable function Υ such that

$$\int_{\mathbf{R}} \Upsilon(\lambda)\, d\nu(\lambda) = 0 \quad \text{and} \quad \int_{\mathbf{R}} |\Upsilon(\lambda)|^p\, d\nu(\lambda) < \infty \tag{15}$$

then

$$H(\lambda) = \int_{\mathbf{R}} H(\lambda)\, d\nu(\lambda)$$

for any $\lambda \in \operatorname{supp}\ \nu$.
(ii) If G is continuous and verifies $|G(\lambda)| \leq C\left(|\lambda|^{p-2}+1\right)$, $p>2$, $C>0$ and

$$\int_{\mathbf{R}} G(\lambda)\Gamma(\lambda)\, d\nu(\lambda) \geq 0$$

for any $\nu-$measurable and positive function Γ such that

$$\int_{\mathbf{R}} (\Gamma(\lambda))^{p/2}\, d\nu(\lambda) < \infty, \tag{16}$$

then

$$G(\lambda) \geq 0$$

for any $\lambda \in \operatorname{supp}\ \nu$.

3 Generalized equilibrium conditions

We study the generalized version of the optimization principle (9) under the hypotheses (5)-(8). The generalized version of (9) is the minimization of the functional

$$\overline{J}(\nu) = \int_{I\times I}\int_{\mathbf{R}\times\mathbf{R}} W(\lambda_1,\lambda_2)\, d\nu_{x_1}(\lambda_1)\, d\nu_{x_2}(\lambda_2)\, dx_1 dx_2, \tag{17}$$

with $\overline{\mathcal{A}}$ as the set of admissibility. The independence on the x_i permits us to simplify the problem. We can restrict $\overline{\mathcal{A}}$ to the subset composed by its homogeneous members. To be more precise, given $\nu = \{\nu_x\}_{x\in I} \in \overline{\mathcal{A}}$ we can consider its homogenization $\overline{\nu}$, also a probability measure in $\overline{\mathcal{A}}$, such that $\overline{J}(\nu) = \overline{J}(\overline{\nu})$. The proof of this statement is easy: recall that $\overline{\nu}$ is defined via the formula

$$\langle \overline{\nu}, \chi \rangle = \fint_I \int_{\mathbf{R}} \chi(\lambda)\, d\nu_x(\lambda)\, dx,$$

for any continuous function χ [2], and consequently

$$\int_{\mathbf{R}} \lambda d\overline{\nu}(\lambda) = \gamma. \tag{18}$$

Then

$$\overline{J}(\nu) = \int_I \int_{\mathbf{R}} L(\lambda_2)\, d\nu_{x_2}(\lambda_2)\, dx_2,$$

where

$$\begin{aligned} L(\lambda_2) &= \int_I \left[\int_{\mathbf{R}} W(\lambda_1,\lambda_2)\, d\nu_{x_1}(\lambda_1) \right] dx_1 \\ &= \int_I (\overline{\nu}, W(\cdot,\lambda_2))\, dx_1 \\ &= \int_I \int_{\mathbf{R}} W(\lambda_1,\lambda_2)\, d\overline{\nu}(\lambda_1)\, dx_1. \end{aligned}$$

In the same way we have

$$\begin{aligned} \overline{J}(\nu) &= \int_I \int_{\mathbf{R}} L(\lambda_2)\, d\nu_{x_2}(\lambda_2)\, dx_2 \\ &= \int_I \int_{\mathbf{R}} L(\lambda_2)\, d\overline{\nu}(\lambda_2)\, dx_2 \\ &= \int_{I\times I} \int_{\mathbf{R}^2} W(\lambda_1,\lambda_2)\, d\overline{\nu}(\lambda_1)\, d\overline{\nu}(\lambda_2)\, dx_1 dx_2 \\ &= \overline{J}(\overline{\nu}). \end{aligned}$$

We analyze (17) assuming that the elements competing in the principle are only the homogeneous Young measures ν of $\overline{\mathcal{A}}$ such that $\int_{\mathbf{R}} \lambda d\nu(\lambda) = \gamma$. We denote this set of admissibility by $\overline{\mathcal{A}}_h^\gamma$. Let the framework be the one introduced before Theorem **??**. We consider the gradients $w_j'(x) = u_j'(x) + t\psi_j'(x)$, where $\{u_j'(x)\}$ generates the homogeneous Young measure ν, a minimizer of the general principle (17). If we take $\{w_j'\}$ and consider μ^t, its homogeneous Young measure in $\overline{\mathcal{A}}_h^\gamma$, we set for $t \in \mathbf{R}$ the function

$$h(t) \doteq \overline{J}(\mu^t) = \int_{\mathbf{R}\times\mathbf{R}} W(\alpha_1,\alpha_2)\, d\mu^t(\alpha_1)\, d\mu^t(\alpha_2).$$

[1] In Young measure theory this is usually written as $\mu_x(d\lambda_1, d\lambda_2) = \mu_x(d\lambda_2|\,\lambda_1) \otimes \nu(d\lambda_1)$, which distinguishes parameters from integration variable

[2] That is the definition for the homogenization of a Young measure; however the average does not affect the integral because $|I| = 1$.

Consequently $h(t)$ coincides with the expression

$$\int_{I\times I}\int_{\mathbf{R}^2\times\mathbf{R}^2} W(S_1,S_2)\ d(\gamma_{x_1}(\lambda_1,\delta_1)\otimes\gamma_{x_2}(\lambda_2,\delta_2))\,dx_1dx_2,$$

where $t\in\mathbf{R}$, $S_i=\lambda_i+t\delta_i$ and $\gamma=\{\gamma_{x_i}(\lambda_i,\delta_i)\}_{x_i\in I}$ is the Young measure generated by the sequences of pairs $\{(u_j'(x_i),\psi_j'(x_i))\}$, $i=1,2$.

The function h has a minimum for $t=0$. By the smoothness assumptions on W we have the classical equilibrium conditions: $h'(0)=0$ and $h''(0)\geq 0$. Henceforth the point is to express these conditions in a more transparent way.

By (14) we have the decomposition

$$\gamma_{x_i}(\lambda_i,\delta_i)=\gamma_{x_i}(\delta_i|\,\lambda_i)\otimes\nu(\lambda_i). \qquad (19)$$

We define the homogenized Young measure of γ by

$$\langle\overline{\gamma},\chi(\lambda_i,\delta_i)\rangle=\fint_I\int_{\mathbf{R}^2}\chi(\lambda_i,\delta_i)\,d\gamma_{x_i}(\lambda_i,\delta_i)\,dx_i.$$

Then

$$\langle\overline{\gamma},\chi(\cdot)\rangle=\int_{\mathbf{R}}\chi(\lambda_i)\,d\nu(\lambda_i)\ =\langle\nu,\chi(\cdot)\rangle,$$

which implies that ν is the canonical projection onto $\mathbf{R}$ of $\overline{\gamma}$ ($\nu(E)=\overline{\gamma}(E\times\mathbf{R})$). Now, defining $\overline{\gamma}(\delta_i|\,\lambda_i)$ via the formula

$$\langle\overline{\gamma}(\delta_i|\,\lambda_i),\chi(\cdot)\rangle=\fint_I\int_{\mathbf{R}}\chi(\delta_i)\,\gamma_{x_i}(\delta_i|\,\lambda_i)\,dx_i$$

(χ continuous), and using (19) we see

$$\begin{aligned}\langle\overline{\gamma},\chi(\lambda_i,\delta_i)\rangle &= \int_{\mathbf{R}}\left[\fint_I\int_{\mathbf{R}}\chi(\lambda_i,\delta_i)\,d\gamma_{x_i}(\delta_i|\,\lambda_i)\right]d\nu(\lambda_i)\\ &= \int_{\mathbf{R}^2}\chi(\lambda_i,\delta_i)\,d(\overline{\gamma}(\delta_i|\,\lambda_i)\otimes\nu(\lambda_i)).\end{aligned}$$

Therefore

$$\overline{\gamma}(\lambda_i,\delta_i)=\overline{\gamma}(\delta_i|\,\lambda_i)\otimes\nu(\lambda_i), \qquad (20)$$

and consequently h can be read as

$$h(t)=\int_{\mathbf{R}^2\times\mathbf{R}^2}W(S_1,S_2)\ d(\overline{\gamma}(\lambda_1,\delta_1)\otimes\overline{\gamma}(\lambda_2,\delta_2)).$$

We should take only into account that $h'(t)=0$ ((7) and (8) permit us the interchange of derivation and integration, see [5] p. 215) and use (20) to arrive at

$$\begin{aligned}0 &= \int_{\mathbf{R}^2}\left\{\frac{\partial W(\lambda_1,\lambda_2)}{\partial A_1}\Upsilon(\lambda_1)+\right.\\ &\left.+\frac{\partial W(\lambda_1,\lambda_2)}{\partial A_2}\Upsilon(\lambda_2)\right\}d\nu(\lambda_1)\,d\nu(\lambda_2),\end{aligned}$$

and by changing variables we get

$$0=\int_{\mathbf{R}}H(\lambda_1)\,\Upsilon(\lambda_1)\,d\nu(\lambda_1)$$

where

$$H(\lambda_1)\doteq\int_{\mathbf{R}}\left\{\frac{\partial W}{\partial A_1}(\lambda_1,\lambda_2)+\frac{\partial W}{\partial A_2}(\lambda_2,\lambda_1)\right\}d\nu(\lambda_2).$$

We observe $\int_{\mathbf{R}}\Upsilon(\lambda_j)\,d\nu(\lambda_j)=0$, and by Jensen's inequality $\int_{\mathbf{R}}|\Upsilon(\lambda_j)|^p\,d\nu(\lambda_j)<\infty$.

Reciprocally, for any field Υ fulfilling (15), we can find a sequence $\{(u_j'(x_i),\psi_j'(x_i))\}$, where $\{u_j'(x)\}$ generates ν and $\{\psi_j(x)\}$ is a bounded sequence in $W_0^{1,p}(I)$, such that its Young measure can be written as

$$\mu(\lambda,\delta)=\mu(\delta|\,\lambda)\otimes\nu(\lambda)$$

and $\int_{\mathbf{R}}\delta d\mu(\delta|\,\lambda)=\Upsilon$ (see [14] and [15] for a complete discussion). Therefore, we can now use Proposition 3 (i) to state the following result.

Theorem 4 *Under the above circumstances*

$$\text{supp}\ \nu\subset\left\{\lambda\in\mathbf{R}:H(\lambda)=\textstyle\int_{\mathbf{R}}H(\lambda_1)\,d\nu(\lambda_1)\right\}.$$

Now, our investigation is concerned with the condition $h''(0)\geq 0$. According to this inequality and by some simple computations we have

$$\begin{aligned}0 \leq &\int_{\mathbf{R}^2}\left\{\frac{\partial^2 W}{\partial A_1^2}(\lambda_1,\lambda_2)\,\Gamma(\lambda_1)+\frac{\partial^2 W}{\partial A_2^2}(\lambda_1,\lambda_2)\,\Gamma(\lambda_2)+\right.\\ &\left.+2\frac{\partial^2 W}{\partial A_1\partial A_2}(\lambda_1,\lambda_2)\,\Upsilon(\lambda_1)\,\Upsilon(\lambda_2)\right\}d\nu(\lambda_1)\,d\nu(\lambda_2),\end{aligned}$$

where

$$\Gamma(\lambda_i)=\int_{\mathbf{R}}\delta_i^2 d\overline{\gamma}(\delta_i|\,\lambda_i).$$

By Jensen's inequality and changing the variables we have

$$\begin{aligned}0\leq &\int_{\mathbf{R}}\left\{\int_{\mathbf{R}}\left\{\frac{\partial^2 W}{\partial A_1^2}(\lambda_1,\lambda_2)+\right.\right.\\ &+\frac{\partial^2 W}{\partial A_2^2}(\lambda_2,\lambda_1)+\frac{\partial^2 W}{\partial A_1\partial A_2}(\lambda_1,\lambda_2)\\ &\left.\left.+\frac{\partial^2 W}{\partial A_1\partial A_2}(\lambda_2,\lambda_1)\right\}d\nu(\lambda_2)\right\}\Gamma(\lambda_1)\,d\nu(\lambda_1)\end{aligned}$$

Obviously $\Gamma(\lambda_i) \geq 0$ and again by Jensen's inequality $(p \geq 2)$ $\int_{\mathbf{R}} |\Gamma(\lambda_i)|^{p/2} d\nu(\lambda_i) < \infty$. As before, given any positive Γ verifying (16), we can built a sequence of pairs whose Young measure is written in the form $\mu(\lambda, \delta) = \mu(\delta|\lambda) \otimes \nu(\lambda)$ and such that the second moment $\int_{\mathbf{R}} \delta^2 d\mu(\delta|\lambda)$ coincides with Γ. So, by Proposition 3 (ii) $G(\lambda_1) \geq 0$ for any $\lambda_1 \in \operatorname{supp}\nu$, where

$$\begin{aligned} G(\lambda_1) \doteq & \int_{\mathbf{R}} \left\{ \frac{\partial^2 W}{\partial A_1^2}(\lambda_1, \lambda_2) + \right. \\ & + \frac{\partial^2 W}{\partial A_2^2}(\lambda_2, \lambda_1) + \frac{\partial^2 W}{\partial A_1 \partial A_2}(\lambda_1, \lambda_2) \\ & \left. + \frac{\partial^2 W}{\partial A_1 \partial A_2}(\lambda_2, \lambda_1) \right\} d\nu(\lambda_2). \end{aligned}$$

Theorem 5

$$\operatorname{supp} \nu \subset \{\lambda \in \mathbf{R} : G(\lambda) \geq 0\}.$$

4 One example

We show how Theorems 4 and 5 may be applied to solve the generalized problem and to decide about the existence of solutions to the original problem (9), in some simple situations.

Let us take

$$W(\lambda_1, \lambda_2) = \left(\lambda_1^2 + \lambda_2^2 - 1\right)^2,$$

and suppose ν is a minimizer for (17). The equilibrium condition of Theorem 4 gives in this case the identity

$$\begin{aligned} &\int_{\mathbf{R}} \int_{\mathbf{R}} \left\{ 8\lambda_1 \left(\lambda_1^2 + \lambda_2^2 - 1\right) \right\} d\nu(\lambda_2)\, d\nu(\lambda_1) \\ &\quad = \int_{\mathbf{R}} \left\{ 8\lambda_1 \left(\lambda_1^2 + \lambda_2^2 - 1\right) \right\} d\nu(\lambda_2) \end{aligned} \tag{21}$$

for any $\lambda_1 \in \operatorname{supp}\nu$. That is This fact shows that at most there exists three different mass points for ν; let us denote them by γ_1, γ_2 and γ_3, and suppose $\nu = \alpha_1 \delta_{\gamma_1} + \alpha_2 \delta_{\gamma_2} + \alpha_3 \delta_{\gamma_3}$, where $\alpha_i \in [0,1]$, $\sum_{i=1}^3 \alpha_i = 1$. If we write (21) in terms of the moments for ν, we obtain

$$\gamma_j^3 + (m_2 - 1)\gamma_j = m_3 + m_1(m_2 - 1), \tag{22}$$

where $m_k = \sum_{j=1}^3 \alpha_j \gamma_j^k$, $k = 1, 2, 3$. We supply the above identity with the constraint $u(x) = \gamma \cdot x$ on $\partial(0,1)$ (see (18)), thus $m_1 = \gamma$ or

$$\sum_{j=1}^3 \alpha_j \gamma_j = \gamma; \tag{23}$$

finally, we arrive at a system of equations whose unknowns are γ_1, γ_2, γ_3, α_1 and α_2. Moreover, by Theorem 5 we have for any γ_j, $j = 1, 2, 3$, the constraints

$$\Upsilon(\gamma_j) = 3\gamma_j^2 + 2m_1\gamma_j + m_2 - 1 \geq 0. \tag{24}$$

Notice that the information about minimizing measures has been reduced to the study of the nonlinear system (22)-(23) with the constraints (24). We get

R-1. If $\gamma \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ we obtain

(i) $\nu = \delta_c$, with $c \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$,

(ii) $\nu = d\delta_{\frac{\sqrt{2}}{2}} + (1-d)\delta_{-\frac{\sqrt{2}}{2}}$, and

(iii) $\nu = d\delta_{\frac{-1}{2}} + (1-d)\delta_{\frac{1}{2}}$.

R-2. If $\gamma > \frac{\sqrt{2}}{2}$ or $\gamma < -\frac{\sqrt{2}}{2}$ we have $\nu = \delta_c$, where $c > \frac{\sqrt{2}}{2}$ or $c < -\frac{\sqrt{2}}{2}$ respectively.

In the range R-1 the unique solution is $\nu = \frac{1+\gamma\sqrt{2}}{2}\delta_{\frac{\sqrt{2}}{2}} + \frac{1-\gamma\sqrt{2}}{2}\delta_{-\frac{\sqrt{2}}{2}}$. We discard $\nu = d\delta_{-\frac{1}{2}} + (1-d)\delta_{\frac{1}{2}}$ because using (24) we necessarily get that $d = \frac{1}{2}$ and $\overline{J}\left(\frac{1}{2}\delta_{-\frac{1}{2}} + \frac{1}{2}\delta_{\frac{1}{2}}\right) > 0$. In R-2 there exists only one solution, $\nu = \delta_\gamma$. Consequently we can write the infimum of the problem as a function of γ :

$$\overline{J}_W(\gamma) = \begin{cases} 0 & \text{if } -\frac{\sqrt{2}}{2} \leq \gamma \leq \frac{\sqrt{2}}{2} \\ \left(2\gamma^2 - 1\right)^2 & \text{otherwise.} \end{cases}$$

There are some other examples for which the optimality conditions can be applied: for instance, we can take

$$W(\lambda_1, \lambda_2) = ((\lambda_1 + \lambda_2)^2 - 1)^2 + (\lambda_1 - \lambda_2)\arctan(\lambda_1 + \lambda_2)^2$$

or

$$W(\lambda_1, \lambda_2) = (\lambda_1^2 + \lambda_2^2 - 1)^2 + \arctan\lambda_1^2 + \arctan\lambda_2^2.$$

5 Final remarks

If $W = W(x_1, ..., x_n, u(x_1), ..., u(x_2), u'(x_1), ..., u'(x_n))$ the analysis exhibited in sections 2 and 3 is not useful to derive selective necessary conditions. In this case a different kind of variational analysis has to be performed. Thus, if $\nu = \{\nu_x\}_{x \in (0,1)}$ is any minimizer of (10)-(12) with $W = W(x_1, x_2, \lambda_1, \lambda_2)$, then

$$\frac{\partial}{\partial x_1}\left\{\int_0^1 [A(x_1, x_2) + B(x_2, x_1)]\, dx_2\right\} = 0 \tag{25}$$

in a weak sense, and

$$\int_0^1 C(x_1, x_2)\, dx_2 \geq 0 \quad \text{a.e. } x_1 \in (0,1), \tag{26}$$

where $\mathbf{x} = (x_1, x_2)$, $\lambda = (\lambda_1, \lambda_2)$,

$$\begin{aligned} A(x_1, x_2) &= \int_{\mathbf{R}^2} \frac{\partial W}{\partial A_1}(x_1, x_2, \lambda_1, \lambda_2)\, d\nu_{x_1}(\lambda_1)\, d\nu_{x_2}(\lambda_2), \\ B(x_1, x_2) &= \int_{\mathbf{R}^2} \frac{\partial W}{\partial A_2}(x_1, x_2, \lambda_1, \lambda_2)\, d\nu_{x_1}(\lambda_1)\, d\nu_{x_2}(\lambda_2), \end{aligned}$$

and

$$C(x_1,x_2)=\int_{\mathbf{R}^2}\left\{\frac{\partial^2 W}{\partial A_1^2}(x_1,x_2,\lambda_1,\lambda_2)+\right.$$
$$+\frac{\partial^2 W}{\partial A_1\partial A_2}(x_1,x_2,\lambda_1,\lambda_2)+\frac{\partial^2 W}{\partial A_1\partial A_2}(x_2,x_1,\lambda_2,\lambda_1)$$
$$\left.+\frac{\partial^2 W}{\partial A_2^2}(x_2,x_1,\lambda_2,\lambda_1)\right\}d\nu_{x_1}(\lambda_1)\,d\nu_{x_2}(\lambda_2).$$

Clearly, the equilibrium conditions (25), (26) do not seem to be the appropriate way to detect generalized minimizers. Even for the homogeneous case, the equilibrium conditions (Theorems 4 and 5) could be imprecise or not explicit enough to obtain solutions. In either case, it seems to be necessary to implement some additional information: for instance, we might, in some problems, complete the obtained necessary conditions with some sort of result limiting the number of Dirac deltas appearing (as a convex combination) in the representation of the solution. This approach has been used in Balder [3] within the frame of nonconvex optimal control problems, Winkler [21] for the minimization of affine functionals defined on moments sets, and also [6] for a problem in optimal design.

Finally, I want to point out that the analysis of Section 3 in higher dimensions becomes tremendously difficult. The crucial point is that we have to consider Young measures $\mu(\lambda_1,\lambda_2)$ coming from sequences of gradients of $\{(u_j(x),\psi_j(x))\}$, $x\in\Omega$, which are vector valued functions. This fact implies profound restrictions on the measure μ, and therefore on the fields Γ and Υ. The difficulties we have to face require a deeper analysis than the one performed here.

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