

# Delay sensitivity of quadratic controllers. A singular perturbation approach.

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## Abstract

We study the variations of the quadratic performance associated to a linear differential system of retarded type for small values of the delays. From an interpretation of delays as singular perturbations of abstract evolution operators, we revisit the usual theory of representation and optimal control of retarded systems. This leads to a new parameterization of associated Riccati operators for which insight is gained in the dependence on the delays. This explicit parameterization of Riccati operators by the delays enables us to prove differentiability at zero for performance viewed as a function of the delays, in the LQ-optimal or  $\mathcal{H}_\infty$  sub-optimal control. The gradient is explicitly computed in terms of the non-negative solution of the finite dimensional Riccati equation associated to the non-delay control problem.

## 1 Introduction

Small time delay often appears as a side effect of network control of physical systems. In the specific case where the sensors, actuators and processors share a single channel, the true open loop system has delayed inputs and outputs, and possibly state delays resulting from delayed output feedback. In view of real time control, such delays are small but strongly dependent on ordering decisions, and it would be desirable for a given control law to be able to have an *a-priori* estimate of the sensitivity of the performance of the controlled system to small delays.

This paper mainly addresses the issue of the variations of the optimal value of a quadratic cost associated to a linear system perturbed by small delays in its inputs and state. Therefore, given some delays  $k$  and  $h$  with  $(k, h) \in [0, K] \times [0, H]$  for  $K, H > 0$ , we consider the following equation

$$\begin{cases} \dot{x}(t) = A_0x(t) + A_1x(t-k) + B_0u(t) + B_1u(t-h), \\ \text{almost everywhere (a.e.) } t \in [0, T], x(0) = x_0 \in \mathbb{R}^n, \\ x = x_0, \text{ a.e. } t \in [-K, 0], x_0 \in L^2(-K, 0; \mathbb{R}^n), \\ u \in L^2(-H, T; \mathbb{R}^m), \quad u = u_0 \text{ a.e. } t \in [-H, 0], \end{cases} \quad (1)$$

where  $A_i \in \mathcal{L}(\mathbb{R}^n)$  and  $B_j \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$  for  $(i, j) \in \{0, 1\}$ . Also, to the solution  $x(\cdot) = x(\cdot; k; h; u)$  of (1), we associate

the following quadratic cost

$$J_T(x) = \frac{1}{2} \langle x(T), G(T)x(T) \rangle_{\mathbb{R}^n} + \frac{1}{2} \int_0^T \{ \|Cx\|_{\mathbb{R}^p}^2 + \langle u, Ru \rangle_{\mathbb{R}^m} \} dt \quad (2)$$

where  $G(T) \in \mathcal{L}(\mathbb{R}^n)$  is non-negative for  $T \in \mathbb{R}_+$  and  $G(\infty) = 0$ ;  $C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ ;  $R$  positive. Moreover, by  $\hat{J}_T(k, h)$  we denote the infimum  $\min_{u \in L^2(0, T; \mathbb{R}^m)} J_T(x(\cdot; k; h; u))$ ,

for  $T \in \mathbb{R}_+ \cup \{+\infty\}$ . In this paper, the central problem of interest is the computation of the gradient of the function  $(k, h) \rightarrow \hat{J}_T(k, h)$  at the point  $(k, h) = (0, 0)$ .

We recall that in a more general framework of systems of evolution with small changes in some parameter that Pritchard[19] has shown that there exists some constant  $0 < c < 1$ , depending on the non-delay optimal feedback and such that  $c\hat{J}_T(0, 0) \leq \hat{J}_T(k, 0)$ , whereas Dontchev [11, 12, 13] with singular perturbation techniques computes  $|\hat{J}_T(k, 0) - \hat{J}_T(0, 0)|$  in terms of the multipliers occurring in the optimality condition. For the state delay case, Clarke & Wolenski [2, Theorem 3.2] prove the differentiability of the optimum of some associated performance and provide a characterization of the derivative in the multipliers occurring in the optimality condition. In this paper, we propose a simplification of these results in the case of quadratic minimization, and extend them to the optimum of the min-max problem associated to  $\mathcal{H}_\infty$  disturbance attenuation via state feedback controller. We prove the existence of the partial derivative at zero under weaker conditions than those of Clarke & Wolenski [2]. Moreover, and this is our major contribution, we provide an explicit gradient formula for the quadratic optimal cost with partial derivatives at zero, simply expressed in terms of the solution of the finite dimensional Riccati equation solving the non-delay optimal control problem, for both finite and infinite horizons. We also state a similar result for the  $\mathcal{H}_\infty$  robust performance. Extended version of this paper as well as complete proofs of the results presented here can be found in [17].

The keynote of this paper is an interpretation of the delays as singular perturbations of some evolution operators that leads to an unusual variant of classical state space representations via semigroup techniques. For the representation of solutions, we refer to Ichikawa [14], Delfour & Karrakchou [7, 8], and Delfour [6]. Then using a well-known compen-

sation trick — see e.g. Kokotović & Yackel [16] for the finite dimensional case — we parameterize Riccati operators with the delays when revisiting optimal control of delay systems in the Pritchard & Salamon Class [20, 21] and provide for Riccati operators some new weak differentiability-like results —. With these differentiability-like results, the problem of the sensitivity analysis of the optimal becomes trivial.

## 2 Main results

**Theorem 2.1** *Let us assume  $T < \infty$ . Then*

(i) *with  $u_0$  continuous on the left side of 0, we have*

$$\frac{\partial \widehat{J}_T}{\partial h}(0,0) = \langle x_{00}, \Delta(P_T)x_{00} \rangle_{\mathbb{R}^n}, \quad (1)$$

where  $\Delta(P_T) = P_T(0)B_1R^{-1}B^*P_T(0)$  for  $P_T$  the non-negative symmetric solution of the Riccati equation

$$\begin{aligned} \frac{dP_T}{dt} + A^T P_T + P_T A - P_T B R^{-1} B^T P_T + C^T C &= 0, \\ P_T(T) &= G(T), \text{ in } C(0, T; \mathcal{L}(\mathbb{R}^n)). \end{aligned} \quad (2)$$

(ii) *With  $x_0$  continuous on the left side of 0, we have*

$$\frac{\partial \widehat{J}_T}{\partial k}(0,0) = -\langle P_T x_{00}, A_1 x_{00} \rangle_{\mathbb{R}^n}. \quad (3)$$

□

**Theorem 2.2** *Assume now the triple  $(A, B, C)$  stabilizable and detectable. Then*

(iii) *with  $u_0$  continuous on the left side of 0, we have*

$$\frac{\partial \widehat{J}_\infty}{\partial h}(0,0) = \langle x_{00}, \Delta(P_\infty)x_{00} \rangle_{\mathbb{R}^n}, \quad (4)$$

where  $\Delta(P_\infty) = P_\infty B_1 R^{-1} B^* P_\infty$  for  $P_\infty$  the non-negative symmetric solution of the Riccati equation

$$A^* P_\infty + P_\infty A - P_\infty B R^{-1} B^* P_\infty + C^* C = 0 \text{ in } \mathcal{L}(\mathbb{R}^n). \quad (5)$$

(iv) *With  $x_0$  continuous on the left side of 0, we have*

$$\frac{\partial \widehat{J}_\infty}{\partial k}(0,0) = -\langle P_\infty x_{00}, A_1 x_{00} \rangle_{\mathbb{R}^n}. \quad (6)$$

□

**Remark 2.1** For the existence of partial derivatives with respect to state delays, we need only continuity of  $x_0$  on the left side of 0, a condition which is weaker than the existence of  $\|\dot{x}_0\|_\infty$  as required in Clarke & Wolenski [2, Theorem 3.2]. □

**Example 2.1 Sensitivity to small input delay.** If  $B_0 = 0$ , then  $\Delta(P_\infty) = P_\infty B_1 R^{-1} B^* P_\infty = K_{opt} R K_{opt} \geq 0$  where the

matrix  $K_{opt} = -R^{-1} B_1^* P_\infty$  is the optimal gain for  $h = 0$ . Therefore formula (4) becomes

$$\frac{\partial \widehat{J}_\infty}{\partial h}(0,0) = \langle K_{opt} x_{00}, R K_{opt} x_{00} \rangle_{\mathbb{R}^n} \geq 0, \quad (7)$$

from where we have  $\widehat{J}_T(0,0) \leq \widehat{J}_T(0,h)$  for a sufficiently small  $h$ . Moreover, we see that the degradation of the optimal performance increases with the optimal gain. ■

## 3 Singular perturbations and product-space approach

First of all we point out that for computation of the partial derivatives at the point  $(k, h) = (0, 0)$ , we only need to compare optimal cost with solely one state or input delay, to non-delay optimal cost. So, in the following, with  $A = A_0 + A_1$  and  $B = B_0 + B_1$ , we shall look separately for state space representations when  $(k = 0, h > 0)$  and when  $(k > 0, h = 0)$ .

### 3.1 Retarded input system: $(k = 0, h > 0)$

First recall that in this case, the solution of equation (1) is given in  $H_{loc}^1(0, \infty; \mathbb{R}^n)$  by the variation of constants formula

$$x(t) = e^{At} x_{00} + \int_0^t e^{A(t-s)} \{B_0 u(s) + B_1 u(s-h)\} ds. \quad (8)$$

But for controller synthesis, it is more convenient to have a description of this solution with a variation of constants formula associated to a non-delay evolution system. A common way is to follow the choice originated by Ichikawa [14] which consists in choosing as a support for description the extended state defined by the pair  $(x(t), v(t, \cdot))' \in \mathcal{H}_h^m = \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^m)$  with the segment function  $v$  given by  $v(t, \theta) = u(t + \theta)$  for a.e.  $t \in [0, T]$  and  $\theta \in [-h, 0]$ . Such a choice leads to an equivalent linear system of evolution in  $\mathcal{H}_h^m$  with unbounded input but bounded solutions as explained in [14] (see also Bensoussan et al. [1]). That line is well known to provide both optimal feedback and cost.

Now we point out that it is nothing but the presence of the delay that has brought the system from finite to infinite dimensions, so the delay may be interpreted as a singular perturbation in a convenient framework. Moreover, in view of sensitivity analysis, it is desirable for simplicity to have a non delay dependent state space. A natural way to do this is to rescale the variable  $\theta$ , in order to chose  $\mathcal{H}_1^m = \mathbb{R}^n \times L^2(-1, 0; \mathbb{R}^m)$  as state space.

Introduce  $u^h$ , and  $\phi^h$ , defined in  $L^2(0, T; L^2(-1, 0; \mathbb{R}^m))$ , and  $L^2(-1, 0; \mathbb{R}^m)$ , respectively, by  $u^h(t, \sigma) = v(t, \sigma h)$ , and  $\phi^h(\sigma) = u_0(\sigma h)$ , for a.e.  $t \in [0, T]$  and  $\sigma \in [-1, 0]$ . Then, using the shorthand  $\mathcal{H}^m = \mathcal{H}_1^m$ , setting  $X_h(t, \cdot) = \begin{pmatrix} x(t) \\ u^h(t, \cdot) \end{pmatrix} \in \mathcal{H}^m$  and  $\mathcal{W} = \mathbb{R}^n \times H^1(-1, 0; \mathbb{R}^m)$ , we get, for any  $h > 0$ , the following equation of evolution follows from standard

transposition techniques:

$$\begin{cases} \frac{\partial X_h}{\partial t} &= \mathcal{A}_h X_h + \mathcal{B}_h u \quad \text{in } L^2(0, T; \mathcal{W}^m)' \\ X_h(0, \cdot) &= \begin{pmatrix} x_{00} \\ \phi^h \end{pmatrix} \in \mathcal{H}^m, \end{cases} \quad (9)$$

where  $\mathcal{A}_h = \begin{pmatrix} A & B_1 \delta_{\sigma=-1} \\ 0 & \frac{1}{h} D \end{pmatrix}$ ,  $\mathcal{D}(\mathcal{A}_h) = \mathbb{R}^n \times \mathcal{D}(D)$  with  $D = \frac{\partial}{\partial \sigma}$  with domain  $\mathcal{D}(D) = \{u \in H^1(-1, 0; \mathbb{R}^m), u(0) = 0\}$  and  $\mathcal{B}_h = \begin{pmatrix} B_0 \\ \frac{1}{h} \delta_{\sigma=0}^* \end{pmatrix} \in \mathcal{L}(\mathbb{R}^m, \mathcal{W}^m)'$ .

Now it clear that  $h = 0$  is a singularity of operators  $\mathcal{A}_h$  and  $\mathcal{B}_h$ . The next step is the well-posedness of that equation. At this stage, observe that if - by construction - the extended state space is  $\mathcal{H}^m$  then the input operator  $\mathcal{B}_h$  is unbounded in  $\mathcal{H}^m$ . Further insight into the characteristics of the operator  $\mathcal{A}_h$  will show that despite that unboundedness, the solution of equation (9) will remain inside  $\mathcal{H}^m$ .

**Proposition 3.1** *Henceforth let the space  $\mathcal{H}^m$  be identified with its dual and take it as a pivot space. Then*

- i)  $\mathcal{A}_h$  is the infinitesimal generator of a  $C_0$ -semigroup  $S_h(t)$  on  $\mathcal{H}^m$ . In addition, the  $\mathcal{H}^m$ -adjoint of the operator  $\mathcal{A}_h$  is given with  $\mathcal{D}(\mathcal{A}_h^*) = \left\{ \begin{pmatrix} y \\ w \end{pmatrix} \in \mathcal{W}^m, w(-1) = hB_1^* y \right\}$  by  $\mathcal{A}_h^* \begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} A_0^* y \\ -\frac{1}{h} D w \end{pmatrix}$ .
- ii) Furthermore, for  $t \geq 0$ , the restriction to  $\mathcal{D}(\mathcal{A}_h^*)$  of  $S_h^*$ , the adjoint of the semigroup  $S_h$ , defines a  $C_0$ -semigroup on  $\mathcal{H}^m$  and then  $S_h(t)$  may be extended to a  $C_0$ -semigroup on the Hilbert space  $\mathcal{D}(\mathcal{A}_h^*)'$ . ■

That means that the operator  $\mathcal{A}_h$  is the infinitesimal generator of a  $C_0$ -semigroup  $S_h(t)$  on Hilbert spaces ordered by dense and continuous injections, namely  $\mathcal{D}(\mathcal{A}_h) \hookrightarrow \mathcal{H}^m \hookrightarrow \mathcal{D}(\mathcal{A}_h^*)'$ . Therefore the weak solution of equation (9) may be defined by the variation of constant formula

$$X_h(t, \cdot) = S_h(t) \begin{pmatrix} x_{00} \\ \phi^h(\cdot) \end{pmatrix} + \int_0^t S_h(t-s) \mathcal{B}_h u(s) ds \quad (10)$$

where, at first sight, it seems that we only have  $X_h \in C(0, T; \mathcal{D}(\mathcal{A}_h^*)')$ . But in fact, due to the structure of the semigroup  $S_h$ , we have additional regularity in view of the following property of the convolution term.

**Lemma 3.1** *The operator  $u \mapsto \{t \rightarrow \int_0^t S_h(t-s) \mathcal{B}_h u(s) ds\}$  is linear continuous from  $L^2(0, T; \mathbb{R}^m)$  to  $C(0, T; \mathcal{H}^m)$ . ■*

Finally, combining Proposition 3.1 with Lemma 3.1 and using e.g. [1, Theorem 3.1, p. 173], we have the following result.

**Theorem 3.1** (i) *The function  $X_h$  given by formula (10) is the unique solution in*

$$\left\{ X \in C(0, T; \mathcal{H}^m) : \frac{dX}{dt} \in L^2(0, T; \mathcal{D}(\mathcal{A}_h^*)') \right\} \quad (11)$$

*of the weak equation*

$$\begin{cases} \frac{dX}{dt} &= \mathcal{A}_h X + \mathcal{B}_h u \quad \text{in } L^2(0, T; \mathcal{D}(\mathcal{A}_h^*)') \\ X(0) &= \begin{pmatrix} x_{00} \\ \phi^h \end{pmatrix} \in \mathcal{H}^m. \end{cases} \quad (12)$$

*In addition, there exists a constant  $c > 0$  such that*

$$\|X_h\|_{C(0, T; \mathcal{H}^m)} + \left\| \frac{\partial X_h}{\partial t} \right\|_{L^2(0, T; \mathcal{D}(\mathcal{A}_h^*)')} \leq c \left[ \left\| \begin{pmatrix} x_{00} \\ \phi^h \end{pmatrix} \right\|_{\mathcal{H}^m} + \|u\|_{L^2(0, T; \mathbb{R}^m)} \right]. \quad (13)$$

- (ii) *Moreover, the first component of this weak solution  $X_h$  is equal to  $x$ , given by formula (8), which is the unique solution of the retarded equation (1). ■*

### 3.2 Retarded state system: ( $k > 0$ , $h = 0$ )

In this case the solution is given in  $H_{loc}^1(0, \infty; \mathbb{R}^n)$  by the variation of constants formula

$$x(t) = e^{A_0 t} x_{00} + \int_0^t e^{A_0(t-s)} \{A_1 x(s-k) + Bu(s)\} ds. \quad (14)$$

It is well known that this solution may be described in a product-space framework by means of the pair  $(x(t), z(t, \cdot))'$  with  $z(t, \theta) = x(t + \theta)$ ,  $t \geq 0$ ,  $\theta \in [-k, 0]$ . Moreover when considering only the homogeneous part of the retarded equation, this pair is generated by a  $C_0$ -semigroup acting on the extended initial condition  $(x_{00}, x_0)'$ . See e.g. Delfour [6] or Staffans [22] for further details.

To emphasize the singular perturbation effect of the delay, we rescale the segment function  $z$  before following the product-space lines for state representation. To this end, introduce the functions defined by  $x^k(t, \sigma) = x(t + \sigma k)$  for  $t \geq 0$  and  $\sigma \in [-1, 0]$ , and  $\psi^k(\sigma) = x_0(\sigma k)$ . Then the solution of the homogeneous part of the retarded equation may be described from  $(x_{00}, \psi^k)'$  with some linear operator given in  $\mathcal{H}^n = \mathbb{R}^n \times L^2(-1, 0; \mathbb{R}^n)$  by

$$S_k(t) \begin{pmatrix} x_{00} \\ \psi^k(\cdot) \end{pmatrix} = \begin{pmatrix} x(t) \\ x^k(t, \cdot) \end{pmatrix} \quad \text{for all } t \geq 0 \text{ and } \begin{pmatrix} x_{00} \\ \psi^k \end{pmatrix} \in \mathcal{H}^n. \quad (15)$$

The next statement is a characterization of  $S_k$  that comes from the usual properties of the extended  $C_0$ -semigroup associated to a delay equation of retarded type ([1, page 61]).

**Proposition 3.2**  *$S_k$  is a  $C_0$ -semigroup on  $\mathcal{H}^n$  which is generated by the operator defined on  $\mathcal{H}^n$  with domain  $\mathcal{D}(\mathcal{A}_k) =$*

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{W}^n = \mathbb{R}^n \times H^1(-1, 0; \mathbb{R}^n), y(0) = x \right\} \text{ as } \mathcal{A}_k = \begin{pmatrix} A_0 & A_1 \delta_{\sigma=-1} \\ 0 & \frac{1}{k} \frac{\partial}{\partial \sigma} \end{pmatrix}. \quad \blacksquare$$

So setting  $X_k = \begin{pmatrix} x(t) \\ x^k(t, \cdot) \end{pmatrix}$  and  $\mathcal{B}_k = \begin{pmatrix} B \\ 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^m, \mathcal{H}^n)$ , we have the following evolution equation

$$\begin{cases} \frac{\partial X_k}{\partial t} &= \mathcal{A}_k X_k + \mathcal{B}_k u, \quad \text{in } L^2(0, T; \mathcal{H}^n) \\ X_k(0, \cdot) &= \begin{pmatrix} x_{00} \\ \psi^k(\cdot) \end{pmatrix} \in \mathcal{H}^n. \end{cases} \quad (16)$$

Now we have a bounded equation of evolution in the state space  $\mathcal{H}^n$ . It is obvious that for any initial condition in  $\mathcal{H}^n$ , its solution satisfies  $X_k \in C(0, T; \mathcal{H}^n)$ . Moreover, equation (16) may be extended by means of the method of transposition. In addition, easy computations lead to the following characterization of the adjoint of  $\mathcal{A}_k$  that will be useful here.

**Lemma 3.2** *The  $\mathcal{H}^n$ -adjoint of the operator  $\mathcal{A}_k$  is given with domain  $\mathcal{D}(\mathcal{A}_k^*) = \left\{ \begin{pmatrix} v \\ w \end{pmatrix} \in \mathcal{W}^n, w(-1) = kA_1^*v \right\}$  as*

$$\mathcal{A}_k^* \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} A_0^*v + \frac{1}{k}\delta_{\sigma=0}w \\ -\frac{1}{k}\frac{\partial}{\partial \sigma}w \end{pmatrix}. \quad \blacksquare$$

Then we have the following theorem.

**Theorem 3.2** *Given any  $t \in [0, T]$ ,  $X_0 \in \mathcal{H}^n$  and  $u \in L^2(0, T; \mathbb{R}^m)$ ,*

$$X(t) = \mathcal{S}_k(t)X_0 + \int_0^t \mathcal{S}_k(t-s)\mathcal{B}_k u(s)ds \in \mathcal{H}^n \quad (17)$$

*is the unique solution in the space*

$$\left\{ X \in C(0, T; \mathcal{H}^n) : \frac{dX}{dt} \in L^2(0, T; \mathcal{D}(\mathcal{A}_k^*))' \right\} \quad (18)$$

*of the transposed adjoint equation*

$$\begin{cases} \frac{dX}{dt} &= \tilde{\mathcal{A}}_k X + \mathcal{B}_k u \quad \text{in } L^2(0, T; \mathcal{D}(\mathcal{A}_k^*))' \\ X(0) &= X_0 \in \mathcal{H}^n \end{cases} \quad (19)$$

where

$$\tilde{\mathcal{A}}_k \in \mathcal{L}(\mathcal{D}(\mathcal{A}_k), \mathcal{H}^n) \cap \mathcal{L}(\mathcal{W}^n, \mathcal{W}^n) \cap \mathcal{L}(\mathcal{H}^n, \mathcal{D}(\mathcal{A}_k^*))' \quad (20)$$

*is the extension of  $\mathcal{A}_k$  given on  $\mathcal{W}^n$  by*

$$\tilde{\mathcal{A}}_k = \begin{pmatrix} A_0 & A_1 \delta_{\sigma=-1} \\ \frac{1}{k} \delta_{\sigma=0}^* & \frac{1}{k} D \end{pmatrix}. \quad (21)$$

*Extension  $\tilde{\mathcal{A}}_k$  generates  $\mathcal{S}_k$  henceforth considered as a  $C_0$ -semigroup on any of the three spaces  $\mathcal{D}(\mathcal{A}_k) \hookrightarrow \mathcal{H}^n \hookrightarrow \mathcal{D}(\mathcal{A}_k^*)'$  which are ordered by continuous and dense injections. Specifically, for  $X_0 = \begin{pmatrix} x_{00} \\ \psi^k \end{pmatrix}$ ,  $X$  coincides with augmented state given from  $x$  by  $\begin{pmatrix} x \\ x^k \end{pmatrix}$ .*  $\blacksquare$

#### 4 Riccati operators, optimal cost and sensitivity

To begin with, we state in the first subsection the solution of linear quadratic minimization problem we obtain as a special case of the general result for systems belonging the abstract class of Pritchard and Salamon [20], see also Bensoussan et al. [1], Delfour & Karakchou [7, 8], Pritchard & Salamon [20, 21]. With the following notation

$$\sigma_s^+[V] = \{P \in \mathcal{L}_s(V, V'), \langle P\phi, \phi \rangle_{V, V'} \geq 0, \forall \phi \in V\}, \quad (22)$$

we can state the next theorem as the special case of [20, Proposition 2.8].

**Theorem 4.1 Finite horizon input delay.** *Assume  $h > 0$  and denote  $\begin{pmatrix} G(T) & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H}^m)$  as  $\mathcal{G}_h(T)$ . Then there exists a unique operator  $\mathcal{P}_{T,h}$  such that*

$$\hat{J}_T(0, h) = \frac{1}{2} \langle X_h(0, \cdot), \mathcal{P}_{T,h}(0)X_h(0, \cdot) \rangle_{\mathcal{H}^m}, \quad (23)$$

*and the optimal control is  $u_{opt}(t) = -R^{-1}\mathcal{B}_h^* \mathcal{P}_{T,h}(t)X_h(t, \cdot)$ , with  $X_h$  given by Theorem 3.1. That operator  $\mathcal{P}_{T,h} \in C_T^1(\mathcal{L}_s(\mathcal{H}^m)) \cap C_T(\sigma_s^+[\mathcal{D}(\mathcal{A}_h^*)]')$  is the unique non-negative self-adjoint solution of the Riccati equation*

$$\begin{aligned} \frac{\partial \mathcal{P}_{T,h}}{\partial t} + \mathcal{A}_h^* \mathcal{P}_{T,h} + \mathcal{P}_{T,h} \mathcal{A}_h - \mathcal{P}_{T,h} \mathcal{B}_h R^{-1} \mathcal{B}_h^* \mathcal{P}_{T,h} + C_h^* C_h &= 0 \\ \mathcal{P}_{T,h}(T) &= \mathcal{G}_h(T). \end{aligned} \quad (24)$$

**Theorem 4.2 Finite horizon state delay.** *Assume  $k > 0$  and denote  $\begin{pmatrix} G(T) & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H}^n)$  as  $\mathcal{G}_k(T)$ . Then there exists a unique operator  $\mathcal{Q}_{T,k}$  such that*

$$\hat{J}_T(k, 0) = \frac{1}{2} \langle X_k(0, \cdot), \mathcal{Q}_{T,k}(0)X_k(0, \cdot) \rangle_{\mathcal{H}^n}; \quad (25)$$

*and the optimal control is  $u_{opt}(t) = -R^{-1}\mathcal{B}_k^* \mathcal{Q}_{T,k}(t)X_k(t, \cdot)$ , where  $X_k$  is given by Theorem 3.2,  $\mathcal{Q}_{T,k} \in C_T^1(\mathcal{L}_s(\mathcal{H}^n)) \cap C_T(\sigma_s^+[\mathcal{D}(\mathcal{A}_k^*)]')$  is the unique non-negative self-adjoint solution of the Riccati equation*

$$\begin{aligned} \frac{\partial \mathcal{Q}_{T,k}}{\partial t} + \tilde{\mathcal{A}}_k^* \mathcal{Q}_{T,k} + \mathcal{Q}_{T,k} \tilde{\mathcal{A}}_k - \mathcal{Q}_{T,k} \mathcal{B}_k R^{-1} \mathcal{B}_k^* \mathcal{Q}_{T,k} + C_k^* C_k &= 0 \\ \mathcal{Q}_{T,k}(T) &= \mathcal{G}_k(T). \end{aligned} \quad (26)$$

Introduce now the shorthand  $\mathcal{E}_q = \mathcal{L}(\mathbb{R}^q) \times \mathcal{L}(L^2(-1, 0; \mathbb{R}^q), \mathbb{R}^q) \times \mathcal{L}(L^2(-1, 0; \mathbb{R}^q))$  for  $q \in \{m, n\}$ . Then we have the following.

**Theorem 4.3 Regularity of Riccati operators** *For any  $h > 0$ ,  $t \in [0, T]$  and  $T < \infty$ , the Riccati operator  $\mathcal{P}_{T,h}(t)$*

has in  $\mathcal{L}_s(\mathcal{H}^m)$  the following decomposition:

$$\mathcal{P}_{T,h}(t) = \begin{pmatrix} \mathcal{P}_{T,h}^1(t) & h\mathcal{P}_{T,h}^2(t) \\ h\mathcal{P}_{T,h}^{2*}(t) & h\mathcal{P}_{T,h}^3(t) \end{pmatrix} \quad (27)$$

where  $\mathcal{P}_{T,h}^1 \in C_T^1(\mathcal{L}_s(\mathbb{R}^n))$ ,  $\mathcal{P}_{T,h}^2 \in C_T^1(\mathcal{L}_s(L^2(-1,0;\mathbb{R}^m),\mathbb{R}^n))$  and  $\mathcal{P}_{T,h}^3 \in C_T^1(\mathcal{L}_s(L^2(-1,0;\mathbb{R}^m)))$  are such that the mapping

$$\Phi : \begin{array}{l} \mathbb{R}_+^* \longrightarrow \mathcal{L}(\mathbb{R}^n) \times \mathcal{E}_m \\ h \longmapsto \left( \frac{d\mathcal{P}_{T,h}^1}{dt}(t), \mathcal{P}_{T,h}^1(t), \mathcal{P}_{T,h}^2(t), \mathcal{P}_{T,h}^3(t) \right) \end{array} \quad (28)$$

satisfies

$$w - \lim_{h \rightarrow 0} \Phi(h) = \left( \frac{dP_T}{dt}(t), P_T(t), P_T B_1(t) I_m, 0 \right) \quad (29)$$

where  $I_m$  defined as  $I_m(y) = -\int_{-1}^0 y(\sigma) d\sigma$  for  $y \in \mathbb{R}^m$ . ■

Note that this type of parameterization was introduced by Kokotović & Yackel [16] for the control of singularly perturbed systems in finite dimensions.

Now the proof of Theorem 2.1 becomes trivial. Indeed, observing that  $\widehat{J}_T(0,0) = \frac{1}{2} \langle x_{00}, P_T(0)x_{00} \rangle_{\mathbb{R}^n}$  whereas  $\widehat{J}_T(0,h) = \frac{1}{2} \langle x_{00}, \mathcal{P}_h^1(0)x_{00} \rangle_{\mathbb{R}^n} + \frac{h}{2} [2 \langle x_{00}, \mathcal{P}_h^2(0)\phi^h \rangle_{L^2} + \langle \phi^h, \mathcal{P}_h^3(0)\phi^h \rangle_{L^2}]$ , it becomes clear from theorem 4.3, that

$$\frac{\partial \widehat{J}_\infty}{\partial h}(0,0) = \lim_{h \rightarrow 0} \langle x_{00}, \mathcal{P}_h^2(0)\phi^h \rangle_{L^2} \quad (30)$$

$$= -\langle B_1^* P_T(0)x_{00}, \lim_{h \rightarrow 0} \int_{-1}^0 u(\sigma h) d\sigma \rangle \quad (31)$$

$$= -\langle B_1^* P_T(0)x_{00}, \lim_{h \rightarrow 0} \frac{1}{h} \int_{-h}^0 u(\theta) d\theta \rangle \quad (32)$$

It follows from the continuity of  $u$  on the left side of 0 that  $\lim_{h \rightarrow 0} \frac{1}{h} \int_{-h}^0 u(\theta) d\theta = u(0) = -R^{-1} B^* P_T(0)x_{00}$ , so finally

$$\frac{\partial \widehat{J}_\infty}{\partial h}(0,0) = \langle B_1^* P_T(0)x_{00}, R^{-1} B^* P_T(0)x_{00} \rangle_{\mathbb{R}^m} \quad (33)$$

More generally, it is possible to prove the other results in section 2.1 with analogues of theorem 4.3 for infinite horizon and/or state delay case. For instance we have the following theorem for state delay case.

**Theorem 4.4** For any  $k > 0$ ,  $t \in [0, T]$  and  $T < \infty$ , the Riccati operator  $Q_{T,k}(t)$  has in  $\mathcal{L}_s(\mathcal{H}^n)$  the following decomposition:

$$Q_{T,k}(t) = \begin{pmatrix} Q_{T,k}^1(t) & kQ_{T,k}^2(t) \\ kQ_{T,k}^{2*}(t) & kQ_{T,k}^3(t) \end{pmatrix} \quad (34)$$

where  $Q_{T,k}^1 \in C_T^1(\mathcal{L}_s(\mathbb{R}^n))$ ,  $Q_{T,k}^3 \in C_T^1(\mathcal{L}_s(L^2(-1,0;\mathbb{R}^n)))$  and  $Q_{T,k}^2 \in C_T^1(\mathcal{L}_s(L^2(-1,0;\mathbb{R}^n), \mathbb{R}^n))$  are such that the mapping

$$\Psi : \begin{array}{l} \mathbb{R}_+^* \longrightarrow \mathcal{L}(\mathbb{R}^n) \times \mathcal{E}_n \\ k \longmapsto \left( \frac{dQ_{T,k}^1}{dt}(t), Q_{T,k}^1(t), Q_{T,k}^2(t), Q_{T,k}^3(t) \right) \end{array} \quad (35)$$

satisfies

$$w - \lim_{k \rightarrow 0} \Psi(k) = \left( \frac{dP_T}{dt}(t), P_T(t), P_T(t) A_1 I_n, 0 \right) \quad (36)$$

■

## 5 Application to sensitivity of $\mathcal{H}_\infty$ robust performance

As a typical example of  $\mathcal{H}_\infty$  sub-optimal control we consider

$$\begin{cases} \dot{x}(t) = Ax(t-k) + B_0 u(t) + B_1 u(t-h) + B_d w(t), \\ z(t) = Cx(t) + Du(t), x(0) = x_{00} \in \mathbb{R}^n, \text{ a.e. } t \geq 0, \\ x = x_0 \text{ a.e. } t \in [-k, 0], x_0 \in L^2(-k, 0; \mathbb{R}^n), \\ u = u_0 \text{ a.e. } t \in [-h, 0], u \in L^2(-h, \infty; \mathbb{R}^m), \\ w \in L^2(0, \infty; \mathbb{R}^q), \end{cases} \quad (37)$$

where  $(A, B_d, B_1) \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^m, \mathbb{R}^n)$  and  $(C, D) \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^p)$ , with the objective of ensuring a given level of disturbance attenuation. We recall, see Tadmor e.g. [24], that for  $\gamma > 0$ , the inequality  $\|z\|_{L^2(0, \infty; \mathbb{R}^p)} < \gamma^2 \|w\|_{L^2(0, \infty; \mathbb{R}^q)}$  is achievable if and only if there exists a control  $u_\gamma \in L^2(0, \infty; \mathbb{R}^m)$  that realizes

$$\widehat{J}_\gamma(k, h) = \sup_{w \in L^2} \inf_{u \in L^2} \frac{1}{2} \left\{ \|z\|_{L^2(0, \infty; Z)}^2 - \gamma^2 \|w\|_{L^2(0, \infty; \mathbb{R}^q)}^2 \right\}. \quad (38)$$

Assume  $u_0$  and  $x_0$  continuous on the left side of 0,  $(A, B_1, C)$  controllable and observable,  $D^*[C D] = [0 R]$  with  $R$  positive definite and moreover that there exists  $\varepsilon > 0$  such that for all  $(\omega, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$  satisfying  $j\omega x = Ax + Bu$  we have  $\|Cx + Du\|_{\mathbb{R}^p} \geq \varepsilon \|x\|_{\mathbb{R}^n}$ . Then it is known from Glover and Doyle [15] that  $\widehat{J}_\gamma(0, 0)$  exists if and only if there exists a non-negative solution to the Riccati equation

$$A^* P_\gamma + P_\gamma A + P_\gamma (\gamma^{-2} B_d B_d^* - B R^{-1} B^*) P_\gamma + C^* C = 0. \quad (39)$$

where  $B = B_0 + B_1$ . Then we have the following statement.

**Theorem 5.1** Take a  $\gamma > 0$  such that there exists a non-negative solution to the Riccati equation (39). Assume that there exists some neighborhood of  $(0, 0)$  where  $(k, h) \rightarrow \widehat{J}_\gamma(k, h)$  is well defined. Then, with  $x_0$  continuous on the left side of 0, we have:

$$\frac{\partial \widehat{J}_\gamma(0, 0)}{\partial k} = \langle P_\gamma x_{00}, A x_{00} \rangle_{\mathbb{R}^n} \quad (40)$$

$$\frac{\partial \widehat{J}_\gamma(0, 0)}{\partial h} = -\langle P_\gamma x_{00}, B_1 R^{-1} B^* P_\gamma x_{00} \rangle_{\mathbb{R}^n}. \quad (41)$$

□

For instance, if  $B_0 = 0$  and  $h$  sufficiently small, then we have:  $\frac{\partial \hat{J}_\gamma(0,0)}{\partial h} = -\langle x_{00}, K_\gamma R K_\gamma x_{00} \rangle_{\mathbb{R}^n} \leq 0$  where  $K_\gamma = -R^{-1} B_1^* P_\gamma$  is the non-delay  $\mathcal{H}_\infty$  sub-optimal gain, so that  $\hat{J}_\gamma(0, h) \leq \hat{J}_\gamma(0, 0) \leq 0$ . This means that *even if there exists a  $\gamma$ -admissible feedback for a small  $h > 0$* , the corresponding closed loop system is less robust than the closed loop system ensuring the disturbance attenuation at the same level without delay. Moreover, for a given  $\gamma$ , the degradation of the robust performance due to the presence of a small input delay is proportional to the square of the non-delay  $\mathcal{H}_\infty$  sub-optimal gain. ■

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