Hybrid Feedback Controls and Stabilization of Linear Differential Systems

Elena Litsyn [1]   Yurii V. Nepomnyashchikh [2]   Arcady Ponosov [3]

Abstract

We study so-called "hybrid feedback stabilizers" for an arbitrarily general system of linear differential equations. We prove that under assumptions of controllability and observability there exists a hybrid feedback output control which makes the system asymptotically stable. The control is designed by making use of a discrete automaton implanted into the system's dynamics. In general, the automaton has infinitely many locations, but it gives rise to an "uniform" (in some sense) feedback control. The approach we propose goes back to the classical feedback control technique combined with some ideas used in the stability theory for equations with time-delay.

AMS Subject classification: 93D15

Keywords: Hybrid feedback control, linear and delay differential equations, stabilization

1 Formulation of the main result

We consider a general system of linear differential equations of the form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx.\end{aligned} \tag{1.1}$$

Here $x \in \mathbf{R}^n$ is a physical state of the system, $y \in \mathbf{R}^m$ is the output, and the control u is an element of $\mathbf{R}^\ell$. A, B, C are given real matrices of the sizes $n \times n$, $n \times \ell$, $m \times n$, respectively. *The control $u(\cdot)$ is assumed to depend on the output y, only.* The nature of this dependence is specified below.

We assume in the sequel that the pair (A, B) is controllable and the pair (A, C) is observable.

We study the following control problem: design a "simplest possible" feedback control u, which asymptotically stabilizes the system (1.1).

It is well-known that if $\operatorname{rank} C = n$, then a control $u = Gy$, where G is a constant $\ell \times m$–matrix, asymptotically stabilizes the system (1.1). If $\operatorname{rank} C < n$, then this may course an absence of such a stabilizer, or even of a nonlinear stabilizer $u = f(\xi) = f(\xi(t))$ (see e.g. [1]).

On the other hand, it is also well-known that there always exists a dynamical stabilizer, which solves the problem [6].

The idea to use a discrete device was suggested in [1], where several stabilizing procedures were tested on examples.

This idea can roughly be described as follows. A discrete device (called *an automaton*) is incorporated into the system (called *a plant*). The device is able to switch on and off certain control functions at certain instances. The time between two consecutive switchings depends on the last observation of y. As it was demonstrated in [1], a careful choice of a design procedure and switching instances can provide asymptotic stability of some systems. Discrete nature of this procedure makes their practical realization easier compared to the dynamical stabilization setup.

In the present paper we prove that these types of feedback controls (called in the sequel "hybrid feedback controls") can stabilize the general system (1.1).

We start with the concept of an automaton basically following the approach presented e.g. in [5], or in [1]. But we also admit automata with infinitely many locations.

A typical automaton we have in mind is described by a triple $\mathcal{A} = (Q, I, M)$, where

(i) Q is a set of all possible automaton states (locations);

(ii) the set I contains the input alphabet;

---

[1] Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, ISRAEL and The Research Institute, The College of Judea and Samaria, Ariel 44837, ISRAEL; Partially supported by the Ministry of Science and the Ministry of Absorption, Center for Absorption in Science, Israel `elenal@wisdom.weizmann.ac.il`

[2] Department of Mechanics & Mathematics, Perm State University, Bukirev str. 15, 614600 Perm, RUSSIA; Supported by the foundation Grants for young scientists (Goscomvuz RF, 1997) and by Grants No. 96-15-96195, 99-01-01278 of the RFBR and of the Research Grant in natural sciences, St.-Petersburg `yuvn@psu.ru`

[3] Institutt for matematiske fag, NLH, Postboks 5035, N-1432 Ås, NORWAY `matap@imf.nlh.no`

(iii) the transition map $M : Q \times I \to Q$ indicates the location after a transition time, based on the previous location q and input i at the time of transition.

We also let our automaton follow solutions of the system (1.1). This can be done by assuming given one more triplet $\mathcal{B} = (T, i, q_0)$. Here

(iv) $T : Q \to (0, \infty)$ is a mapping which sets a period $T(q)$ between transition times, and $\inf\limits_{q \in Q} T(q) > 0$,

(v) $i : \mathbf{R}^m \to I$ is a function providing the element $i(y)$ of the alphabet I for any output y of the system (1.1),

(vi) and $q_0 = q(\tau_0)$ is a state of the automaton at the initial time τ_0.

Later on we will assume, without loss of generality, that $\tau_0 = 0$.

Definition 1.1 *By an automaton we mean a 6-tuple* $\Delta = (Q, I, M, T, i, q_0)$.

For arbitrary sets X, Y (topological spaces X, Y) we denote by $\mathbf{P}(X, Y)$ (resp. $\mathbf{C}(X, Y)$) the set of all functions (resp. continuous functions) from X to Y.

Now, for any automaton Δ satisfying (i)-(vi) we define by induction a Volterra operator $F_\Delta : \mathbf{C}([0, \infty), \mathbf{R}^m) \to \mathbf{P}([0, \infty), Q)$. For each $y \in \mathbf{C}([0, \infty), \mathbf{R}^n)$ F_Δ is given by:

1. $(F_\Delta y)(0) = q(0)$; $\tau_1 = T(q(0))$; $(F_\Delta y)(t) \equiv q(0), \quad t \in [0, \tau_1)$;

2. $(F_\Delta y)(\tau_1) = M(q(0), i(y(\tau_1))) := q(\tau_1)$; $\tau_2 = \tau_1 + T(q(\tau_1))$;

$$(F_\Delta y)(t) = q(\tau_1), \quad t \in [\tau_1, \tau_2);$$

3. If $\tau_0, \tau_1, \ldots, \tau_k$ and the values $(F_\Delta y)(t)$ for $t \in [0, \tau_k)$ are already known, then τ_{k+1} and $(F_\Delta y)(t)$ for $t \in [\tau_k, \tau_{k+1})$ are defined by the equalities $(F_\Delta y)(\tau_k) = M(q(\tau_{k-1}), i(y(\tau_k))) := q(\tau_k)$; $\tau_{k+1} = \tau_k + T(q(\tau_k))$; $(F_\Delta y)(t) \equiv q(\tau_k), \quad t \in [\tau_k, \tau_{k+1})$.

Definition 1.2 *A control* $u(\cdot)$ *of the type*

$$u(t) = \varphi(y(t), (F_\Delta y)(t)), \tag{1.2}$$

where $\varphi : \mathbf{R}^m \times Q \to \mathbf{R}^\ell$ *is a certain function, will be addressed as a* **hybrid feedback control** *(abbr. HFC).*

Now we define the class of HFCs which we will use stabilizers.

Definition 1.3 *A hybrid feedback control* $u(\cdot)$ *given by (1.2) is called* **uniform** *if*

1) T *is a constant function*

2) the function φ *does not depend on the first argument, i.e.* $u(t) = \varphi((F_\Delta y)(t))$.

The set of all uniform HFCs will in the sequel be denoted by $\mathcal{H}$.

The main result of our paper is

Theorem 1.1 *The system (1.1) is* $\mathcal{H}-$*stabilizable under assumptions of controllability of* (A, B) *and observability of* (A, C).

2 Proof of the main result

We will use the following notation:

– $\mathcal{M}_{j,k}$ is the set of all real $j \times k$-matrices;

– $\mathcal{M}_k = \mathcal{M}_{k,k}$ is the set of all real $k \times k$-matrices;

– $\mathcal{L}(\mathbf{R}^j, \mathbf{R}^k)$ is the set of all linear operators from $\mathbf{R}^j$ to $\mathbf{R}^k$;

– $\mathcal{L}(\mathbf{R}^k) = \mathcal{L}(\mathbf{R}^k, \mathbf{R}^k)$ is the set of all linear operators in the space $\mathbf{R}^k$;

– $|\cdot|$ is the Euclidean norm in the spaces $\mathbf{R}^k$ (for any k);

– $\|\cdot\|$ is the corresponding operator-norm in the spaces $\mathcal{L}(\mathbf{R}^j, \mathbf{R}^k)$;

– I_k is the identity $k \times k$-matrix;

– χ_E is the characteristic function of a set E;

– $\mathbf{C}$ is the space of all continuous functions from $[0, \infty)$ to $\mathbf{R}^n$;

– $\mathbf{L}_0$ is the space of all Lebesgue measurable functions from $[0, \infty)$ to $\mathbf{R}^n$.

– we write $[A]_B$ instead of the $n \times \ell n$-matrix $[B \ AB \ \ldots \ A^{n-1}B]$ and ${}_C[A]$ instead of the $mn \times n$-matrix $([A^\top]_{C^\top})^\top$.

We need two lemmas to prove the theorem 1.1.

Lemma 2.1 *There exists a* $\ell \times n-$*matrix* G *such that the matrix* $A+BG$ *is stable and the matrix pair* $(A+BG, C)$ *is observable.*

Proof: We first choose a matrix $G_0 \in \mathcal{M}_{\ell,n}$, for which $A+BG_0$ is stable. Clearly, there exists $\varepsilon > 0$ such that for any $t \in (1-\varepsilon, 1+\varepsilon)$ a matrix $A+tBG_0$ is stable. Consider a function $\psi : \mathbf{R} \to \mathbf{R}$, which to each t associates the sum of squares of all $n-$th order minors of the matrix ${}_C[A+tBG_0]$.

Due to Kalman's criterion $\mathcal{P} := \{t \in \mathbf{R} \mid \ (A+tBG, C) \ \text{ is observable}\} = \psi^{-1}(\mathbf{R} \setminus \{0\})$. Evidently, ψ is a polynomial, and moreover, $\psi \not\equiv 0$. The last inequality follows immediately from the observability of the pair (A, C), which implies $\psi(0) \neq 0$. Thus, ψ has a finite number of zeros and, therefore, the exists $t^* \in \mathcal{P} \bigcap (1-\varepsilon, 1+\varepsilon)$. Hence, the matrix $G = t^* G_0$ satisfies the required conditions. ■

Our next lemma is a version of the Lyapunov stability theorem for a class of *functional differential equations.*

The Cauchy formula we will use is classical (see e.g. [2] and [3]):

$$x(t) = \mathcal{X}(t)x(0) + \int_0^t \mathcal{C}(t,s)f(s)ds, \qquad t \geq 0. \quad (2.1)$$

Here $\mathcal{C}(t,s)$ is the Cauchy matrix of the equation

$$Lx \equiv \dot{x}(t) - \sum_{k=1}^{n} \tilde{a}_k(t)x[h_k(t)] = f(t), \qquad t > 0, \quad (2.2)$$

and $\mathcal{X}(t)$ is the fundamental matrix of the corresponding homogeneous equation.

Lemma 2.2 *Let* $\tilde{a}_k : [0,\infty) \to \mathcal{L}(\mathbf{R}^n)$ *be locally Lebesgue-integrable and* $h_k : [0,\infty) \to \mathbf{R}$ *Lebesgue measurable functions. Assume that there exist* $N, \beta, \tau > 0$ *such that* $\zeta(t) \leq h_k(t) \leq t$, $t \in [0,\infty)$, $k = 1, \dots, n$, *where* $\zeta(t) = \max\{0, t-\tau\}$ *and*

$$\|\mathcal{C}(t,s)\| \leq Ne^{-\beta(t-s)}. \quad (2.3)$$

Then there exists $\varepsilon > 0$ *such that for any operator* $V : \mathbf{C} \to \mathbf{L}_0$ *satisfying*

$$|(Vx)(t)| \leq \varepsilon \max_{\zeta(t) \leq s \leq t} |x(s)|, \qquad t \in [0,\infty), \quad x \in \mathbf{C}, \quad (2.4)$$

the zero solution of the perturbed equation

$$Lx = Vx \quad (2.5)$$

is globally exponentially stable:

$$|x(t)| \leq N_0 e^{-\beta_0 t}|x(0)|, \qquad t \geq 0 \quad (2.6)$$

for any solution $x(t)$ *of (2.6) and for some positive constants* N_0, β_0 *independent of* $x(t)$.

Proof: From the Cauchy formula (2.1) as well as from (2.3) and (2.4) it immediately follows that

$$|x(t)| \leq Ne^{-\beta t}|x(0)| + N\varepsilon \int_0^t e^{-\beta(t-s)} \max_{\zeta(s) \leq \xi \leq s} |x(\xi)| ds$$

for $t \geq 0$. Putting $z(t) := \max\limits_{\zeta(t) \leq s \leq t} |x(t)|e^{\beta t}$ we obtain the estimate

$$z(t) \leq N_0|x(0)| + N_0\varepsilon \int_0^t z(s)ds, \qquad t \geq 0$$

with $N_0 = Ne^{\beta\tau}$.

By the Gronwall-Bellman inequality,

$$z(t) \leq N_0|x(0)|e^{N_0\varepsilon}, \ t \geq 0,$$

so that

$$|x(t)| \leq N_0 e^{(N_0\varepsilon - \beta)t}|x(0)|, t \geq 0.$$

Then for any $\varepsilon < \beta/N_0$ we obtain the estimate (2.6). ■

Now, we are ready to start proving our central result.

I. By the lemma 2.1, we find a matrix $G \in \mathcal{M}_{m,n}$ such that the matrix $A+BG$ is stable and the pair $(A+BG, C)$ is observable.

Let us fix an arbitrary $\tau > 0$ and put

$$A(t) = \chi_{[0,\tau]}A_0 + \chi_{[\tau,\infty)}A_1, \qquad \tau \in [0,\infty),$$

where $A_0 = A, \quad A_1 = A+BG$.

Denote by $X(t,s)$ the Cauchy matrix of the ordinary differential equation $\dot{x} = A(t)x$. We do not assume in the sequel that $t \geq s$ (defining $X(s,t) = X^{-1}(t,s)$ if necessary). Evidently,

$$\begin{array}{ll} X(t,s) = e^{A_1(t-s)}, & t \geq \tau, \ s \geq \tau \\ X(t,s) = X(t,\tau-0)X(\tau,s) & \\ = e^{A_0(t-\tau)}e^{A_1(\tau-s)}, & 0 \leq t \leq \tau \leq s \end{array} \quad (2.7)$$

(we do not describe here all possible cases here restricting ourselves to those formulae which we are going to use in the proof).

II. Consider the following equation for the unknown matrix-valued function

$$X(t) := \ [X_1(t), \dots, X_n(t)] \in \mathcal{M}_{n,mn}$$

with

$$X_k : [\tau,\infty) \to \mathcal{M}_{n,m}$$

given by

$$\sum_{k=1}^{n} X_k(t)CX(p\tau + kh, t) = I_n. \quad (2.8)$$

Here $t \in [(p+1)\tau, (p+2)\tau) := E_p$ and $h = \tau/n$ and $p = 0, 1, 2, \ldots$ Our next aim is to prove existence of a solution $X(t)$ to this equation.

a) Let us first assume that $t \in E_0$. Then $0 \le kh \le \tau \le t, \quad k = 1, \ldots, n$, and due to (2.7)

$$X(kh, t) = e^{A_0(kh-\tau)} e^{A_1(\tau - t)} = T_0^{k-1} U_0, \qquad (2.9)$$

where $k = 1, \ldots, n$, $T_0 = e^{A_0 h}$, $U_0 = e^{A_0(h-\tau)} e^{A_1(\tau - t)}$.

b) Let t belong to $E_p, \quad p = 1, 2, \ldots$ Then $0 < \tau \le p\tau + kh \le t, \quad k = 1, \ldots, n$, and according to (2.7)

$$X(p\tau + kh, t) = T_1^{k-1} U_p, \quad k = 1, \ldots, n, \qquad (2.10)$$

where $T_1 = e^{A_1 h}$, $U_p = e^{A_1(p\tau + h - t)}$. Notice that from the observability of the pairs (A_i, C) and Kalman's criterion it follows that $\operatorname{rank}({}_{C}[A_i]) = n$, $i = 0, 1$. Hence, for generic h (see e.g. [4]),

$$\operatorname{rank}({}_{C}[T_i]) = n \qquad (2.11)$$

as well. In what follows we shall always assume without loss of generality that our h satisfies the conditions (2.11).

For an arbitrary ordered n-tuple $\mathcal{K} = \{k_1, \ldots k_n\} \subset \{1, 2, \ldots, mn\}$ ($k_j < k_{j+1}$, $j = 1, \ldots, n-1$) we define now two operators $R_{\mathcal{K}} : \mathcal{M}_{mn,n} \to \mathcal{M}_n$, $S_{\mathcal{K}} : \mathcal{M}_n \to \mathcal{M}_{mn,n}$ in the following manner:

– j-th row in the matrix $R_{\mathcal{K}} Z$ coincides with k_j-th row of the matrix Z;

– k_j-th column of the matrix $S_{\mathcal{K}} Z$ coincides with j-th column of the matrix Z;

– the columns with numbers $k \notin \mathcal{K}$ of the matrix $S_{\mathcal{K}} Z$ are all null-columns.

From (2.11) it follows that for some ordered n-tuples $\mathcal{K}_i \subset \{1, 2, \ldots, mn\}$ the matrices $R_{\mathcal{K}_i}({}_{C}[T_i])$ $(i = 1, 2)$ are nonsingular. Due to (2.9), (2.10), it is straightforward that the matrix $X(t)$, defined by

$$X(t) = S_{\mathcal{K}_i}\left(U_p^{-1} \left(R_{\mathcal{K}_i} \left({}_{C}[T_i]\right)^{-1} \right) \right), \qquad (2.12)$$

satisfies the equation (2.9), where it is assumed that $i = p = 0$ for $t \in E_0$ and $i = 1$ for $t \in E_p$, $p = 1, 2, \ldots$

III. For $p = 0, 1, 2, \ldots$ we put

$$u_1(t) = \begin{cases} 0, & 0 \le t < \tau \\ G \left(\sum\limits_{k=1}^{n} X_k(t) y(p\tau + kh) \right), & t \in E_p, \end{cases} \qquad (2.13)$$

where $y = Cx$ and $X(t) := [X_1(t), \ldots, X_n(t)]$ is defined by (2.12). Due to (2.8),

$$u_1(t) = G \left(\sum_{k=1}^{n} X_k(t) C x(p\tau + kh) \right) =$$

$$G \left(\sum_{k=1}^{n} X_k(t) C X(p\tau + kh, t) x(t) \right) = G x(t),$$

$$t \in E_p, \quad p = 0, 1, 2, \ldots$$

Hence, (1.1) coincides with the exponentially stable linear equation $\dot{x} = A_1 x$ in the interval $t \in [\tau, \infty)$. Therefore, the system (1.1) is u_1-stabilizable.

But u_1 is not yet a hybrid feedback control in the sense of the definition 1.3.

According to the formula (1.2) we need now to find suitable discrete approximations for $A_i(t)$ and $Cx(\tau h + kh)$. To do it, we will use a technique based on the preservation of the asymptotic stability for functional differential equations with respect to small perturbations both in coefficients and time-delays [3].

IV. Let us first of all rewrite the system (1.1) involving the control u_1 in the form of the following delay-equation:

$$\dot{x}(t) - \sum_{k=0}^{n} a_k(t) x[h_k(t)] = 0, \quad t \in [0, \infty). \qquad (2.14)$$

Here

$$a_0(t) = \chi_{[0,\tau)}(t) A, \quad a_k(t) = \chi_{[\tau,\infty)}(t) B G X_k(t) C,$$

$$h_0(t) = t, \qquad h_k(t) = \sum_{p=0}^{\infty} \chi_{E_p}(t)(p\tau + kh),$$

and $k = 1, \ldots, n$. As shown in **III**, the equation (2.14) is exponentially stable. From (2.12), it also follows that

$$a_k(t) = BG\, S_{\mathcal{K}_i} \left(U_p^{-1} \left(R_{\mathcal{K}_i} \left({}_{C}[T_i]\right)^{-1} \right) \right) \Lambda_k C,$$

$$k = 1, \ldots, n, \quad t \in E_p, \quad p = 0, 1, 2, \ldots, \qquad (2.15)$$

where the block matrices $\Lambda_k \in \mathcal{M}_{mn,n}$ are defined by

$$\Lambda_k = (\underbrace{\theta \ \ldots \ \theta}_{m(k-1)} \ I_m \underbrace{\theta \ \ldots \ \theta}_{m(n-k)})^\top, \qquad k = 1, \ldots, n$$

(θ is m-dimensional zero vector-column), and $i = 0$ for $p = 0$, $i = 1$ for $p > 0$.

The representation (2.15) implies that each of the functions $a_k : [0, \infty) \to \mathcal{L}(\mathbf{R}^n)$ is piecewise continuous with possible jumps at the points $p\tau$, $p = 1, 2, \ldots$, only.

Since for $t \in [\tau, \infty)$

$$\|a_k(t)\| \le \max\Bigg\{ \|A\|, \ \|BG\| \cdot \|C\| \times$$

$$\max_{i=1,2} \left\| \left(S_{\mathcal{K}_i}({}_{C}[T_i]) \right)^{-1} \right\| \cdot e^{(\|A_0\| + \|A_1\|)\tau} \Bigg\} < \infty,$$

the functions a_k are bounded. Then, according to Corollary 2 in [3, p. 173], there exists $\sigma > 0$ such that for all locally Lebesgue-integrable functions $\tilde{a}_k : [0,\infty) \to \mathcal{L}(\mathbf{R}^n)$, satisfying

$$\max_k \limsup_{t\to\infty} \|a_k(t) - \tilde{a}_k(t)\| < \sigma, \tag{2.16}$$

the Cauchy matrix $\mathcal{C}(t,s)$ of the equation (2.2) has the exponential estimate (2.3).

The equalities (2.15) and (2.10) imply a periodicity of a_k with the period τ for $t \geq 2\tau$. Now let us approximate $X(t)$ by a step function on $[2\tau, 3\tau]$. For the sake of convenience, we may assume that the points $2\tau + kh$ are included in the set of possible jump-points of the step function. Then we extend this function τ-periodically to the interval $[3\tau,\infty)$. Finally, we approximate $X(t)$ on $[\tau, 2\tau]$ by a suitable step function. Let us notice that such approximations can be found with a prescribed accuracy. Our output will be a function $\tilde{X}(t) = [\tilde{X}_1(t), \ldots, \tilde{X}_n(t)]$ of the form

$$\tilde{X}_k(t) = \sum_{j=0}^{J-1} \chi_{E_{1j}}(t)\tilde{c}_{kj} + \sum_{p=1}^{\infty}\sum_{j=0}^{J-1} \chi_{E_{pj}}(t)c_{kj}, \quad t \in E_{pj}, \tag{2.17}$$

where J is a natural constant, $\delta = \tau/J$, $E_{pj} = [(p+1)\tau + j\delta, (p+1)\tau + (j+1)\delta)$, and $\tilde{c}_{kj}, c_{kj} \in \mathcal{M}_{n,m}$ are some matrices as well. Here $k = 1, \ldots, n$, $j = 0, 1, \ldots, J-1$, and $p = 0, 1, \ldots$ By construction, $\|\tilde{X}_k(t) - X_k(t)\| < \sigma \cdot (\|BG\| \cdot \|C\|)^{-1}$.

We set now $a_0(t) \equiv A$, $\tilde{a}_k(t) = BG\tilde{X}_k(t)C$, $k = 1, \ldots, n$. Then

$$\max_k \sup_{t\in[0,\infty)} \|\tilde{a}_k(t) - a_k(t)\| < \sigma,$$

so that (2.16) holds.

Consequently, we obtain that the Cauchy matrix of the equation (2.2) admits the exponential estimate (2.3).

V. According to the lemma 2.2 one can choose a positive ε such that for any (as general as necessary) nonlinear operator $V : \mathbf{C} \to \mathbf{L}_0$ satisfying the condition (2.4) every solution x of the perturbed equation (2.5) has the exponential estimate (2.6) with certain positive constants N_0, β_0 (independent of x).

By (2.17),

$$\varepsilon_1 := \varepsilon \left[\|BG\| \cdot \|C\| \cdot \sup_{t\in[0,\infty)} \sum_{k=1}^{n} \|\tilde{X}_k(t)\| \right]^{-1} > 0.$$

A multivalued function $\Phi : \mathbf{R}^m \to 2^{\mathbf{Q}^m}$ ($\mathbf{Q}$ is the set of rational numbers) defined by

$$\Phi(v) = \{ r \in \mathbf{Q}^m \mid |v - r| \leq \varepsilon_1 |v| \},$$

has then nonempty images: $\Phi(v) \neq \emptyset$, $v \in \mathbf{R}^m$. We take an arbitrary selector ϱ of the multivalued function Φ and define V as follows:

$$(Vx)(t) = \sum_{k=1}^{n} BG\tilde{X}_k(t) \left(\varrho(Cx[h_k(t)]) - Cx[h_k(t)] \right). \tag{2.18}$$

It is easy to check that V acts from $\mathbf{C}$ to $\mathbf{L}_0$ and satisfies the inequality (2.4). Hence the equation (2.5) with the operator V just defined becomes asymptotically stable, and the stability is uniform w.r.t. compact subsets.

By construction, (2.5) has the form

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{n} BG\tilde{X}_k(t)\varrho(Cx[h_k(t)]).$$

This equation is, in turn, equivalent to the original system (1.1), where the corresponding control u is defined by

$$u(t) = \begin{cases} 0, & t \in [0,\tau) \\ G\left(\sum\limits_{k=1}^{n} \tilde{c}_{kj}\varrho(y(kh)) \right), & t \in E_{0j}, \\ G\left(\sum\limits_{k=1}^{n} c_{kj}\varrho(y(p\tau + kh)) \right), & t \in E_{pj}, \end{cases} \tag{2.19}$$

where $j = 0, 1, \ldots, J-1$, $p = 1, 2, \ldots$ (recall that $y = Cx$ is the output).

The system (1.1) controlled by u from (2.19) is therefore asymptotically stable, and the stability is uniform w.r.t. any compact subset of $\mathbf{R}^n$.

VI. We have not yet shown that $u \in \mathcal{H}$.

Put $N_J = \{0, 1, \ldots, J-1\}$ and denote by Ω a subset of $\mathcal{M}_{\ell,mn}$ consisting of the block matrices

$$\tilde{c}_j = (G\tilde{c}_{1j}, \ldots, G\tilde{c}_{nj}), \quad c_j = (Gc_{1j}, \ldots, Gc_{nj}),$$

$$\bar{z} = (z, \ldots, z), \quad j = 1, \ldots, J-1,$$

where $z \in \mathcal{M}_{\ell,mn} \setminus \{G\tilde{c}_{kj}, Gc_{kj} \mid k = 1, \ldots, n, \quad j = 1, \ldots, J-1\}$ is arbitrary. Put $\mathcal{Y} \subset \Omega \times N_J$. The elements of $\mathcal{Y}$ are $(\bar{z}, j)$, $(\tilde{c}_j, j)$ and (c_j, j). Let us define a mapping $M_0 : \mathcal{Y} \to \mathcal{Y}$ by:

$$\begin{aligned} &M_0(\bar{z}, j) = (\bar{z}, j+1) && (j = 1, \ldots, J-2), \\ &M_0(\bar{z}, J-1) = (\tilde{c}_0, 0), \\ &M_0(\tilde{c}_j, j) = (\tilde{c}_{j+1}, j+1) && (j = 0, 1, \ldots, J-2), \\ &M_0(\tilde{c}_{J-1}, J-1) = (c_0, 0), \\ &M_0(c_j, j) = (c_{j+1}, j+1) && (j = 0, 1, \ldots, J-2), \\ &M_0(c_{J-1}, J-1) = (c_0, 0). \end{aligned}$$

Let P be the set of all row-vectors of the form $q = (q_1, \ldots, q_n)$, where $q_k \in \mathbf{Q}^m$. We extend now M_0 to the set $\Omega \times N_J$ and define two mappings $M_- : P \times N_J \times P \to P$ and $M_+ : P \times P \times N_J \to P$ as follows

$$M_-(q, j, i) =$$

$$\begin{cases} q, & j \neq J-1,\ j \neq \frac{kJ}{n}, \\ (q_1,\dots,q_{k-1},i,q_{k+1},\dots,q_n), & j = \frac{kJ}{n}, \\ (q_1,\,\dots\,,q_{n-1},i), & j = J-1, \end{cases}$$

where $q = (q_1,\,\dots\,,q_n)$ and $k = 1,\dots,n-1$,

$$M_+(q^+,q^-,j) = \begin{cases} q^+, & \text{if} \quad j \neq 0 \\ q^-, & \text{if} \quad j = 0 \end{cases} .$$

We first describe an automaton $\Delta = (Q, I, M, T, i, q_0)$. Let the (countable) set $Q = P \times P \times (\Omega \times N_J)$ contain 4-tuples $(q^+, q^-, (c,j))$ and the (countable) alphabet I be equal to $\mathbf{Q}^m$. The mapping $M : Q \times I \to Q$ is defined then by

$$M((q^+,q^-,(c,j),i) = \\ (M_+(q^+, M_-(q^-,j,i),j), M_-(q^-,j,i), M_0(c,j)).$$

Assume that $T \equiv \delta$ and define $i : \mathbf{R}^m \to I$ and q_0 by

$$i(s) = \varrho(s), \qquad q_0 = (\Theta, \Theta, (\bar{z}, 0)),$$

where Θ is zero in P.

Finally, we choose a function $\varphi : Q \to \mathbf{R}^\ell$ as follows

$$\varphi((q_1,\,\dots\,,q_n), q^-, ((d_1,\,\dots\,,d_n),j)) =$$

$$\begin{cases} 0, & \text{if} \quad (d_1,\,\dots\,,d_n) = \bar{z} \\ \sum\limits_{k=1}^{n} d_k q_k, & \text{if} \quad (d_1,\,\dots\,,d_n) \neq \bar{z} \end{cases} .$$

By construction, the control u given by (2.19) is of the form $u(\cdot) = \varphi((F_\Delta y)(\cdot))$ and belongs to the class $\mathcal{H}$. The control u is therefore a uniform HFC in the sense of the definition 1.3. Moreover according to part **V** of the proof, the control u stabilizes the system (1.1). The proof of the theorem 1.1 is completed.

Remark 1 *Consider the linear control system*

$$\dot{\xi} = \eta + u, \quad \dot{\eta} = \xi, \quad u(t) = \varphi(\xi(t), (F_\Delta \xi)(t)), \qquad (2.20)$$

We claim that there is no linear HFC stabilizing system (2.20). To prove it let us observe that for an arbitrary $\eta_0 > 0$ *the set* $(\xi(t), \eta(t)) \in [0,\infty) \times [\eta_0,\infty) := \mathcal{D}$ *is flow-invariant whatever a linear control strategy is. This excludes asymptotic stability. This example shows that the use of affine HFC, as suggested in the course of the proof of Theorem 1.1, is essential.*

Remark 2 *It is easy to show that the design of the HFC from Theorem 1.1 can easily be implemented. We omit, however, examples here.*

In conclusion we would like to thank Professor Zvi Artstein for introducing us the problem and his very helpful comments.

References

[1] Z. Artstein, *Example of stabilization with hybrid feedback*, in Hybrid Systems III. Verification and Control. Springer-Verlag, Berlin, Lecture Notes in Computer Scient., 1066 (1996), pp. 173–185.

[2] N. V. Azbelev and V. P. Maksimov and L. F. Rakhmatullina, *Introduction to the theory of functional differential equations*, World Federation Publishers Company, Inc., Atlanta, 1995.

[3] L. Berezansky and E. Braverman, *Preservation of the exponential stability under perturbation of linear delay impulsive differential equations*, J. Anal. and its Appl., 14 (1995), No. 1, pp. 157–174.

[4] G. Kreisselmeier, *On sampling without loss of observability/controllability* , IEEE Trans. Automat. Contr, 44 (1999), pp. 1021-1025.

[5] A. Nerode and W. Kohn, *Models for hybrid systems: automata, topologies, controllability, observability*, in Hybrid Systems, R. L. Grossman et al eds., Springer-Verlag, Berlin, Lecture Notes in Computer Scient., 736 (1993), pp. 317–356.

[6] E. Sontag, *Mathematical control theory*, Springer-Verlag, New York, 1989.