A High-Order Local Maximum Principle for Abnormal Extremals - Examples[1]

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Abstract

We illustrate a generalized local Maximum Principle published earlier [6] which gives necessary conditions for optimality of abnormal trajectories in optimal control problems. In this theorem the multiplier associated with the objective is non-zero.

1 Introduction

We consider an optimal control problem on $\mathbb{R}^n$ with fixed terminal time and terminal constraints. The control set is a closed and convex subset of $\mathbb{R}^m$ with nonempty interior. In [6] we developed necessary conditions for optimality of *abnormal extremals*, i.e for admissible input-trajectory pairs which satisfy the conditions of the Pontryagin Maximum Principle with a vanishing multiplier at the objective. The main reason why conventional necessary conditions fail for abnormal cases is that they, like the Maximum Principle, only use linear approximations for the equality constraint described by an operator F between Banach spaces. Linear approximations are woefully inadequate near abnormal points z_* when the kernel of the operator $F'(z_*)$ contains many directions which are *not* tangent to the equality constraint. Generalizing a result of Avakov [2] in [4] we derived a high-order generalization of the classical Lyusternik Theorem which for arbitrary $p \in \mathbb{N}$ describes the structure of the actual p-order tangent directions to an operator equality constraint in a Banach space also for non-regular operators. Based on the results in [4] p-order tangent cones to the equality constraint can explicitly be calculated along critical directions which comprise the low order terms. Combining these cones with standard constructions of high-order cones of decrease for the functional and high-order feasible cones to inequality constraints, all taken along critical directions, generalized necessary conditions for optimality for extremum problems in Banach spaces can be derived which allow to incorporate the objective with a nonzero multiplier. Characteristic of these results is that they are parametrized by critical directions as it is *natural* near abnormal points.

In [6] we derived a generalized version of the so-called weak or local version of the Maximum Principle. The theory developed in [6] gives necessary conditions for optimality for increasingly more degenerate structures. Here we give some illustrative examples. However, due to space limitations we need to refer the reader to the full paper [6] for a development of the theoretical aspects. There the reader will also find a more complete list of related references and discussion of the literature on the topic.

2 Problem formulation

We consider an optimal control problem in Bolza form with fixed terminal time: **(OC)** Minimize the functional

$$I(x,u) = \int_0^T L(x(t),u(t),t)dt + \ell(x(T)) \qquad (1)$$

subject to the constraints

$$\dot{x}(t) = f(x(t),u(t),t), \qquad (2)$$

$$x(0) = 0, \qquad q(x(T)) = 0, \qquad (3)$$

$$u(\cdot) \in \mathcal{U} = \{u \in L^r_\infty(0,T):\ u(t) \in U \text{ a.e.}\}. \qquad (4)$$

The terminal time T is fixed and we make the following **regularity assumptions** on the data: $L : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}^n$ are C^∞ in (x,u) for every $t \in [0,T]$; both functions and their derivatives are measurable in t for every (x,u) and the functions and all partial derivatives are bounded on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m \times [0,T]$; $\ell : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^n \to \mathbb{R}^k$ are C^∞ and the rows of the Jacobian matrix q_x (i.e. the gradients of the equations defining the

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terminal constraint) are linearly independent; $U \subset \mathbb{R}^m$ is a closed and convex set with nonempty interior.

The equality constraints can be modelled as $\mathcal{F} = \{(x,u) \in \overline{W}_{11}^n(0,T) \times L_\infty^m(0,T) : F(x,u) = 0\}$ where $F : \overline{W}_{11}^n(0,T) \times L_\infty^m(0,T) \to \overline{W}_{11}^n(0,T) \times \mathbb{R}^k$ is defined as

$$F(x,u) = \left(x(\cdot) - \int_0^{(\cdot)} f(x(s),u(s),s)ds,\ q(x(T)) \right) \tag{5}$$

The operator F gives the state-space representation of the dynamical system for the optimal control problem. Equivalently, and this is used for instance by Agrachev and Sarychev in [1], one could use the input-output map $Q : L_\infty^m(0,T) \to \mathbb{R}^k$, $u \mapsto q(x(T))$ where x is the solution to $\dot{x} = f(x,u(t),t)$, $x(0) = 0$. These formulations are equivalent. The input-output map has the advantage that it is clear that the image of $Q'(u)$ is closed, in fact finite-dimensional. The state-representation leads to calculations which, although not intrinsic, are quite transparent and only require simple differentiations of data directly given.

The operator F is called *non-regular* at an input-trajectory pair (x_*, u_*) if its Fréchet derivative $F'(x_*, u_*)$ is not onto. Non-regularity of F can be characterized in terms of the conditions of the so-called local Maximum Principle. In fact, the codimension of $F'(x_*, u_*)$ is given by the number of linearly independent solutions to $\dot{\lambda}(t) = -\lambda(t) f_x(x_*(t), u_*(t), t)$ which satisfy $\lambda(t) f_u(x_*(t), u_*(t), t) \equiv 0$ on $[0,T]$ and for which $\lambda(T)$ is orthogonal to $\ker q_x(x_*(T))$.

3 Critical directions

We need to briefly describe the set of critical directions along which high-order tangent approximations to the equality constraint $\mathcal{F}$ can be set up. Details are given in [6].

Let $F : Z \to Y$ be an operator between Banach spaces which is sufficiently often continuously Fréchet differentiable in a neighborhood of $z_* \in Z$. The Taylor-expansion of F along a curve $\gamma(\varepsilon) = z_* + \sum_{i=1}^m \varepsilon^i h_i$ is given by

$$F(z_*) + \sum_{i=1}^m \varepsilon^i \nabla^i F(z_*)(h_1, \dots, h_i) + \widetilde{r}(\varepsilon) \tag{6}$$

where $\nabla^i F(z_*)(h_1, \dots, h_i)$ is defined as

$$\sum_{r=1}^i \frac{1}{r!} \left(\sum_{j_1 + \dots + j_r = i} F^{(r)}(z_*)(h_{j_1}, \dots, h_{j_r}) \right) \tag{7}$$

and $\widetilde{r}(\varepsilon)$ is a function of order $o(\varepsilon^m)$ as $\varepsilon \to 0$. Note that $\nabla^i F(z_*)(h_1, \dots, h_i)$ simply collects the ε^i-terms in this expansion. Define linear operators $G_k = G_k[F](z_*; H_{k-1})$, $k \in \mathbb{N}$, depending on the derivatives up to order k of F in the point z_* (the k-jet of F in z_*) and on vectors $H_{k-1} = (h_1, \dots, h_{k-1})$, with $G_k(v)$ given by

$$\sum_{r=1}^{k-1} \frac{1}{r!} \left(\sum_{j_1 + \dots + j_r = k-1} F^{(r+1)}(z_*)(h_{j_1}, \dots, h_{j_r}, v) \right). \tag{8}$$

These operators $G_k[F](z_*; H_{k-1})$ are simply the Fréchet-derivatives of the $(k-1)$-th directional derivative of F at z_* along H_{k-1}. Note that these terms are homogeneous of degree $k-1$. For simplicity of notation we often suppress the arguments. Also define operators $R_q = R_q[F](z_*; H_\ell)$ by

$$\sum_{r=2}^q \frac{1}{r!} \left(\sum_{\substack{j_1 + \dots + j_r = q, \\ 1 \le j_k \le \ell,\ 1 \le k \le r}} F^{(r)}(z_*)(h_{j_1}, \dots, h_{j_r}) \right). \tag{9}$$

Thus $R_q(H_\ell)$ consists of the terms which are homogeneous of degree q, but only involve vectors from H_ℓ, i.e., the indices of the vectors h_j only range from 1 to ℓ. Note that the remainders only have contributions from derivatives of at least order two. These operators allow to formalize high-order approximations to an equality constraint at non-regular points [5].

We now describe the set of critical directions along which high-order tangent approximations to the equality constraint $\mathcal{F}$ can be set up. For a given admissible process $z_* = (x_*, u_*) \in \mathcal{A}$ and a finite sequence $H_{p-1} = (h_1, \dots, h_{p-1}) \in Z^{p-1}$ let

$$Y_i = \sum_{k=1}^i \operatorname{Im} G_k[F](z_*); H_{k-1}), \quad i = 1, \dots, p. \tag{10}$$

We need to impose the following conditions:

(i) the first $p-1$ directional derivatives of F along H_{p-1} vanish,

$$\nabla^i F(z_*)(H_i) = 0 \quad \forall\ i = 1, \dots, p-1; \tag{11}$$

(ii) and the compatibility conditions

$$R_{p-1+i}[F](z_*; H_{p-1}) \in Y_i, \ \ i = 1, \dots, p-1, \tag{12}$$

are satisfied.

In these equations all partial derivatives of f are evaluated along the reference trajectory. Both conditions (i) and (ii) are necessary for the existence of a p-order tangent vector along H_{p-1}. Conditions (i) and (ii) are sufficient for the existence of p-order approximations along H_{p-1} under the following regularity condition:

Definition 3.1 *Let $F : Z \to Y$ be an operator between Banach spaces. We say the operator F is* p-regular *at z_* in direction of the sequence $H_{p-1} \in Z^{p-1}$ if the following conditions are satisfied:*
(A1) *$F : Z \to Y$ is $(2p-1)$-times continuously Fréchet differentiable in a neighborhood of z_*.*
(A2) *The subspaces Y_i, $i = 1, \dots, p$, are closed.*
(A3) *The map $\mathcal{G}_p = \mathcal{G}_p[F](z_*; H_{p-1})$*

$$\begin{aligned} \mathcal{G}_p : Z &\to Y_1 \times Y_2/Y_1 \times \cdots \times Y/Y_{p-1} \\ v &\mapsto \mathcal{G}_p(v) = \left(G_1(v), \pi_1 G_2(v), \dots, \pi_{p-1} G_p(v)\right), \end{aligned} \tag{13}$$

where $\pi_i : Y_{i+1} \to Y_{i+1}/Y_i$ denotes the canonical projection onto the quotient space, is onto.

In the sense of this definition 1-regularity corresponds to the classical Lyusternik condition while 2-regularity is similar to Avakov's definition [2]. It is shown in [4, Thm.1] and [5, Cor.3.3] that the set of all directions h_p for which $H_p = (h_1, \dots, h_{p-1}; h_p)$ is tangent to $\{z \in Z : F(z) = 0\}$ at z_* is a non-empty affine subspace if F is p-regular in direction of $H_{p-1} = (h_1, \dots, h_{p-1})$ and H_{p-1} satisfies (i) and (ii).

The critical directions for the objective I in the least degenerate case are given by

(iii) $I'(z_*)$ is not identically zero and $\nabla^i I(z_*)(H_i) = 0$ for $i = 1, \dots, p-1$

and the critical directions for the inequality constraint $\mathcal{U}$ in the optimal control problem consist of all vectors $H_{p-1} = ((\eta_1, \xi_1); \dots, (\eta_{p-1}, \xi_{p-1}))$ such that

(iv) the p-order feasible cone $FS^{(p)}(\mathcal{U}; u_*, V_{p-1})$ to the convex inequality constraint $\mathcal{U}$ at u_* in direction of $V_{p-1} = (\xi_1), \dots, \xi_{p-1})$ is non-empty.

Definition 3.2 *We call a direction H_{p-1} a* p-regular critical direction *for the extremum problem at z_* if the operator F is p-regular at z_* along H_{p-1} and if conditions (i)-(iv) are satisfied.*

4 The p-order local Maximum Principle

Theorem 4.1 below [6, Thm. 6.1] gives a generalized p-order version of the Maximum Principle. Notice that we write covectors like ψ as row vectors. Also we denote partial derivatives by subscripts. For instance, if $\nabla^i f(H_i)$ denotes the i-th directional derivative of $f = f(x, u, t)$ with respect to the sequence H_i, then $\left(\nabla^i f(H_i)\right)_x$ denotes its partial derivative in x.

Theorem 4.1 ***(p-Order Local Maximum Principle, [6])*** *Suppose the admissible process (x_*, u_*) is optimal for the optimal control problem (OC). Then for every p-regular critical direction H_{p-1} there exist a number $\nu_0 = \nu_0(H_{p-1}) \geq 0$, vectors $a_i = a(H_{p-1}) \in (\mathbb{R}^k)^*$, $i = 0, 1, \dots, p-1$, and absolutely continuous functions $\psi(\cdot) = \psi(H_{p-1})(\cdot)$ and $\rho_i(\cdot) = \rho_i(H_{p-1})(\cdot)$, $i = 1, \dots, p-1$, from $[0, T]$ into $(\mathbb{R}^n)^*$, which satisfy the following conditions along the optimal trajectory $(x_*(t), u_*(t), t)$:*
(a) nontriviality condition: *ν_0 and the functional $\lambda : L^m_\infty(0, T) \to \mathbb{R}$ given by*

$$\lambda(\xi) = \int_0^T \left\langle \nu_0 L_u + \psi f_u + \sum_{i=1}^{p-1} \rho_i \left(\nabla^i f(H_i)\right)_u, \xi \right\rangle dt \tag{14}$$

do not both vanish identically.
(b) extended adjoint equation:

$$\dot{\psi}(t) = -\nu_0 L_x - \psi(t) f_x - \sum_{i=1}^{p-1} \rho_i(t) \left(\nabla^i f(H_i)\right)_x \tag{15}$$

with terminal condition (evaluated at $x_*(T)$)

$$\psi(T) = \nu_0 \ell_x + a_0 q_x + \sum_{i=1}^{p-1} a_i \left(\nabla^i q(H_i)\right)_x. \tag{16}$$

(c) orthogonality conditions on the additional multipliers: *The functions $\rho_i(\cdot)$, $i = 1, \dots, p-1$, satisfy*

$$\begin{aligned} \dot{\rho}_i(t) &= -\rho_i(t) f_x, \quad \rho_i(t) f_u \equiv 0, && (17) \\ \rho_i(T) &= a_i q_x(x_*(T)) && (18) \end{aligned}$$

and for $j = 1, \dots, i-1$, the following conditions are satisfied for a.e. $t \in [0, T]$:

$$\rho_i(t) \left(\nabla^j f(H_j)\right)_x = 0, \tag{19}$$

$$\rho_i(t) \left(\nabla^j f(H_j)\right)_u = 0, \tag{20}$$

$$a_i \left(\nabla^j q((x_*(1); H_j)\right)_x = 0 \tag{21}$$

(d) separation condition: *we have for all $\xi \in FS^{(p)}(\mathcal{U}; u_*, V_{p-1})$ that*

$$\begin{aligned} 0 \leq{} & \nu_0 R_p[\ell] + a_0 R_p[q] + \sum_{i=1}^{p-1} a_i R_{p+i}[q] && (22) \\ &+ \int_0^T \left\langle \nu_0 L_u + \psi f_u + \sum_{i=1}^{p-1} \rho_i(t) \left(\nabla^i f(H_i)\right)_u, \xi \right\rangle dt \\ &+ \int_0^T \left(\nu_0 R_p[L] + \psi(t) R_p[f] + \sum_{i=1}^{p-1} \rho_i(t) R_{p+i}[f] \right) dt \end{aligned}$$

5 Examples

In this section we give several examples which illustrate how Theorem 4.1 can be used to eliminate abnormal candidates from optimality. In these examples the order p will be arbitrary. However, we use the same equality constraint so that the analysis of high-order tangent directions only needs to be done once. We consider the problems to minimize the functional

$$I_{\pm}(x,u) = \int_0^T \left[(x_1-1)^2 + x_2^p + (x_3 \pm 1)^2 - 2 \right] dt$$

over all $(x,u) \in \overline{W}^3_{11}(0,T) \times L^2_\infty(0,T)$ subject to the dynamics

$$\dot{x}(t) = \begin{pmatrix} 0 \\ x_1^p \\ \alpha x_2^{p-1} x_3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{23}$$

initial condition $x(0) = 0$ and terminal constraints $x_1(T) = 0$ and $x_3(T) = 0$. Here p is an integer, $p \geq 2$, and α is an arbitrary real number. For simplicity we have not imposed any control constraints.

It is easily seen that the reference trajectory $\Gamma = (x_*, u_*) \equiv (0,0)$ is an abnormal extremal for each problem. Furthermore, the corank of Γ is 1 for the problem to minimize I_+, i.e., there exists a unique multiplier which satisfies the conditions of the Maximum Principle. This multiplier is given by $\lambda_0 = 0$ and $\lambda(t) \equiv (\nu, 0, \nu)$ for some non-zero constant ν. For the problem to minimize I_-, however, Γ has 2 linearly independent multipliers, one normal, given by $\lambda_0 = 1$ and $\lambda(t) \equiv (2(t-T), 0, 2(t-T))$, the other abnormal, given by $\lambda_0 = 0$ and $\lambda(t) \equiv (1,0,1)$. It is shown in [6] that for the problem to minimize I_+ and with $p = 2$ the optimality of Γ can be excluded either by using the Goh-condition [3] or using a more recent result of Agrachev and Sarychev [1, Thm. 3.4]. For $p > 2$, however, these conditions become inconclusive and for the problem to minimize I_- they no longer apply at all since both normal and abnormal multipliers exist.

We now show how Theorem 4.1 can be used to obtain additional information. We first consider the problem to minimize I_+ and show how to eliminate the optimality of Γ for any $p \geq 2$ using Theorem 4.1. For this we first need to find an appropriate p-regular critical direction

$$H_{p-1} = ((\eta_1, \xi_1); \ldots ; (\eta_{p-1}, \xi_{p-1})) .$$

Since the components of f are homogeneous polynomials of degree p and the terminal constraints are linear, it follows that for $i = 1, \ldots, p-1$, the directional derivatives $\nabla^i F(z_*)(H_i)$ are simply given by

$$\nabla^i F(z_*)(H_i) = F'(0,0)(\eta_i, \xi_i) \tag{24}$$

and thus these derivatives vanish provided $(\eta_i, \xi_i) \in \ker F'(0,0)$ for $i = 1, \ldots, p-1$. Since the dynamics involves only p-order terms it seems reasonable to consider only first and p-th order approximations. We therefore choose H_{p-1} of the form

$$H_{p-1} = ((\eta_1, \xi_1); (0,0); \ldots ; (0,0)) \tag{25}$$

with $(\eta_1, \xi_1) \in \ker F'(0,0)$. With this choice of directions the compatibility conditions (ii) simplify considerably and reduce to the first condition only,

$$\begin{aligned} R_p[F](\Gamma; H_{p-1}) &= \frac{1}{p!} F^{(p)}(0,0)\left((\eta_1, \xi_1); \ldots ; (\eta_1, \xi_1)\right) \\ &\in \operatorname{Im} F'(0,0). \end{aligned} \tag{26}$$

The conditions for $i = 2, \ldots, p-1$, are automatically satisfied since $R_{p-1+i}[F](\Gamma; H_{p-1}) = 0$. Condition (26) is equivalent to

$$(1,0,1) \cdot \int_0^T D_x^{(p)} A(0)(\eta_1, \ldots, \eta_1) ds = 0 \tag{27}$$

where $D_x^{(p)} A(0)(\eta_1, \ldots, \eta_1)$ denotes the 3-vector whose i-th entry is given by the values of the multi-linear form $D_x^{(p)} A_i(0)$ acting on the vector η_1 in every component. In particular, since $A_1(x) \equiv 0$ in the example, we must therefore have

$$\int_0^T D_x^{(p)} A_3(0)(\eta_1, \ldots, \eta_1) ds = 0. \tag{28}$$

If we denote the components of η_1 by $\eta_1^{[j]}$, $j = 1,2,3$, then for $A_3(x) = \alpha x_2^{p-1} x_3$ this requires that

$$\int_0^T \left(\eta_1^{[2]}\right)^{p-1} \left(\eta_1^{[3]}\right) ds = 0. \tag{29}$$

We satisfy this by choosing $\eta_1^{[3]} = -\eta_1^{[1]} \equiv 0$. Then choosing a nonzero $\eta_1^{[2]}$ with zero boundary conditions defines a nontrivial vector H_{p-1} of the form (25) for which conditions (i) and (ii) in the definition of p-regular critical directions are satisfied. The operator F is p-regular in this direction if the operator $\mathcal{G}_p(\Gamma; H_{p-1})$ is onto. Because of the homogeneity properties of A the p-th component G_p of $\mathcal{G}_p$ acting on $(\widetilde{\eta}, \widetilde{\xi})$ is given by

$$\frac{1}{(p-1)!} F^{(p)}(0,0)\left((\eta_1, \xi_1); \ldots ; (\eta_1, \xi_1); (\widetilde{\eta}, \widetilde{\xi})\right)$$

while all operators G_i for $i = 2, \ldots, p-1$, vanish. Thus $\mathcal{G}_p(\Gamma; H_{p-1})$ is onto if there exists a direction $(\widetilde{\eta}, \widetilde{\xi})$ such that $G_p[F](\Gamma; H_{p-1})(\widetilde{\eta}, \widetilde{\xi}) \notin \operatorname{Im} F'(0,0)$. Like above, this is equivalent to

$$\int_0^T D_x^{(p)} A_3(0)(\eta_1, \ldots, \eta_1, \widetilde{\eta}) ds \neq 0. \tag{30}$$

Equivalently, the following integral must not vanish

$$\int_0^T \left(\eta_1^{[2]}\right)^{p-2} \left[\left(\eta_1^{[2]}\right)\left(\widetilde{\eta}^{[3]}\right) + (p-1)\left(\eta_1^{[3]}\right)\left(\widetilde{\eta}^{[2]}\right) \right] ds. \tag{31}$$

Since we have $\eta_1^{[3]} \equiv 0$ in our chosen direction H_{p-1} this can simply be satisfied by choosing $\widetilde{\eta}^{[3]} = \eta_1^{[2]}$ if p is even or $\widetilde{\eta}^{[3]} = \left(\eta_1^{[2]}\right)^2$ if p is odd. Hence F is p-regular in direction of H_{p-1} at Γ. Finally, these directions are also critical for the objective: we have $I'_+(0,0)(\eta_1, \xi_1) = 0$ and furthermore

$$\nabla^2 I_+(0,0)(H_2) = \int_0^T \left(\eta_1^{[1]}\right)^2 + \left(\eta_1^{[3]}\right)^2 ds = 0$$

provided $p > 2$. Since no other I_+-derivatives arise in the directional derivatives $\nabla^i I_+(0,0)(H_i)$ for $i = 3, \ldots, p-1$, the direction $H_{p-1} = ((\eta_1, \xi_1); (0,0); \ldots; (0,0))$ with $\eta_1^{[1]} = \eta_1^{[3]} \equiv 0$ and a nonzero $\eta_1^{[2]}$ is a nonzero p-regular critical direction for the problem to minimize I_+ subject to $F = 0$ for any $p \geq 2$.

We thus can apply Theorem 4.1. Since there are no control constraints we can normalize the multipliers so that $\nu_0 = 1$. The additional multipliers ρ_i, $i = 1, \ldots, p-1$, are associated with elements in the dual spaces of the quotients Y_{i+1}/Y_i. But here $Y_i = \operatorname{Im} F'(0,0)$ for $i = 1, \ldots, p-1$, and Y_p is the full space. Thus we have $\rho_i \equiv 0$ for $i = 1, \ldots, p-2$, and the only non-zero multipliers are ψ and ρ_{p-1} which for simplicity of notation we just call ρ. Now (18) of Theorem 4.1 states that ρ is an adjoint multiplier for which the conditions of the weak local Maximum Principle for an abnormal extremal are satisfied. Since this multiplier is unique we must have $\rho(t) = (\nu, 0, \nu)$, but $\nu \in \mathbb{R}$ could now be zero. To write down the extended adjoint equation and minimum condition we need to evaluate the directional derivatives $\nabla^{p-1} f(x,u)(H_i)$ where $f(x,u) = A(x) + Bu$. Note that all partial derivatives of order at least two which contain one u-derivative vanish. Thus we only need to calculate the actual x-partials. It can be shown that

$$\left(\nabla^{p-1} f(0,0)(H_i)\right)_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\eta_1^{[2]}\right)^{p-1} \end{pmatrix} \tag{32}$$

and

$$\left(\nabla^{p-1} f(0,0)(H_i)\right)_u \equiv 0. \tag{33}$$

Thus the extended minimum condition reduces to $\psi B \equiv 0$, the minimum condition of the weak maximum principle. Hence $\psi_2(t) \equiv 0$ and $\psi_1(t) = \psi_3(t)$. The extended adjoint equation is given by

$$\dot{\psi}(t) = (2, 0, -2) - \rho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\eta_1^{[2]}\right)^{p-1} \end{pmatrix} \tag{34}$$

and thus

$$4 = \dot{\psi}_1(t) - \dot{\psi}_3(t) - \nu \left(\eta_1^{[2]}(t)\right)^{p-1} = -\nu \left(\eta_1^{[2]}(t)\right)^{p-1}.$$

But we can certainly choose $\eta_1^{[2]}$ non-constant to violate this condition. This contradiction proves that Γ cannot be optimal for the problem to minimize I_+ for any $p \geq 2$.

We now consider the problem to minimize I_- for which the codimension of $\operatorname{Im}(F'(0,0), I'_-(0,0))$ is 2. This becomes a in some aspects harder, but in other aspects actually easier problem. Essentially, since there exist two linearly independent multipliers which satisfy the weak local Maximum Principle, conditions (a)-(c) of Theorem 4.1 can be satisfied in a straightforward way using these multipliers where (ν_0, ψ) is the normal multiplier and $\rho \equiv 0$. Hence these conditions will give nothing new. On the other hand, since the codimension is 2, the restrictions on the critical directions are less stringent since any direction $(\eta, \xi) \in \ker F'(0,0)$ is automatically critical for the objective as well. Hence more directions are critical and the separation condition (d) becomes stronger. Now, for the details.

We first assume $p > 2$ and consider the same p-regular tangent directions as above. Note that these directions remain p-order critical for the objective and thus are p-regular critical directions. The only difference to the analysis above is that the extended adjoint equation now reads

$$\dot{\psi}(t) = (2, 0, 2) - \rho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta_1^{[2]} \end{pmatrix} \tag{35}$$

and thus we get

$$0 = \dot{\psi}_1(t) - \dot{\psi}_3(t) - \nu \eta_1^{[2]}(t) = -\nu \eta_1^{[2]}(t). \tag{36}$$

But this equation can be satisfied with $\nu = 0$ implying $\rho \equiv 0$. The separation condition (d) reduces to

$$0 \leq \int_0^T R_p[L](H_{p-1}) + \psi(t) R_p[f](H_{p-1}) dt. \tag{37}$$

Since $\eta_1^{[1]} \equiv \eta_1^{[3]} \equiv 0$ we have

$$R_p[f](H_{p-1}) = \left(0, \left(\eta_1^{[2]}\right)^p, 0\right)^T \tag{38}$$

and this term will be annihilated by $\psi_2(t) \equiv 0$. Also $R_p[L](H_{p-1})$ only generates the term $\left(\eta_1^{[2]}(t)\right)^p$ as nonzero term. Thus it is a necessary condition for optimality of Γ that

$$0 \leq \int_0^T \left(\eta_1^{[2]}(t)\right)^p dt \tag{39}$$

for any p-regular critical directions which satisfies $\eta_1^{[1]} \equiv \eta_1^{[3]} \equiv 0$. But we can multiply these directions by -1 and thus Γ is not optimal if p is odd.

If p is even, (39) will always be satisfied and these directions do not give rise to better values of the objective.

In fact, it can be shown that no improvement of the objective is possible for any tangent direction. This strongly suggests that indeed Γ is locally optimal if p is even and $p \geq 4$. However, we are not aware of sufficient conditions for optimality which would apply to this situation.

The case $p = 2$ is the most interesting one. As above, in this case 2-regular critical directions for which $\eta_1^{[1]} \equiv \eta_1^{[3]} \equiv 0$ cannot be used to exclude the optimality of Γ. However, now any direction in $\ker F'(0,0)$ is critical and thus any nontrivial $(\eta, \xi) \in \ker F'(0,0)$ which satisfies the compatibility condition (ii)

$$\int_0^T \eta^{[2]}(s)\eta^{[3]}(s)ds = 0. \tag{40}$$

defines a 2-regular critical direction. It is easily seen from (31) that the operator F is 2-regular at Γ in direction of (η, ξ) by choosing $\widetilde{\eta}^{[3]} = \eta_1^{[2]}$ and $\widetilde{\eta}^{[2]} = \eta_1^{[3]}$. Hence any direction $(\eta, \xi) \in \ker F'(0,0)$ which satisfies (40) is a 2-regular critical direction. The fact that $\operatorname{co\,dim}(\operatorname{Im} I'_-(0,0), \operatorname{Im} F'(0,0)) = 2$ allows for this *large* class of critical directions H_1. The separation condition (d) of Theorem 4.1 and the fact that $\rho \equiv 0$ therefore imply that the quadratic form

$$\mathcal{Q} = \frac{1}{2}\int_0^T L^{(2)}(0,0)(H_1, H_1) + \psi(t) f^{(2)}(0,0)(H_1, H_1)dt$$

is positive semi-definite on the set of all 2-regular critical directions H_1. Note that $\mathcal{Q}$ takes the form of the accessory problem for the normal multiplier $(1, \psi)$, but that the domain is restricted to the actual tangent directions to the equality constraint. In this case we have $\psi_3(t) = \nu + 2(t - T)$, $\nu = \psi_3(T)$. One degree of freedom in the multipliers is taken by the multipliers $(\nu, 0, \nu)$ from the annihilator of $\operatorname{Im} F'(0,0)$. But, as stated in [6, Cor. 7.2], this freedom does not enter into the value of the quadratic form:

$$\mathcal{Q} = \int_0^T \left(\eta^{[2]}\right)^2 + 2\left(\eta^{[3]}\right)^2 + 2t\alpha\eta^{[2]}\eta^{[3]}dt. \tag{41}$$

Regardless of the value of ν, the quadratic form $\mathcal{Q}$ takes the same value for all possible multipliers because of the compatibility condition (40).

Now we pick a suitable subset of 2-regular critical directions H_1. Let C denote the space of all twice continuously differentiable functions $h : [0, T] \to \mathbb{R}$ which satisfy zero boundary conditions $h(0) = h(T) = 0$. It is easy to see that C is isomorphic to a subspace of critical directions defined by

$$\eta^{[3]}(t) = -\eta^{[1]}(t) = h(t), \quad \eta^{[2]}(t) = \dot{h}(t) \tag{42}$$

and

$$\xi^{[1]}(t) = \ddot{h}(t), \quad \xi^{[2]}(t) = -\dot{h}(t). \tag{43}$$

Hence it is a necessary condition for optimality that the quadratic form

$$\begin{aligned} \mathcal{Q} &= \int_0^T \left(\dot{h}(t)\right)^2 + 2\left(h(t)\right)^2 + 2\alpha t\dot{h}(t)h(t)dt \\ &= \int_0^T \left(\dot{h}(t)\right)^2 + (2 - \alpha)\left(h(t)\right)^2 dt \end{aligned} \tag{44}$$

is positive semi-definite on C. It follows from the Jacobi-equation that $\mathcal{Q}$ is positive definite on C for $\alpha < 2 + \left(\frac{T}{\pi}\right)^2$, positive semi-definite for $\alpha = 2 + \left(\frac{T}{\pi}\right)^2$ and indefinite for $\alpha > 2 + \left(\frac{T}{\pi}\right)^2$. Thus Γ is not optimal for $\alpha > 2 + \left(\frac{T}{\pi}\right)^2$.

6 Conclusion

The results in [6] establish a hierarchy of necessary conditions for optimality for various levels of approximations and regularity. As the example presented here demonstrates these results are fairly effective and allow further analysis than other known results. Both classical results like the Goh-condition or the more recent results of Agrachev and Sarychev only give a conclusive answer for the problem to minimize I_+ and with $p = 2$, while our results eliminate the zero trajectory from optimality for I_+ for all $p \geq 2$. In the case of minimizing I_- when classical results are not even applicable we can eliminate the zero trajectory from optimality in certain cases for $p = 2$ and in general for an odd $p \geq 3$. In the remaining cases of even p when our results remain inconclusive there is strong reason to suspect that the zero trajectory is indeed optimal, but we have no proof as of yet.

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