OBSERVABILITY INEQUALITIES FOR SHALLOW SHELLS [1]

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Abstract

We consider some observability inequalities from boundary for a general shallow shell with a middle surface, any shape. At first, an estimate is established by the geometric multiplier method in the case that no boundary conditions are imposed under some checkable geometric conditions. Then our results yield continuous observability estimates for two kinds of boundary conditions which have physical meaning with an explicit observability time; hence, by duality, exact controllability results.

1 Introduction

The purpose of this paper is to establish some observability estimates for the shallow shell from which some boundary exact controllability results can be derived. This problem has been well understood in the cases of wave equations and plates, and, in particular, in the constant coefficient case, Komornik [9], Lagnese, Lions [11]. It is, in general, hard to handle the variable coefficient case in which some special tools are often needed in addition to the classical multiplier method, for instance, the microlocal analysis method Bardos, Lebeau, Rauch [1], the pseudo-differential method Tataru [14], and the geometric method Yao [18]. In the case of thin shells with a middle surface of any shape, very little is apparently known in the context of control/stabilization theory partly because thin shell problems are always of variable coefficient (at least on space variables). Generally, direct adaptation of the techniques, traditionally developed in assuming that the middle surface is defined by one coordinate, would not be fully adequate when dealing with some observability estimates since the presence of the Christoffel symbols Γ_{jk}^k can often make computing too complicated.

First, we here shall briefly introduce the classical shallow model in a version, produced by Yao [19], in which the middle surface is viewed as a Riemann manifold with the induced metric in $\mathbb{R}^3$. One of advantages in doing this is to build a bridge to the modern geometry. For instance, the Bochner technique, which can not be applied once fixed in one coordinate, consists of the key ingredient throughout this paper to be used to overcome the complexity of computation when we deal with all the estimates. This technique, which describes a method initiated by Bochner some fifty years ago for proving some identities of geometric interest, is not so easily described, but it offers the greatest computational simplification on some variable coefficient problems. For details, we refer to book Wu [17].

Next, we set up an assumption (H2) on the middle surface under which an estimate for the shallow model is established in the case that no boundary conditions are imposed (Theorem 2.1). In fact, this assumption also works for some observability inequalities of Naghdi's models, Yao [22]. It is easy to prove that the main assumption (H2) always holds locally for the middle surface of any shape by the geometry method. In particular, several examples of the middle surface that verify the main assumption (H2) are presented at the end of Section 2.

Finally, the estimates in Theorem 2.1 will produce continuous observability inequalities in the both Dirichlet and Neumann cases (Theorems 2.2 and 2.3), respectively, by a compactness/uniqueness argument to absorb the lower order terms as in Bardos and Lebeau and Rauch [1] for the wave equation. Fortunately, all the uniqueness results we here need in the both cases can be derived from some old uniqueness issues, Hörmander [6] or Shirota [14]. In addition, regularities of solutions to all the shallow equations, needed here, should be an intrinsic issue. Since we are mainly concerned with estimates of inequalities, it is assumed that all the regularities of solutions we need hold in this paper.

Finally, we mention that some works have been done on control problems of some special shallow shells, for example, Chen, Coleman, and Liu [4] for circular cylindrical shells and Lasiecka, Triggiani, Valente [12], and Triggiani [16] for spherical shells.

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2 Main Results

2.1. Model Denote the usual inner product in $I\!R^3$ by $\langle\cdot,\cdot\rangle$, i.e., the dot product. Let M be a surface in $I\!R^3$. For simplicity, M is assumed to be smooth. Surface M produces a natural Riemannian manifold of dimension 2 with the induced metric in $I\!R^3$. We denote this induced metric on surface M by g or by $\langle\cdot,\cdot\rangle$, as convenient.

Let us assume that the middle surface of the shell occupies a bounded region Ω of surface M in $I\!R^3$. Denote by N the normal of surface M. Let Γ be the boundary of the middle surface Ω. The shell, a body in $I\!R^3$, is defined by

$$\mathcal{S} = \{\, p \,|\, p = x + zN(x),\, x \in \Omega,\, -h/2 < z < h/2\,\}. \tag{2.1}$$

where h is the thickness of the shell, "small", Ciarlet and Paumier [5]. Denote $\zeta(x)$ the displacement vector of point x of the middle surface. We decompose displacement vector ζ into sums

$$\zeta(x) = W(x) + w(x)N(x),$$
$$x \in \Omega,\ W(x) \in M_x, \tag{2.2}$$

i.e., W and w are components of ζ on the tangent plane and on the normal of the undeformed middle surface Ω, respectively. Let D be the Levi-Civita connection of manifold M. The linearized strain tensor and the change of curvature tensor of the middle surface Ω are then given by

$$\Upsilon(\zeta) = \frac{1}{2}(DW + D^*W) + w\Pi \tag{2.3}$$

and

$$\rho(\zeta) = -D^2 w \tag{2.4}$$

in a form coordinates free, respectively, where Π is the second fundamental form of surface M and D^2w the Hessian of w, which are justified for a shallow shell. For (2.3) and (2.4), we refer to Niordson [13, p. 355] or to Koiter [7, p.27].

The shell strain energy associated to a displacement field ζ of the middle surface Ω can be written as

$$\mathcal{B}_1(\zeta,\zeta) = \frac{Eh}{1-\mu^2}\int_\Omega B(\zeta,\zeta)\,dx, \tag{2.5}$$

where

$$B(\zeta,\zeta) = a(\Upsilon(\zeta),\Upsilon(\zeta)) + \gamma a(\rho(\zeta),\rho(\zeta)), \quad \gamma = h^2/12, \tag{2.6}$$

$$a(\Upsilon(\zeta),\Upsilon(\zeta)) = (1-\mu)\langle\Upsilon(\zeta),\Upsilon(\zeta)\rangle_{T_x^2} + \mu(tr\Upsilon(\zeta))^2, \tag{2.7}$$

for $x \in \Omega$, where E, μ respectively denote Young's modulus and Poison's coefficient of the material. For (2.5), we refer to Bernadou and Boisserie [2, p. 15]. Thus, with expression (2.5), we are able to associate the following symmetric bilinear form, directly defined on the middle surface Ω:

$$\mathcal{B}(\zeta,\eta) = \int_\Omega B(\zeta,\eta)\,dx, \tag{2.8}$$

where ζ is given in (2.2) and

$$\eta = U + uN, \quad U(x) \in M_x,\ x \in \Omega. \tag{2.9}$$

Denote by H and by k the mean curvature and the Gauss curvature of surface M, respectively. From Yao [19], we have the following Green formula for a shallow shell.

Formula I. Let bilinear form $\mathcal{B}(\cdot,\cdot)$ be given in (2.8). For $\zeta = (W,w)$, $\eta = (U,u) \in H^1(\Omega,\Lambda)\times H^2(\Omega)$, we have

$$\mathcal{B}(\zeta,\eta) = (\mathcal{A}\zeta,\eta)_{L^2(\Omega,\Lambda)\times L^2(\Omega)} + \int_\Gamma \partial(\mathcal{A}\zeta,\eta)\,d\Gamma, \tag{2.10}$$

where

$$\begin{aligned}\partial(\mathcal{A}\zeta,\eta) &= B_1(W,w)\langle U,n\rangle + B_2(W,w)\langle U,\tau\rangle + \\ &+\gamma[(\Delta w + (1-\mu)B_3 w)\frac{\partial u}{\partial n} \\ &-(\frac{\partial \Delta w}{\partial n} + (1-\mu)B_4 w)u],\end{aligned} \tag{2.11}$$

n, τ are the normal and the tangential along curve Γ, respectively,

$$\mathcal{A}\zeta = \begin{pmatrix} -\mathbf{\Delta}_\mu W - (1-\mu)kW - \mathcal{F}(w) \\ \gamma[\Delta^2 w - (1-\mu)\delta(kdw)] \\ +(H^2 - 2(1-\mu)k)w + \mathcal{G}(W)\end{pmatrix}, \tag{2.12}$$

$\mathbf{\Delta}_\mu$ is of the Hodge-Laplacian type, applied to 1-forms (or equivalently vector fields), defined by

$$\mathbf{\Delta}_\mu = -(\frac{1-\mu}{2}\delta d + d\delta), \tag{2.13}$$

d the exterior differential, δ the formal adjoint of d, Δ the Laplacian on manifold M,

$$\begin{cases}\mathcal{F}(w) = (1-\mu)l_{dw}\Pi + \mu H dw + w dH, \\ \mathcal{G}(W) = (1-\mu)\langle DW,\Pi\rangle_{T_x^2} - \mu H\delta W,\end{cases} \tag{2.14}$$

and

$$\begin{cases} B_1(W,w) = (1-\mu)\Upsilon(\zeta)(n,n) + \mu(wH - \delta W), \\ B_2(W,w) = (1-\mu)\Upsilon(\zeta)(n,\tau), \\ B_3 w = -D^2 w(\tau,\tau), \\ B_4 w = \dfrac{\partial}{\partial\tau}(D^2 w(\tau,n)) + k(x)\dfrac{\partial w}{\partial n}.\end{cases} \tag{2.15}$$

By the "Principle of Virtual Work" and Formula I, we obtain the following displacement equations for a shallow shell (see Yao [18]) after changing t to t/λ with $\lambda^2 E/(1-\mu^2) = 1$:

Formula II. We assume that there are no external loads on the shell and that the shell is clamped along a portion Γ_0 of Γ and free on Γ_1, where $\Gamma_0 \cup \Gamma_1 = \Gamma$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Then the displacement vector $\zeta = (W, w)$ satisfies the following boundary value problem:

$$\begin{cases} W_{tt} - [\mathbf{\Delta}_\mu W + (1-\mu)kW + F(w)] = 0, \\ w_{tt} - \gamma\Delta w_{tt} + \gamma\left(\Delta^2 w - (1-\mu)\delta(kdw)\right) + \\ \qquad +(H^2 - 2(1-\mu)k)w + \mathcal{G}(W)] = 0, \\ \zeta(0) = \zeta_0, \quad \zeta_t(0) = \zeta_1, \end{cases} \quad \text{in } Q_\infty, \tag{2.16}$$

$$\begin{cases} W = 0, \\ w = \dfrac{\partial w}{\partial n} = 0, \end{cases} \quad \text{on } \Sigma_{0\infty} \tag{2.17}$$

$$\begin{cases} B_1(W,w) = B_2(W,w) = 0, \\ \Delta w + (1-\mu)B_3 w = 0, \\ \dfrac{\partial \Delta w}{\partial n} + (1-\mu)B_4 w - \dfrac{\partial w_{tt}}{\partial n} = 0, \end{cases} \quad \text{on } \Sigma_{1\infty}, \tag{2.18}$$

where

$$Q_\infty = \Omega \times (0,\infty), \quad \Sigma_{0\infty} = \Gamma_0 \times (0,\infty), \quad \Sigma_{1\infty} = \Gamma_1 \times (0,\infty). \tag{2.19}$$

Remark 2.1. *In the literature for some special cases the displacement equations are expressed in terms of three displacement components of the shell and their derivatives such as spherical shells, Lasiecka, Trigginia, Valente* [12]*, and circular cylindrical ones, Chen, Coleman, Liu* [4]*. For a shell with a general middle surface of any shape, this method may not be possible (see some comments by Koiter* [8] *p.33) and we have to draw support from Formula II.*

Remark 2.2. *If the shell is flat, a plate, equations* (2.16) *are uncoupled. The equation on component w is the same as in Lagnese* [10, *pp.* 15 − 16]*, a Kirchhoff plate (see Yao [19]).*

2.2. Observability inequalities In obtaining observability inequalities, the ellipticity of the shell strain energy is necessary, which is assumed throughout, that is, there is constant $\lambda_0 \geq 1$ such that (**H.1**)

$$\lambda_0 \mathcal{B}(\zeta,\zeta) \geq \|DW\|^2_{L^2(\Omega,T^2)} + \gamma\|D^2 w\|^2_{L^2(\Omega,T^2)} \tag{2.20}$$

for $\zeta = (W,w) \in H^1(\Omega,\Lambda) \times H^2(\Omega)$. The above inequality is established if Π and $D\Pi$ are small enough by Bernadou, Oden [3]; proved if there is some information on the curvature of the middle surface by Yao [19].

Main Assumption (H.2). Suppose that there is a vector field $V \in \mathcal{X}(M)$ such that

$$DV(X,X) = b(x)|X|^2, \quad X \in M_x, \, x \in \overline{\Omega}, \tag{2.21}$$

where b is a function on Ω. Set

$$a(x) = \frac{1}{2}\langle DV, \mathcal{E}\rangle_{T^2_x}, \quad x \in \overline{\Omega}, \tag{2.22}$$

where $\mathcal{E}$ is the volume element of M. Moreover, suppose that b and a meet inequality

$$2\min_{x\in\overline{\Omega}} b(x) > \lambda_0(1+\mu)\max_{x\in\overline{\Omega}} |a(x)|. \tag{2.23}$$

We say that middle surface Ω satisfies assumption (H2) if there is a vector field V such that conditions (2.21) and (2.23) hold.

Set

$$\sigma_0 = \max_{x\in\Omega} |V|,$$

$$\sigma_1 = \min_{x\in\Omega} b(x) - \frac{\lambda_0(1+\mu)}{2}\max_{x\in\Omega}|a(x)|; \tag{2.24}$$

$$Q = \Omega \times (0,T), \quad \Sigma_0 = \Gamma_0 \times (0,T), \quad \Sigma_1 = \Gamma_1 \times (0,T). \tag{2.25}$$

Remark 2.3. *Geometric condition (2.21) is used in Yao* [20] *for some observability inequalities of the Euler-Bernoulli equation with variable coefficients. If the shell is flat, a plate, then $M = I\!R^2$. For any $x^0 \in I\!R^2$, set $V = x - x^0$. It is easily checked that*

$$DV = g, \quad x \in I\!R^2,$$

where g is the dot product in R^2, with $b = 1$ and $a = 0$. For any M, we can prove that there always exists a vector field to meet condition (2.21) *on Ω and that assumption $(H2)$ always holds locally. Indeed, we can also show that there are a vector field V and $\epsilon > 0$ such that*

$$2\min_{x\in B(\epsilon)} b(x) > \lambda_0(1+\mu)\max_{x\in B(\epsilon)}|a(x)|$$

where $B(\epsilon)$ is the geodesic ball with radius ϵ and centered at x_0. The above inequality means that assumption $(H2)$ is true if middle surface $\Omega \subset B(\epsilon)$. Next, let surface M be of constant curvature or of revolution. It is then easy to prove that there exists a vector field V such that relation (2.21) *holds on the whole surface M with $a(x) = 0$ for all $x \in M$. So if middle surface $\Omega \subset M$ such that $b(x) \neq 0$ for all $x \in \overline{\Omega}$, then assumption $(H2)$ holds for Ω.*

The total energy of the shell is to be defined by

$$E(t) = \frac{1}{2}(\|W_t\|^2_{L^2(\Omega,\Lambda)} + \|w_t\|^2_{L^2(\Omega)} + \gamma\|Dw_t\|^2_{L^2(\Omega,\Lambda)} + \mathcal{B}(\eta,\eta)). \tag{2.26}$$

For $\eta = (W,w)$, we set

$$\eta_1 = (W,0); \quad \eta_2 = (0,w); \tag{2.27}$$

$$L(t) = \|W(t)\|^2_{L^2(\Omega,\Lambda)} + \|w(t)\|^2_{L^2(\Omega)} + \gamma\|w_t(t)\|^2_{L^2(\Omega)} + \|Dw(t)\|^2_{L^2(\Omega,\Lambda)}. \tag{2.28}$$

Theorem 2.1 *Let assumptions* $(H.1)$ *and* $(H.2)$ *hold. Let* $\eta = (W, w)$ *solve problem*

$$\eta_{tt} - \gamma(0, \Delta w_{tt}) + \mathcal{A}\eta = 0 \tag{2.29}$$

such that all the terms on the left hand side of inequality (2.30) *below are well defined. Given* $T > 0$. *Then, for any* $\epsilon > 0$, *there is* $C_\epsilon > 0$, *independent of* η, *such that*

$$\begin{aligned} & SB|_\Sigma + C_\epsilon[L(0) + L(T) + \int_0^T L(t)dt] \\ & +(\sigma_0\lambda_0 + \epsilon)[E(0) + E(T)] \\ & \geq \sigma_1 \int_0^T E(t)dt, \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} SB|_\Sigma = & \frac{1}{2}\int_\Sigma [|\eta_t|^2 + \gamma|Dw_t|^2 - B(\eta,\eta)]\langle V, n\rangle \, d\Sigma + \\ & + \int_\Sigma [\partial(\mathcal{A}\eta, \, m(\eta) - \frac{1}{2}b\eta_2 + \frac{1}{2}h\eta_1) \\ & +\gamma(V(w) - \frac{1}{2}bw)\frac{\partial w_{tt}}{\partial n}] \, d\Sigma; \end{aligned} \tag{2.31}$$

$$m(\eta) = (D_V W, V(w)); \quad h = 2b - \sigma_1. \tag{2.32}$$

Dirichlet control. First, we consider the Dirichlet mixed problem in unknown $\zeta = (\Phi, \phi)$

$$\begin{cases} \zeta_{tt} - \gamma(0, \Delta\phi_{tt}) + \mathcal{A}\zeta = 0, \quad \text{in} \quad \Omega \times (0,T), \\ \zeta(0) = \zeta^0, \quad \zeta_t(0) = \zeta^1, \quad \text{on} \quad \Omega, \\ \Phi|_{\Gamma_1} = 0, \quad \Phi|_{\Gamma_0} = U, \quad 0 < t < T, \\ \phi|_{\Gamma_1} = \dfrac{\partial\phi}{\partial n}|_{\Gamma_1} = 0, \quad 0 < t < T, \\ \phi|_{\Gamma_0} = u, \quad \dfrac{\partial\phi}{\partial n}|_{\Gamma_0} = v, \quad 0 < t < T, \end{cases} \tag{2.33}$$

with control functions U, u, and v. Its dual version in $\eta = (W, w)$

$$\begin{cases} \eta_{tt} - \gamma(0, \Delta w_{tt}) + \mathcal{A}\eta = 0, \quad \text{in} \quad Q, \\ \eta(0) = \eta^0, \quad \eta_t(0) = \eta^1, \quad \text{on} \quad \Omega, \\ W = 0, \quad \text{on} \quad \Sigma, \\ w = \dfrac{\partial w}{\partial n} = 0, \quad \text{on} \quad \Sigma. \end{cases} \tag{2.34}$$

Remark 2.4 *In the flat case, for the normal component, one control function* $\frac{\partial\phi}{\partial n}|_{\Gamma_0} = v$ *is enough, Lagnese and Lions* [11]. *We here add another control function* $\phi|_{\Gamma_0} = u$ *for problem* (2.33) *in order to avoid the following uniqueness assumption: problem*

$$\begin{cases} \lambda^2\eta - \lambda^2\gamma(0, \Delta w) + \mathcal{A}\eta = 0 \quad on \quad \Omega, \\ W = 0 \quad on \quad \Gamma, \\ w = \dfrac{\partial w}{\partial n} = 0 \quad on \quad \Gamma, \\ D_n W = \Delta w = 0 \quad on \quad \Gamma_0, \end{cases} \tag{2.35}$$

admits the unique zero solution. The above uniqueness result dose not fall into a class of systems to which the Homgren Theorem may be applied even if the coefficients are analytic since it is not the Cauchy problem for component w *(it does for component* W*). For the flat case, it has been proved in Lagnese and Lions* [11].

Continuous observability inequality in Dirichlet case. From Proposition 2.7 of subsection 2.2, it is easily checked that if $\eta = (W, w)$ solves problem (2.34), then

$$SB|_\Sigma = \frac{1}{2}\int_\Sigma B(\eta,\eta)\langle V, n\rangle \, d\Sigma. \tag{2.36}$$

The exact controllability of problem (2.33) then leads to the following observability inequality: to seek constant $T_0 > 0$ such that, for any $T > T_0$, there is $c > 0$ satisfying

$$\int_{\Sigma_0} [B(\eta,\eta) + \gamma\left(\frac{\partial\Delta w}{\partial n}\right)^2] \, d\Sigma \geq cE(0), \tag{2.37}$$

where $\eta = (W, w)$ is a solution to problem (2.34) with the initial data $(\eta^0, \eta^1) \in \big(L^2(\Omega, \Lambda) \times H_0^1(\Omega)\big) \times \big(H^{-1}(\Omega, \Lambda) \times L^2(\Omega)\big)$, and

$$\begin{aligned} \int_{\Sigma_0} B(\eta,\eta) \, d\Sigma = & \gamma\int_{\Sigma_0} (\Delta w)^2 \, d\Sigma \\ & + \int_{\Sigma_0} (DW(n,n))^2 \, d\Sigma \\ & + \frac{1-\mu}{2}\int_{\Sigma_0} (DW(\tau,n))^2 \, d\Sigma. \end{aligned} \tag{2.38}$$

We have the following

Theorem 2.2 *(Dirichlet case) Let assumptions* $(H.1)$ *and* $(H.2)$ *hold. Then for any* $T > T_0$, *there exits* $c > 0$ *such that observability inequality* (2.37) *holds, where*

$$T_0 = 2\lambda_0\sigma_0/\sigma_1; \tag{2.39}$$

$$\Gamma_0 = \{\, x \,|\, x \in \Gamma, \, V(x) \cdot n(x) > 0 \,\}. \tag{2.40}$$

Neumann control. Here we let $\overline{\Gamma_1} \neq \emptyset$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, and consider problem $\zeta = (\Phi, \phi)$

$$\begin{cases} \zeta_{tt} - \gamma(0, \Delta\phi_{tt}) + \mathcal{A}\zeta = 0 \quad \text{in} \quad Q, \\ \zeta(0) = \zeta^0, \quad \zeta_t(0) = \zeta^1 \quad \text{on} \quad \Omega. \end{cases} \tag{2.41}$$

We can act on $\Sigma_1 = \Gamma_1 \times (0,T)$ by

$$\begin{cases} \Phi = 0 \quad \text{on} \quad \Sigma_1, \\ \phi = \dfrac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad \Sigma_1, \end{cases} \tag{2.42}$$

and we can act Σ_0 by

$$\begin{cases} B_1(\Phi,\phi) = u_1 \quad B_2(\Phi,\phi) = u_2 \quad \text{on} \quad \Sigma_0, \\ \Delta\phi + (1-\mu)B_3\phi = v_1 \quad \text{on} \quad \Sigma_0, \\ \dfrac{\partial\Delta\phi}{\partial n} + (1-\mu)B_4\phi - \dfrac{\partial\phi_{tt}}{\partial n} = v_2 \quad \text{on} \quad \Sigma_0. \end{cases} \tag{2.43}$$

The dual problem for the above is the following in $\eta = (W, w)$

$$\begin{cases} \eta_{tt} - \gamma(0, \Delta w_{tt}) + \mathcal{A}\eta = 0 \quad \text{in} \quad Q, \\ \eta(0) = \eta^0, \quad \eta_t(0) = \eta^1 \quad \text{on} \quad \Omega, \end{cases} \tag{2.44}$$

subject to the boundary condition

$$\begin{cases} W = 0 \quad \text{on} \quad \Sigma_1, \\ w = \dfrac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Sigma_1, \end{cases} \tag{2.45}$$

$$\begin{cases} B_1(W, w) = B_2(W, w) = 0 \quad \text{on} \quad \Sigma_0, \\ \Delta w + (1-\mu)B_3 w = 0 \quad \text{on} \quad \Sigma_0, \\ \dfrac{\partial \Delta w}{\partial n} + (1-\mu)B_4 w - \dfrac{\partial w_{tt}}{\partial n} = 0 \quad \text{on} \quad \Sigma_0. \end{cases} \tag{2.46}$$

Continuous observability inequality in Neumann case. Let $\eta = (W, w)$ solve problem (2.44)–(2.46). It is easy to check from boundary conditions (2.45) and (2.46) that

$$\begin{aligned} SB|_\Sigma &= \frac{1}{2}\int_{\Sigma_0} [|\eta_t|^2 + \gamma|Dw_t|^2 - B(\eta, \eta)]\langle V, n\rangle \, d\Sigma \\ &+ \frac{1}{2}\int_{\Sigma_1} B(\eta, \eta)\langle V, n\rangle \, d\Sigma. \end{aligned} \tag{2.47}$$

It follows from (2.47) that to obtain the observability inequality is to seek $T_0 > 0$ such that for any $T > T_0$, there is $c > 0$ satisfying

$$\int_{\Sigma_0} [|\eta_t|^2 + \gamma|Dw_t|^2] \, d\Sigma \geq cE(0), \tag{2.48}$$

for all initial data $(\eta^0, \eta^1) \in \big(H^1_{\Gamma_1}(\Omega, \Lambda) \times H^2_{\Gamma_1}(\Omega)\big) \times \big(L^2(\Omega, \Lambda) \times L^2(\Omega)\big)$ for which the left hand side of (2.48) is finite. We have the following

Theorem 2.3 *(Neumann case) Let assumptions* $(H.1)$ *and* $(H.2)$ *hold. Then for any* $T > T_0$*, there is* $c > 0$ *such that inequality* (2.48) *holds, where* T_0 *and* Γ_0 *are defined by* (2.39) *and* (2.40)*, respectively.*

Remark 2.5. *Exact controllability results (in suitable function spaces) for* $T > T_0$ *follow from* (2.37) *and* (2.48) *and duality.*

Remark 2.6 *Let the shell be flat, that is,* $M = \mathbb{R}^2$. *Then system* (2.33) *becomes two systems in which one is the wave equation and the other the plate equation. We may take* $\lambda_0 = 1$. *If we set* $V = x - x_0$, x_0 *a fixed point in* $\mathbb{R}^2$*, then inequalities* (2.37) *and* (2.48) *on component* w *are exactly the same as in Lagnese and Lions* [11]. *In this case* $\sigma_1 = 1$. *It follows that* $T_0 = 2diameter(\Omega)$ *which is the best for wave component* W*, Komornik* [9]. *In this sense,* T_0 *in* (2.39) *is the best.*

2.3 Two examples We here give two examples that verify assumption (H2).

Example 2.1. Let middle surface Ω be of constant curvature. Suppose that the curvature of manifold (M, g) is constant k. Given $x_0 \in M$. Let ρ be the distance function from $x \in M$ to x_0 on (M, g), i.e., $\rho(x) = dis(x_0, x)$. Set $V = h(\rho)D\rho$, where $h(\rho)$ is defined by

$$h(\rho) = \begin{cases} \sin(\sqrt{k}\rho(x)), & k > 0, \\ \rho(x), & k = 0, \\ \sinh(\sqrt{-k}\rho(x)), & k < 0. \end{cases} \tag{2.49}$$

We can prove

$$DV = b(x)g \quad \text{and} \quad a(x) = 0 \tag{2.50}$$

where

$$b(x) = \begin{cases} \sqrt{k}\cos(\sqrt{k}\rho(x)), & k > 0, \\ 1, \quad k = 0, \\ \sqrt{-k}\cosh(\sqrt{-k}\rho(x)), & k < 0. \end{cases} \tag{2.51}$$

It follows that assumption (H2) holds with vector field V if and only if $\min_{x\in\Omega} b(x) > 0$. By expression (2.51), we have the following conclusions: (a) if $k > 0$, assumption (H2) holds when $\overline{\Omega}$ is contained a geodesic ball with radius $\pi/(2\sqrt{k})$; (b) if $k \leq 0$, assumption (H2) holds for any $\Omega \subset M$.

Example 2.2. Consider a helicoid, defined by

$$M = \{\, \alpha(t, s) \,|\, (t, s) \in \mathbb{R}^2, \, t > 0 \,\}, \tag{2.52}$$

where

$$\alpha(t, s) = (t\cos s, t\sin s, c_0 s), \quad c_0 > 0. \tag{2.53}$$

The Gauss curvature is $-c_0^2/(t^2 + c_0^2)^2$. We set

$$E_1 = \alpha_t = (\cos s, \sin s, 0); \tag{2.54}$$

$$E_2 = \frac{1}{\sqrt{t^2 + c_0^2}}\alpha_s = \frac{1}{\sqrt{t^2 + c_0^2}}(-t\sin s, t\cos s, c_0). \tag{2.55}$$

Then E_1, E_2 makes up a frame field on the whole surface M. We may obtain

$$D_{E_1}E_1 = 0; \quad D_{E_2}E_1 = \frac{t}{t^2 + c_0^2}E_2; \tag{2.56}$$

$$D_{E_1}E_2 = 0; \quad D_{E_2}E_2 = \frac{-t}{t^2 + c_0^2}E_1, \tag{2.57}$$

where D is the Levi-Civita connection of surface M.

For any $c > 0$, we set

$$V_c = f_c E_1 + hE_2, \tag{2.58}$$

where

$$f_c = \sqrt{t^2 + c_0^2}(\int_0^t \frac{dt}{\sqrt{t^2 + c_0^2}} + c); \quad h = \sqrt{t^2 + c_0^2}s. \tag{2.59}$$

We then have

$$DV_c = b_c g + a\mathcal{E}, \quad \text{for} \quad c > 0, \tag{2.60}$$

where g is the induced metric of M in $I\!R^3$, $\mathcal{E}$ is the volume element of M, and

$$b_c = 1 + \frac{t}{t^2 + c_0^2} f_c; \quad a = -\frac{3st}{\sqrt{t^2 + c_0^2}}. \tag{2.61}$$

It is clear that, for any $\Omega \subset M$ bounded with $\overline{\Omega} \subset M$ and for any constant $c_1 > 0$, there is $c > 0$ large enough such that

$$\min_{x\in\Omega} b_c \geq c_1 \max_{x\in\Omega} |a|, \tag{2.62}$$

that is, for any $\Omega \subset M$, we can find a vector field V_c ($c > 0$ large enough), defined by (2.58), to meet geometric conditions (2.21) and (2.23).

Remark 2.7. *It is easy to check by the curvature information and Yao* [19] *that all Examples 2.1-2.3 satisfy assumption* $(H1)$, *too.*

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