# On the Strong Stabilizability of MIMO $n$-dimensional Linear Systems

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**Abstract**

A plant is *strongly stabilizable* if there exists a *stable* compensator to stabilize it. This paper presents necessary conditions for the strong stabilizability of complex and real $n$-D multi-input multi-output (MIMO) shift-invariant linear plants. For the real case, the condition is a generalization of the *parity interlacing property* of Youla *et al.* for the strong stabilizability of a real 1-D MIMO plant. These conditions are also sufficient for the cases of $n$-D plants with a single output (MISO) or with a single input (SIMO). For general $n$-D MIMO plants, we do not know if the conditions are sufficient or not. A useful sufficient, but not necessary, condition for the strong stabilizability of a class of $n$-D ($n \geq 2$) MIMO plants is given.

**Key words:** multidimensional system, feedback stabilization, strong stabilizability, sign condition

## 1 Introduction

Let $\bar{U}^n = \{z \in \boldsymbol{C}^n |\ |z_1| \leq 1, \cdots, |z_n| \leq 1\}$ be the closed unit polydisc in $\boldsymbol{C}^n$. In this paper a polynomial in $z$ is said to be *Hurwitz* if it is free from 0 in $\bar{U}^n$. A rational function with a Hurwitz denominator is regular (analytic) over $\bar{U}^n$, and will be said to be *stable*.

An $n$-dimensional ($n$-D) multi-input multi-output (MIMO) linear shift invariant plant with $l$ inputs and $m$ outputs can be described by a transfer matrix with entries of rational functions in $z = (z_1, \cdots, z_n)$:

$$P(z) = \begin{bmatrix} p_{11}(z) & \cdots & p_{1l}(z) \\ \vdots & \cdots & \vdots \\ p_{m1}(z) & \cdots & p_{ml}(z) \end{bmatrix}. \qquad (1)$$

The system is called **real** if the entries are real rational functions and **complex** if they are complex rational functions.

$P(z)$ is **stable** by definition if all its entries are stable. In some feedback configuration with compensator $C(z)$ [8, 5], the stability of the feedback system is equivalent to the stability of the following system

$$H(P, C) = \begin{bmatrix} I - P(I + CP)^{-1}C & -P(I + CP)^{-1} \\ (I + CP)^{-1}C & (I + CP)^{-1} \end{bmatrix} \qquad (2)$$

$P$ is said to be **stabilizable** if such a $C$ (complex or real) exists to make $H(P, C)$ stable. Stabilizability conditions and stabilizing compensator construction method can be found in [5]. For real stabilizable systems, the compensators can always be constructed real.

In the case that the compensator $C(z)$ itself can be chosen stable, $P$ is said to be **strongly stabilizable**. Furthermore, for a real system $P$, if $C$ can be chosen real and stable, we say that $P$ is **real strongly stabilizable**.

For an MIMO 1-D linear system described by a real rational transfer matrix, Youla *et al.* [12] in the seventies gave a constructive condition for the existence of a stable *real* compensator. The extension of Youla *et al.*'s result to $n$-D systems has been a long standing open problem [8, Section 8.3]. For the problem of strong stabilization of $n$-D linear systems, recently a topological condition for strong stabilizability of an $n$-D SISO complex system was given by Shankar [6], and a computable equivalent was given by Ying [9]. By introducing a concept of "sign" of real functions on complex varieties, Ying gave a necessary and sufficient condition for real strong stabilizability of a real $n$-D SISO system [9].

In this paper we present some results concerning strong stabilizability of MIMO $n$-D systems. Section 2 presents some mathematical facts used in later sections. In Section 3, necessary conditions for strong stabilizability of $n$-D MIMO systems are presented. Some examples are given at the end. In Section 4, a useful sufficient, but not necessary, condition for the strong stabilizability of a complex MIMO system is given.

The proofs of the theorems will be omitted, which can be found in [10].

## 2 Some Mathematical Facts

In this section we solve the following problems:

Let $g(z)$, $\alpha_1(z), \cdots, \alpha_M(z)$ be complex polynomials.

i. What is the condition for the existence of $M$ stable complex rational functions $h_1(z), \cdots, h_M(z)$ such that
$$g(z)+h_1(z)\alpha_1(z)+\cdots+h_M(z)\alpha_M(z) \neq 0 \text{ on } \bar{U}^n? \quad (3)$$

ii. When $g(z)$, $\alpha_1(z), \cdots, \alpha_M(z)$ are real polynomials and the above inequality (4) has a solution, what is the condition for $h_1, \cdots, h_M$ to be real?

### 2.1 Complex Stable Rational Functions

Recall that the winding number of a cycle (a closed curve) $\gamma$ in $\boldsymbol{C}^* = \boldsymbol{C}\backslash\{0\}$ is defined as
$$W(\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{d\xi}{\xi} = \frac{1}{2\pi i}\int_\gamma d(\log \xi).$$
It is the number of times that $\gamma$ winds around the origin counterclockwise. A single-valued logarithmic function $\log \xi$ can be defined in some subset $\Sigma$ of $\boldsymbol{C}^*$ if and only if $\Sigma$ does not contain any cycle with a non-zero winding number.

In general, for a subset $\Sigma$ in some complex space, an analytic function $g : \Sigma \longrightarrow \boldsymbol{C}^*$ has a single-valued logarithmic function $\log g$ on $\Sigma$ if and only if $g$ maps any cycle in $\Sigma$ into a cycle with winding number 0 in $\boldsymbol{C}^*$, i.e., $W(g(\gamma)) = 0$. This property is equivalent to that $g$ induces a 0 homomorphism from the first homology group of $\Sigma$ to that of $\boldsymbol{C}^*$ (see, e.g., [2]). If this is satisfied we say that $g$ is **0-homotopic** on $\Sigma$.
Let $I$ be the ideal generated by $\alpha_1, \cdots, \alpha_M$ in $\boldsymbol{C}[z]$, the ring of complex polynomials, and let
$$V(I) \cap \bar{U}^n = \left\{z \in \bar{U}^n | \ f(z) = 0, \ \forall f(z) \in I\right\}.$$

**Theorem 1** *Let $g(z), \alpha_1(z), \cdots, \alpha_M(z)$ be complex polynomials. A necessary and sufficient condition for the existence of analytic functions $h_1, \cdots, h_M$ on $\bar{U}^n$, such that*
$$g(z) + \sum_{1\le j\le M} h_j\alpha_j \neq 0 \text{ on } \bar{U}^n, \quad (4)$$
*is that $g(z)$ is 0-homotopic on $V(I) \cap \bar{U}^n$; or equivalently, $g(z)$ has a single-valued logarithmic function $\log g(z)$ on $V(I) \cap \bar{U}^n$.*

### 2.2 Real Stable Rational Functions

Suppose that $g(z), \alpha_1(z), \cdots, \alpha_M(z)$ are real and that there exist real stable rational functions $h_1(z), \cdots, h_M(z)$ such that
$$g(z) + h_1(z)\alpha_1(z) + \cdots + h_M(z)\alpha_M(z) \neq 0 \text{ on } \bar{U}^n.$$
This is a real valued continuous mapping on $\bar{U}^n \cap \boldsymbol{R}^n$ and
$$g(z) + h_1(z)\alpha_1(z) + \cdots + h_M(z)\alpha_M(z) \neq 0 \text{ on } \bar{U}^n \cap \boldsymbol{R}^n.$$
It must have an invariant sign on the connected set $\bar{U}^n \cap \boldsymbol{R}^n$. Therefore $g$ has an invariant sign, either + or -, over $V(I) \cap \bar{U}^n \cap \boldsymbol{R}^n$.

The concept of the sign of $g$ can be extended on a component of $V(I) \cap \bar{U}^n$ in the complex space. See [10] for details. For the strong stabilizability problem, it suffices here to give the following definition.

**Definition 1** *$g(z)$ is said to have a positive (negative, respectively) sign on $V(I)\cap\bar{U}^n$ if there exists an analytic function $G$ on $V(I) \cap \bar{U}^n$ such that*
$$e^{G(z)} = g(z) \ \left(e^{G(z)} = -g(z), \ \textit{respectively}\right) \quad \textit{and}$$
$$\overline{G(z)} = G(\bar{z}) \quad \forall z \in V(I) \cap \bar{U}^n.$$

**Theorem 2** *Let $g(z), \alpha_1(z), \cdots, \alpha_M(z)$ be real polynomials. Let $I$ denote the ideal generated by $\alpha_1(z), \cdots, \alpha_M(z)$ in $\boldsymbol{C}[z]$. A necessary and sufficient condition for the existence of real stable rational functions $h_1, \cdots, h_M$, such that*
$$g(z) + \sum_{1\le j\le M} h_j\alpha_j \neq 0 \text{ on } \bar{U}^n \quad (5)$$
*is that $g(z)$ is 0-homotopic on $V(I) \cap \bar{U}^n$ and that $g(z)$ has an invariant sign on all components of $V(I) \cap \bar{U}^n$.*

## 3 Necessary Conditions for Strong Stabilizability of MIMO Systems

In this section we will adopt the *Matrix Fraction Description* (MFD) approach for describing a system. A *left* MFD of $P$ is defined as
$$P(z) = D^{-1}(z)N(z),$$
where $D$ is an $m\times m$ and $N$ an $m\times l$ polynomial matrix:
$$D = \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \cdots & \vdots \\ d_{m1} & \cdots & d_{mm} \end{bmatrix}, \ N = \begin{bmatrix} n_{11} & \cdots & n_{1l} \\ \vdots & \cdots & \vdots \\ n_{m1} & \cdots & n_{ml} \end{bmatrix}. \quad (6)$$
where $d_{jk}$ and $n_{jk}$ are polynomials in $z$ with real or complex coefficients, corresponding to a real or a complex system, respectively. Let
$$F = [DN] = \begin{bmatrix} d_{11} & \cdots & d_{1m} & n_{11} & \cdots & n_{1l} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ d_{m1} & \cdots & d_{mm} & n_{m1} & \cdots & n_{ml} \end{bmatrix}. \quad (7)$$

Let $M = \begin{pmatrix} m+l \\ l \end{pmatrix}$. Let $\alpha_1, \alpha_2, \cdots, \alpha_M$ denote the $M$ maximal order minors of $F$, with $\alpha_1 = \det D$.

An MFD is said to be **minor coprime** if the $\alpha_i$'s have no non-unit common factor. Similarly, a minor coprime *right* MFD is defined as a matrix fraction $YX^{-1}$ such that the maximal order minors of $\begin{bmatrix} X \\ Y \end{bmatrix}$ have no common non-unit factor. A minor coprime MFD (either left or right) can always be constructed for a 1-D and a 2-D transfer matrix, but not generally for an $n$-D ($n \geq 3$) matrix [4].

### 3.1 Necessary Conditions for Strong Stabilizability MIMO Systems

Let $P$ be an $n$-D transfer matrix with a (not necessarily minor coprime) left MFD $D^{-1}N$. Let $F = [D\ N]$.

Let $\alpha_1, \alpha_2, \cdots, \alpha_M$ denote the $M = \begin{pmatrix} m+l \\ l \end{pmatrix}$ maximal order minors of $F$, with $\alpha_1 = \det D$. Let $d$ be the greatest common divisor of the $\alpha_i'$s. The polynomials

$$b_1 = \frac{\alpha_1}{d}, \cdots, b_M = \frac{\alpha_M}{d} \tag{8}$$

are called **generating polynomials** of $P$ and are independent of the choice of $F$ up to a non-zero constant [4].

**Proposition 1** *[4] $P$ is stable if and only if the first generating polynomial $b_1$ is free from 0 in $\bar{U}^n$.*

**Proposition 2** *[5] $P$ is stabilizable if and only if the generating polynomials do not share common zero in $\bar{U}^n$.*

In the following we assume the stabilizability of $P(z)$.

Let $e_1, \cdots, e_M$ be the generating polynomials of the compensator $C$. It is shown in [5] that the first generating polynomial of the resultant feedback system is

$$b_{H1} = r \sum_{j=1}^{M} b_j e_j, \tag{9}$$

where $r$ is a non-zero constant.

If $P = D^{-1}N$ is strongly stabilizable, then the compensator $C$ can be chosen stable. This implies that both $e_1$ and $b_{H1}$ in the above equation are free from 0 in $\bar{U}^n$. Dividing the equation by $e_1$ and $r$, we have

$$\frac{b_{H1}}{re_1} = b_1 + \sum_{j=2}^{M} b_j \frac{e_j}{e_1}. \tag{10}$$

This leads to the following:

**Theorem 3** *(i) Let $b_j$, $j = 1, \cdots, M$, $M = \begin{pmatrix} m+l \\ m \end{pmatrix}$, be the generating polynomials of $P = D^{-1}N$. A necessary condition for $P$ to be strongly stabilizable is that there exist stable rational functions $h_2, \cdots, h_M$ such that*

$$b_1 + \sum_{2 \leq j \leq M} h_j b_j \neq 0 \ on\ \bar{U}^n.$$

*(ii) If $P = D^{-1}N$ is strongly stabilizable and is real, then a necessary condition for the existence of a real stable compensator is that the above $h_j$'s can be chosen real.*

In view of Theorems 1 and 2 of Section 2, we have

**Theorem 4** *Let $b_1, b_2, \cdots, b_M$ be the generating polynomials of $P = D^{-1}N$, and $I$ be the ideal generated by $b_2, \cdots, b_M$ in $\boldsymbol{C}[z]$.*

*(i) A necessary condition for $P$ to be strongly stabilizable is that $b_1$ is 0-homotopic on $V(I) \cap \bar{U}^n$.*

*(ii) If $P$ is strongly stabilizable and is real, then a necessary condition for the existence of a real stable compensator is that $b_1$ has an invariant sign on $V(I) \cap \bar{U}^n$.*

If $P = D^{-1}N$ is a *minor coprime* MFD, then the generating polynomials are exactly the maximal order minors of $[D\ N]$. The following lemmas say that, if $P = D^{-1}N$ is minor coprime, then, instead of $V(I)$, it is equivalent to investigate the behavior of $\det D(z)$ on another variety $V(J)$, where $J$ is the ideal generated by $n_{jk}(z)$, $1 \leq j \leq m, 1 \leq k \leq l$, the entries of $N(z)$.

**Lemma 1** *Assume $D^{-1}N$ is a minor coprime MFD for a stabilizable plant. Let $\alpha_1 = \det D, \alpha_2, \cdots, \alpha_M$ be the maximal order minors of $[D\ N]$, let $I$ be the ideal generated by $\alpha_2, \cdots, \alpha_M$, and $J$ be the ideal generated by $n_{ij}$, $1 \leq j \leq m, 1 \leq k \leq l$, the entries of $N(z)$. Then*

$$V(I) \backslash V(\det D) = V(J) \backslash V(\det D).$$

**Lemma 2** *With the same notations and assumptions as in Lemma 1, we have*

$$V(I) \cap \bar{U}^n = V(J) \cap \bar{U}^n.$$

The following theorem is obvious from Lemma 2 and Theorem 4.

**Theorem 5** *Suppose that $P$ has a minor coprime MFD $D^{-1}N$. Let $J$ be the ideal generated by the entries of $N$ in $\boldsymbol{C}[z]$.*

*(i) A necessary condition for $P$ to be strongly stabilizable is that* $\det D(z)$ *is 0-homotopic on* $V(J) \cap \bar{U}^n$.

*(ii) If $P$ is strongly stabilizable and is real, then a necessary condition for the existence of a real stable compensator is that* $\det D$ *has an invariant sign on* $V(J) \cap \bar{U}^n$.

For the real case, the condition is a generalization of the *parity interlacing property* of Youla *et al.*, which is also sufficient, for the strong stabilizability of a real 1-D MIMO plant [12, 8].

### 3.2 Examples

[Example 1] (real strongly stabilizable)

$$F = [D\ N] = \begin{bmatrix} z_2^2 & 1 & z_1 & z_2 \\ -1 & z_3^2 & z_3 & 0 \end{bmatrix}.$$

The minors are $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (1 + z_2^2 z_3^2, z_1 + z_2^2 z_3, z_2, z_3 - z_1 z_3^2, -z_2 z_3^2, -z_2 z_3)$. Obviously $D^{-1}N$ is minor coprime. In accordance with our previous notations, $I = (\alpha_1, \cdots, \alpha_6),\ J = (z_1, z_2, z_3)$. We have

$$V(I) \cap \bar{U}^3 = V(J) \cap \bar{U}^3 = \{(0, 0, 0)\}.$$

The necessary conditions of Theorem 4 and Theorem 5 are satisfied. For this plant we have the following stable real compensator $C = YX^{-1}$ with

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ z_2 & 0 \\ -z_1 & z_3 \end{bmatrix}.$$

[Example 2] (strongly stabilizable, but not real strongly stabilizable)

$$F = \left[z_1, z_1^2 + z_2 z_3 - z_2 - 2, z_3\right].$$

$V(J) \cap \bar{U}^3 = V(z_1^2 + z_2 z_3 - z_2 - 2, z_3) \cap \bar{U}^3 = \{(-1, -1, 0), (1, -1, 0)\}$ is a discrete point set, and the condition (i) for strong stabilizability of Theorem 5 is trivial. By trial, we found a *complex* stable compensator $C = YX^{-1}$,

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0.5 z_1 + i \\ -1 \\ z_2 \end{bmatrix}.$$

In fact, it is not difficult to check that the polynomial $(0.5 z_1 + i) z_1 - (z_1^2 + z_2 z_3 - z_2 - 2) + z_2 z_3 = -0.5 z_1^2 + i z_1 + z_2 + 2$ is free from 0 in $\bar{U}^2$ hence in $\bar{U}^3$. However, $z_1$ has opposite signs at the two discrete points of $\bar{U}^3 \cap V(J)$, thus violates condition (ii) in Theorem 5. There is no real stable compensator. Indeed, though we can find a real compensator $\begin{bmatrix} z_1 \\ -1 \\ z_2 \end{bmatrix}$, which is, however, not stable.

[Example 3] (not minor coprime MFD)

$$F = \begin{bmatrix} d(z_1, z_2, z_3) & 0 & z_1 z_2 \\ 0 & d(z_1, z_2, z_3) & 1 \end{bmatrix},$$

$$d = z_1^2 + z_2^2 + z_3^2 - 1.$$

The minors are

$$\alpha_1 = d^2,\ \alpha_2 = d,\ \alpha_3 = -d z_1 z_2.$$

$D^{-1}N$ is not minor coprime. The generating polynomials are

$$(b_1, b_2, b_3) = (d,\ 1,\ -z_1 z_2).$$

We have

$$1 \cdot b_1 + 5 \cdot b_2 = z_1^2 + z_2^2 + z_3^2 + 4,$$

a Hurwitz polynomial. Using the method of [3, 5], one can find a compensator $C = YX^{-1}$,

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} d + 5 & -z_1 z_2 \\ 0 & d \\ 0 & 5d \end{bmatrix}.$$

$YX^{-1} = [0\ 5]$ is a right MFD of a stable system.

[Example 4] (stabilizable, but not strongly stabilizable)

$$P = \frac{1 - 4 z_1 z_2}{z_1} \begin{bmatrix} z_2 + 1 & z_2 \\ z_1 & z_1 \end{bmatrix},$$

$$F = \begin{bmatrix} z_1 & 0 & (z_2 + 1) f & z_2 f \\ 0 & z_1 & z_1 f & z_1 f \end{bmatrix},$$

where $f = 1 - 4 z_1 z_2$. The $2 \times 2$ minors are $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (z_1^2, z_1^2 f, z_1^2 f, -z_1(z_2 + 1) f, -z_1 z_2 f, z_1 f^2)$. The generating polynomials are

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (z_1, z_1 f, z_1 f, -(z_2 + 1) f, -z_2 f, f^2).$$

The plant is stabilizable because

$$V(b_1, b_6) \cap \bar{U}^2 = \phi.$$

But

$$V(b_2, \cdots, b_6) \cap \bar{U}^2 = V(1 - 4 z_1 z_2) \cap \bar{U}^2$$

has a cycle

$$\gamma : t \longrightarrow \left(\frac{1}{2} e^{i 2\pi t}, \frac{1}{2} e^{-i 2\pi t}\right), t \in [0, 1]$$

which is mapped by $z_1$ to a cycle $\{\frac{1}{2} e^{i 2\pi t},\ 0 \le t \le 1\}$ in $\boldsymbol{C}^*$, which has winding number 1 around the origin. Thus the system is not strongly stabilizable.

## 4 On Sufficient Conditions for Strong Stabilizability of MIMO Systems

For two special classes of $n$-D systems: the MISO and SIMO systems, the necessary conditions for strong stabilizability given in Theorems 4 and 5 are actually sufficient [10]. For general $n$-D MIMO plants, we do not know if the conditions are sufficient or not. In this section, we give a condition which is sufficient, but not necessary, for the strong stabilizability of a certain class of complex MIMO $n$-D plants.

**Theorem 6** *Let $D(z)$ and $N(z)$ be two complex polynomial $m \times m$ matrices. A sufficient condition for the existence of a complex polynomial $m \times m$ matrix $X(z)$ such that*

$$\det[D(z) + N(z)X(z)] \neq 0 \quad on \ \bar{U}^n \qquad (11)$$

*is that $D(z)$ has a single-valued logarithmic matrix function $\log D(z)$ on $V(\det N(z)) \cap \bar{U}^n$.*

**Corollary 1** *Let $D(z)^{-1}N(z)$ be a minor coprime MFD of a complex $m \times l$ plant $P(z)$, $l \geq m$. If there is some $m \times m$ submatrix $N_0(z)$ of $N(z)$, such that $D(z)$ has a single-valued logarithmic matrix function $\log D(z)$ on $V(\det N_0(z)) \cap \bar{U}^n$, then $P(z)$ is strongly stabilizable.*

In Example 1 of Section 3.2,

$$[D \ N] = \begin{bmatrix} z_2^2 & 1 & z_1 & z_2 \\ -1 & z_3^2 & z_3 & 0 \end{bmatrix}.$$

It can be verified (e.g., by using the theory of operational calculus [1], see Appendix A of this paper) that $D(z) = \begin{bmatrix} z_2^2 & 1 \\ -1 & z_3^2 \end{bmatrix}$ has a single-valued analytic logarithmic function over $V(\det N) \cap \bar{U}^3 = V(-z_2 z_3) \cap \bar{U}^3$.

If $D(z)$ =diag$(d_1(z), \cdots, d_m(z))$, a diagonal matrix, then its logarithmic function can be defined as

$$\log D(z) = \mathrm{diag}(\log d_1(z), \cdots, \log d_m(z)).$$

This yields the following simpler criterion for strong stabilizability test for a class of plants that have a special form of minor coprime MFD.

**Corollary 2** *Let $D(z)^{-1}N(z)$ be a minor coprime MFD of a complex $m \times l$ plant $P(z)$, $l \geq m$ with $D(z) =$ diag$(d_1(z) \cdots d_m(z))$, a diagonal matrix. If there is some $m \times m$ submatrix $N_0(z)$ of $N(z)$, such that, for each $i = 1, \cdots, m$, $d_i(z)$ has a single-valued logarithmic function $\log d_i(z)$ on $V(\det N_0(z)) \cap \bar{U}^n$; or equivalently, each $d_i(z)$ is 0-homotopic on $V(\det N_0(z)) \cap \bar{U}^n$, then $P(z)$ is strongly stabilizable.*

[Example 5]

$$[D \ N] = \begin{bmatrix} 1 - 4z_1 z_2 & 0 & z_2 + 1 & z_2 \\ 0 & 1 & z_1 & z_1 \end{bmatrix}.$$

$\det N = z_1$. The condition of Corollary 2 is satisfied. We actually have a stable compensator $C = YX^{-1}$ with

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 4z_1 & 0 \end{bmatrix}.$$

However, both conditions of the above corollaries are not necessary, as demonstrated by the following example.

[Example 6]

$$F = [D \ N] = \begin{bmatrix} z_1 & 0 & 1 & 0 \\ 0 & z_2 & 0 & 1 - 4z_1 z_2 \end{bmatrix}.$$

We have $\det D(z) + \frac{1}{4} \det N(z) = z_1 z_2 + \frac{1}{4}(1 - 4z_1 z_2) = \frac{1}{4}$. But it is not possible to define a single-valued $\log D(z)$ on $V(\det N(z)) \cap \bar{U}^2$ (see Appendix A). Nevertheless, for this plant we have the following stable real compensator $C = YX^{-1}$ with

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3z_1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Actually, $\det\left([D \ N]\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \det[DX + NY] = \det\begin{bmatrix} 4z_1 & 1 \\ -1 + 4z_1 z_2 & z_2 \end{bmatrix} = 1$.

Note that while $\det\begin{bmatrix} 4z_1 & 1 \\ -1 + 4z_1 z_2 & z_2 \end{bmatrix} = 1$, and has a logarithmic function 0 on $\bar{U}^2$, $\begin{bmatrix} 4z_1 & 1 \\ -1 + 4z_1 z_2 & z_2 \end{bmatrix}$ does not have a logarithmic function on $\bar{U}^2$, even though $\bar{U}^2$ is simply connected. In fact, it can be shown that $\begin{bmatrix} 4z_1 & 1 \\ 0 & z_2 \end{bmatrix}$ does not have a logarithmic function on $V(-1 + 4z_1 z_2) \cap \bar{U}^2$ (see Appendix A). Theorem 6 actually requires the resulting matrix have a logarithmic function. This is stronger than what we need for strong stabilizability.

## 5 Conclusion

We conclude this paper with some open problems.

1. For the case that the ideal $J$ is generated by one polynomial, computational procedures for testing the criterion in Theorem 1 and the sign invariance condition in Theorem 2 have been given based on the cylindrical algebraic decomposition [9, 11]. The extension

to the general case ($J$ be generated by a finite number of polynomials) is straightforward. Unfortunately, our method is not constructive in that it does not give a solution to the inequality (3) in Section 2. The development of an algorithmic method to solve inequality (3) remains open.

2. Based on the fact that all stable complex (or real) rational functions in one variable form a Euclidean domain, the necessary conditions in Theorem 5 have been constructively proven to be sufficient too for 1-D systems [12, 8]. The construction of a stable compensator for a 1-D system relies essentially on the Euclidean division algorithm, and on the Smith-McMillan canonical form description of the transfer matrix. But the domains of stable rational functions in two or more variables are no longer Euclidean, thus neither the Euclidean algorithm nor the Smith-McMillan form method can be applied. It is an interesting topic for future research to establish a new algebraic framework (e.g., a framework analogous to the Smith-McMillan form for the 1-D systems) in which adequate algorithms can be applied to solve the problems remaining unsolved in this paper.

## Appendix A. Operational Calculus on Matrix

For some $z$, consider $D(z)$ as a linear operator on a finite dimensional linear space. Let $\sigma(D(z)) \subset \boldsymbol{C}^*$ denote the set of its eigenvalues. As an inverse of the exponential map, an analytic logarithmic function of $D(z)$ can be defined by the following integral

$$\log D(z) = \frac{1}{2\pi i}\int_B \log(\lambda)(\lambda I - D(z))^{-1} d\lambda, \qquad (12)$$

where $B$ is the boundary of a domain which contains the closure of some open set containing $\sigma(D(z))$, and consists of a finite number of closed rectifiable Jordan curves. Clearly, $\log D(z)$ is single-valued if the domain with boundary $B$ does not contain the origin in $\boldsymbol{C}^*$.

In Example 1, $D(z) = \begin{bmatrix} z_2^2 & 1 \\ -1 & z_3^2 \end{bmatrix}$ and $N(z) = \begin{bmatrix} z_1 & z_2 \\ z_3 & 0 \end{bmatrix}$. The eigenvalues of $D(z)$ are

$$\lambda = \frac{1}{2}\left(z_2^2 + z_3^3 \pm \sqrt{(z_2^2 - z_3^3)^2 - 4}\right). \qquad (13)$$

It is easy to see that the union of the sets of eigenvalues $\cup_{z \in V(z_2 z_3) \cap \bar{U}^3} \sigma(D(z))$ does not intersect with the real axis, and can be contained in a simply connected open domain not containing the origin in $\boldsymbol{C}^*$. With $B$ being the boundary of this domain, $\log D(z)$ defined in (12) is a single-valued analytic logarithmic function in $z$ because $\lambda$ in (13) varies analytically in $z$.

On the other hand, if $\log D(z)$ is defined as a single-valued matrix function, then by the *spectral mapping theorem* [1, p.569], the eigenvalues of $D(z)$ have single-valued logarithms which are the eigenvalues of $\log D(z)$.

In Example 6, the eigenvalues of $D(z)$ are $z_1$ and $z_2$, which are not 0-homotopic on $V(\det N(z)) \cap \bar{U}^2 = V(1 - 4z_1 z_2) \cap \bar{U}^2$. Therefore $D(z)$ does not have a single-valued logarithm on $V(\det N(z)) \cap \bar{U}^2$. For the same reason, $\begin{bmatrix} 4z_1 & 1 \\ -1 + 4z_1 z_2 & z_2 \end{bmatrix}$ does not have a logarithm on $V(1 - 4z_1 z_2) \cap \bar{U}^2$.

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