ON ALGORITHMS FOR ATTITUDE ESTIMATION USING GPS

Assaf Nadler[#], Itzhack Y. Bar-Itzhack[*] and Haim Weiss[♦]
Faculty of Aerospace Engineering, Technion,
Haifa 32000, Israel
Tel. +972-4-829-3196
Fax. +972-4-823-1848
E-mail: Ibaritz@tx.Technion.ac.il

Abstract

This paper discusses algorithms for attitude determination using GPS differential phase measurements, assuming that the cycle integer ambiguities are known. The problem of attitude determination is posed as a parameter optimization problem where a new quaternion-based cost function is used. Unlike the cost function associated with the vectorized measurements, the new cost function is not a simple quadratic form and therefore Davenport's q-Method is not applicable in this case. Three algorithms for finding the optimal quaternion are derived, two of which are discrete. The third one is a continuous version of the Newton-Raphson algorithm. This continuous version is new and has a guaranteed exponential convergence to the closest local minimum located on the gradient direction in regions where the associated Hessian matrix is positive definite. The algorithms presented in this paper can handle cases of planar antenna arrays and thus cover a deficiency in earlier algorithms. The efficiency of the new algorithms is demonstrated through numerical examples.

I. INTRODUCTION

Attitude determination using GPS carrier signals has been given a considerable attention in the last decade[1,2]. Much attention was given to concept, hardware, and algorithm development as well as to testing. Algorithms for GPS attitude determination given differential phase measurements can be broken into integer resolution and attitude calculations. Several methods for integer resolution are presented in the literature (see e.g. Refs. 1 and 3). In this work we assume that the integer ambiguity is solved and we are concerned only with attitude calculation. The problem of attitude determination can be expressed as the problem of minimizing the following cost function with respect to the attitude matrix D_a^e.

$$\rho\left(D_a^e\right) = \sum_i^n p_i \sum_j^2 \left| B_{ji} - \mathbf{a}_j^T \, D_a^e \, \mathbf{s}_i \right|^2 \qquad (1)$$

The cost function is defined for n satellites and a planar array of 3 antennae where $\mathbf{s}_i$ is a unit vector in the direction of an observed GPS satellite, which is designated as satellite number i, *e* is the reference (earth) coordinate system in which $\mathbf{s}_i$ is resolved, *a* is the body coordinate system, $\mathbf{a}_j$ is the $j^{\underline{th}}$ axis of the latter system, D_a^e is the transformation matrix from *e* to *a*, and B_{ji} is the processed phase measurement, which is defined as the projection of $\mathbf{s}_i$, the unit vector to satellite i, on the body coordinate system axis j. The transpose is denoted by T and p_i is a normalized weight given to the measurement related to the $i^{\underline{th}}$ satellite, where $\sum_{i=1}^{n} p_i = 1$. It is assumed that the components of $\mathbf{s}_i$ in *e* are known. Note that we are considering a planar problem in which only two components of B are measured for each satellite. These components are the projections of processed phase measurements on the body coordinate axes

It was shown[4] in the past that the phase measurements could easily be converted into vector measurements, and then the least squares fit could be found using one of the available algorithms like QUEST. Another possible approach is based on a least squares fit of the attitude quaternion to the basic GPS phase measurements. Following this approach[4] the cost function can be expressed as a function of the quaternion of rotation. This was motivated by the success attained in quaternion fitting to vector measurements, which was achieved using the q-Method solution. However, unlike the case of vector measurements, where the cost function reduces to a quadratic form of a symmetric matrix, in the case of phase measurements, the cost function is a sum of quartic forms. Therefore the q-Method solution is not applicable in this case. A possible solution to the problem of finding the optimal quaternion, which minimizes this cost function, is an iterative one. Indeed, such a solution was presented in the literature[4]. That solution used the gradient projection technique to develop a steepest descent search for the local minimum of the cost function. It was found that this iterative process converged slowly. Therefore a faster converging algorithm was sought in the present work.

II. ATTITUDE DETERMINATION USING GPS VECTORIZED OBSERVATIONS

Several efficient algorithms for attitude determination based on a least squares fit of the attitude to *vector* measurements were introduced in the past. To make use of these algorithms, the

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[#] Graduate student.

[*] Sophie and William Shamban Professor of Aerospace Engineering.
Member, Technion Asher Space Research Institute. IEEE Fellow, AIAA Fellow.
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founded by The Israel Academy of Sciences and Humanities – The Jack Adler Foundation.

[♦] Adjunct Senior Teaching Fellow, Lady Davis Visiting Scientist. On Sabbatical leave from RAFAEL,
P.O.B. 2250, Dept. 35, Haifa 31021, Israel.

phase measurements have to be converted into vector measurements in the body coordinate system as follows.

$$\mathbf{s}_{ia} = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \left(1 - B_{1i}^2 - B_{2i}^2\right)^{1/2} \end{bmatrix} \tag{2}$$

Note that the third component of $\mathbf{s}_{ia}$ is chosen to be the positive root of the expression in parentheses. This was done since only the signals of those GPS satellites which are above the antenna plane, and thus in the positive direction of the $\mathbf{a}_3$ axis, are received by the antennae. The vector $\mathbf{s}_{ia}$, resolved in earth reference coordinates, is denoted by $\mathbf{s}_{ie}$, which is equivalent to $\mathbf{s}_i$. The latter is easily computed since both the satellite and the vehicle positions are known in earth coordinates. With the pairs $\mathbf{s}_{ia}$, $\mathbf{s}_{ie}$ on hand, $i = 1,2\ldots,n$, one can replace Eq. (1) by the following cost function introduced by Wahba[5]

$$\rho'(D_a^e) = \frac{1}{2}\sum_i^n p_i \left|\mathbf{s}_{ia} - D_a^e \mathbf{s}_{ie}\right|^2 \tag{3}$$

and use QUEST[6], similar algorithms that use the q-Method[7], or other algorithms, to obtain a weighted least squares attitude quaternion fit which minimizes ρ'. For the sake of comparison between QUEST, which operates on measured vectors and the algorithms that will be developed later, which operate on phase measurements, a short description of QUEST is given next.

Since D_a^e is a known function of the attitude quaternion[8] $\mathbf{q}$, then $\rho'\left(D_a^e\right)$ can be replaced by w($\mathbf{q}$) where

$$w(\mathbf{q}) = \frac{1}{2}\sum_i^n p_i \left|\mathbf{s}_{ia} - D_a^e(\mathbf{q})\mathbf{s}_{ie}\right|^2 \tag{4}$$

It can be shown that $\mathbf{q}$*, the $\mathbf{q}$ which minimizes w($\mathbf{q}$), is the same $\mathbf{q}$ which maximizes the cost function

$$\eta(\mathbf{q}) = \mathbf{q}^T K \mathbf{q} \tag{5}$$

where

$$K = \left[\begin{array}{c|c} S - \sigma I & z \\ \hline z^T & \sigma \end{array}\right] \tag{6}$$

and

$$\sigma = \sum_{i=1}^n p_i s_{ia}^T s_{ie} \quad (7.a) \qquad B = \sum_{i=1}^n p_i s_{ia} s_{ie}^T \quad (7.b)$$

$$S = B + B^T \quad (7.c) \qquad z = \sum_{i=1}^n p_i (s_{ia} \times s_{ie}) \quad (7.d)$$

The matrix I is the third order identity matrix. It turns out that $\mathbf{q}$* is the eigenvector which corresponds to the largest eigenvalue of K. QUEST[6] is an algorithm which yields this $\mathbf{q}$*.

III. PHASE-RELATED COST FUNCTION CONVERSION TO A QUARTIC FORM

Recall Eq. (1)

$$\rho(D_a^e) = \sum_i^n p_i \sum_j^2 \left|B_{ji} - \mathbf{a}_j^T D_a^e \mathbf{s}_i\right|^2 \tag{8}$$

where one wishes to find D_a^e which minimizes $\rho(D_a^e)$. Note that in contrast to Eq. (4) where the cost $w(\mathbf{q})$ was formulated as a function of vector measurements, here the cost function is based on the raw phase measurements. Since, as mentioned earlier, D_a^e is a known function of the attitude quaternion $\mathbf{q}$, then $\rho(D_a^e)$ can be replaced by J($\mathbf{q}$) where

$$J(\mathbf{q}) = \sum_i^n p_i \sum_j^2 \left|B_{ji} - \mathbf{a}_j^T D_a^e(\mathbf{q})\mathbf{s}_i\right|^2 \tag{9}$$

In order to facilitate the search for the quaternion $\mathbf{q}$* which minimizes J($\mathbf{q}$) the latter is now converted into a function of a quartic quaternion form. To meet this end define

$$C_{ji} = \mathbf{s}_i \mathbf{a}_j^T \quad (10.a) \qquad E_{ji} = C_{ji} + C_{ji}^T \quad (10.b)$$

$$\mathbf{p}_{ji} = \mathbf{a}_j \times \mathbf{s}_i \quad (10.c) \qquad \mu_{ji} = \mathbf{a}_j^T \mathbf{s}_i \quad (10.d)$$

and then define

$$L_{ji} = \left[\begin{array}{c|c} E_{ji} - \mu_{ji} I & \mathbf{p}_{ji} \\ \hline \mathbf{p}_{ji}^T & \mu_{ji} \end{array}\right] \tag{11}$$

It can be shown that[4]

$$\mathbf{a}_j^T D_a^e(\mathbf{q})\mathbf{s}_i = \mathbf{q}^T L_{ji} \mathbf{q} \tag{12}$$

Substitution of Eq. (12) into Eq. (9) yields

$$J(\mathbf{q}) = \sum_i^n p_i \sum_j^2 \left|B_{ji} - \mathbf{q}^T L_{ji} \mathbf{q}\right|^2 \tag{13}$$

Define

$$\Phi_{ji} = B_{ji} I \tag{14}$$

Then since $\mathbf{q}^T\mathbf{q} = 1$, one can write

$$B_{ji} = q^T \Phi_{ji} q \tag{15}$$

Therefore

$$B_{ji} - \mathbf{q}^T L_{ji} \mathbf{q} = \mathbf{q}^T [\Phi_{ji} - L_{ji}] \mathbf{q} \tag{16}$$

Let

$$M_{ji} = \sqrt{p_i} \cdot (\Phi_{ji} - L_{ji}) \tag{17}$$

where M_{ji} is 4x4 symmetric matrix.

Then using Eqs. (16) and (17) the following is obtained

$$J(\mathbf{q}) = \sum_i^n \sum_j^2 \left|\mathbf{q}^T M_{ji} \mathbf{q}\right|^2 \tag{18}$$

Eq. (18) can also be written as

$$J(\mathbf{q}) = \mathbf{q}^T \left[\sum_i^n \sum_j^2 M_{ji} \mathbf{q}\mathbf{q}^T M_{ji}\right] \mathbf{q} \tag{19}$$

Obviously, the problem of finding the matrix D_a^e which minimizes $\rho(D_a^e)$, defined in Eq. (1), has been transformed into finding $\mathbf{q}$ that minimizes J($\mathbf{q}$) of either Eq. (18) or Eq. (19). Unfortunately J($\mathbf{q}$), is quartic in $\mathbf{q}$ whereas the cost function which has to be optimized when solving Wahba's problem is only quadratic in $\mathbf{q}$. For this reason the q-Method[7] solution *is not suitable* in the present case. One needs to use some other methods for minimizing J($\mathbf{q}$). An iterative solution was suggested[4], which was based on the *gradient projection technique*[9]. That algorithm converged quite slowly therefore faster algorithms were sought.

IV. MINIMIZATION OF THE QUARTIC FORM COST FUNCTION

Define $C(q)$ as follows

$$C(\mathbf{q}) = \sum_i^n \sum_j^2 M_{ji} \mathbf{q} \ \mathbf{q}^T M_{ji} \tag{20}$$

Then, the quartic cost function of Eq. (19) can be written as

$$J(\mathbf{q}) = \mathbf{q}^T C(\mathbf{q}) \mathbf{q} \tag{21}$$

We wish to minimize J with respect to $\mathbf{q}$ where the latter has to satisfy the normality constraint

$$g(\mathbf{q}) \equiv \mathbf{q}^T\mathbf{q} - 1 = 0 \quad (22)$$

To accomplish this, define the following Lagrange function

$$L(\mathbf{q}, \lambda) = J(\mathbf{q}) - 2\lambda(\mathbf{q}^T\mathbf{q} - 1) \quad (23)$$

The **q,** which, minimizes J subject to the constraint on **q**, as well as the Lagrange multiplier, λ, are stationary points of L, therefore they satisfy the equations

$$\frac{\partial L}{\partial \mathbf{q}} = 0 \quad (24.a) \qquad \frac{\partial L}{\partial \lambda} = 0 \quad (24.b)$$

When performing the differentiation and after some elaborate manipulations one obtains the following corresponding equations[10]

$$C(\mathbf{q})\mathbf{q} = \lambda\mathbf{q} \quad (25)$$

$$\mathbf{q}^T\mathbf{q} = 1 \quad (26)$$

where $\mathbf{q} \in \Re^4$ and $\lambda \in \Re$. Eqs. (25) and (26) define a set of 5 nonlinear algebraic equations. The sought q^* satisfies these equations which determine a necessary condition for q^* to be a minimum. Observe that $\lambda = \mathbf{q}^T C(\mathbf{q})\mathbf{q} = J(\mathbf{q}) \geq 0$ and hence $\min_{\mathbf{q}} J(\mathbf{q}) = \lambda_{min}$.

V. EIGENPROBLEM SOLUTION OF THE NONLINEAR EQUATIONS

The nature of Eqs. (25) and (26) calls for an iterative solution. One such solution that immediately comes to mind is the following. Guess an initial $\mathbf{q}$ and use it in Eq. (20) to compute C. Then, find the eigenvalues and eigenvectors of C. It was shown in Ref. 10 that J is minimal when **q** is the unit eigenvector, which corresponds to the smallest eigenvalue of C. Therefore select this eigenvector of C and use it as **q** for the following iteration. This algorithm, however, is problematic. Experiments showed that its convergence was slow and near the end it alternated between two values, none of which is the correct solution[10]. It was observed, however, that the two values were almost symmetric about the correct solution. Therefore the algorithm was modified in the following way. The solutions obtained from two successive iterations were averaged. The average solution was then fed into the iterative algorithm, which was run twice again. The results of these two iterations were averaged again and so on.

A step-by-step description of the algorithm is as follows:

1. Guess $\mathbf{q}_k$, and set $k = 0$.
2. Compute $C_k(\mathbf{q}_k)$.
3. Find the eigenvalues and eigenvectors of $C_k(\mathbf{q}_k)$.
4. Set $\mathbf{q}_{k+1}$ to the eigenvector corresponding to the smallest eigenvalue of $C_k(\mathbf{q}_k)$.
5. Go once more to Step 2, repeat Steps 3 and 4.
6. Replace $\mathbf{q}_{k+1}$ by the average of the last two $\mathbf{q}_{k+1}$'s.
7. If $|q_{k+1} - q_k| \leq \delta$ where δ is a pre-determined constant, then stop, otherwise increase the argument by 1 and go back to step 2.

VI. NEWTON-RAPHSON SOLUTION OF THE NONLINEAR EQUATIONS

VI.1The Discrete Newton-Raphson Algorithm[11,12]

Define

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} C(\mathbf{q})\mathbf{q} - \lambda\mathbf{q} \\ \mathbf{q}^T\mathbf{q} - 1 \end{bmatrix} \quad (27)$$

where $\mathbf{x}^T = [\mathbf{q}^T, \lambda]$. To solve the equation $\mathbf{f}(\mathbf{x}) = 0$ iteratively, we first compute the Jacobian J and then iterate the equation

$$\mathbf{x}_{k+1} = \mathbf{x}_k - J^{-1}(\mathbf{x}_k) \cdot \mathbf{f}(\mathbf{x}_k) \quad (28)$$

where

$$J = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \quad (29)$$

For **f** defined in Eq. (27) J takes the form[10]

$$J = \left[\begin{array}{c|c} 2C(\mathbf{q}) + D(\mathbf{q}) - \lambda I & -\mathbf{q} \\ 2\mathbf{q}^T & 0 \end{array}\right] \quad (30)$$

where

$$D(\mathbf{q}) = \sum_{i=1}^{n}\sum_{j=1}^{2} \mathbf{q}^T M_{ji}\mathbf{q}M_{ji} \quad (31)$$

When convergence occurs the first four elements of **x** constitute the sought **q**. Observe that $J^{-1}(\mathbf{x}_k)$ exists in the case of a planar array of antennae with at least 2 nonparallel baselines and with measurements from at least 2 satellites. Also observe that $\lambda_0 = 0$ can be used to initialize the algorithm near the minimum point.

The discrete Newton-Raphson algorithm[11,12] requires that the initial point $\mathbf{x}_0$ be at the vicinity of the local minimum, and that the Hessian matrix, $J(\mathbf{x}_k)$, at that point be positive definite. In general, the Hessian matrix may be non-positive definite, therefore the executed search direction will not necessarily be a descending one. Moreover, the Hessian matrix may be singular and then its inverse does not exist.

Notice that near the minimum point a convergence of order 2 is achieved for a quadratic approximation of the function to be minimized. In the case where the function is essentially different from that approximation, the iterative search may wander and even diverge. Although in the discussed case the existence of $J^{-1}(\mathbf{x}_k)$ is guaranteed, one cannot guarantee the prevention of a wandering search or, a convergence to a non-desired point, which is not in the vicinity of the initial point $\mathbf{x}_0$. The example in the next section demonstrates such a situation. It is possible, however, to devise Newton algorithms, involving second order derivatives that possess local quadratic convergence quality without the aforementioned disadvantages. Such an algorithm is presented next.

VI.2 The Continuous Newton-Raphson Algorithm

The new algorithm is based on the gradient flow concept[13]. This Newton-Raphson gradient flow algorithm converges fast in an exponential manner from every initial condition. It is, in fact, a continuous version of the Newton-Raphson algorithm based on a solution of an ordinary differential equation (ODE). The steady state solution of the associated ODE is the desired solution of the nonlinear algebraic equation. Moreover, as will be shown later, the iterative Newton-Raphson algorithm is a special case of the continuous version obtained by a special selection of the integration step using the Euler integration scheme.

Consider the problem of finding the solution, $\mathbf{x}$, of the equation $\mathbf{f}(\mathbf{x}) = 0$ where $\mathbf{x} \in \Re^n$, $\mathbf{f}(\mathbf{x}) : \Re^n \to \Re^n$, and $\mathbf{f}(\mathbf{x})$ has a first derivative with respect to all elements of $\mathbf{x}$. Define

$$v = \frac{1}{2}\mathbf{f}^T\mathbf{f} > 0 \tag{32}$$

then

$$\dot{v} = \mathbf{f}^T\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\dot{\mathbf{x}} \tag{33}$$

Assuming that $\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^{-1}$ exists, it is always possible to define

$$\dot{\mathbf{x}} = -\frac{1}{2}\eta\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^{-1}\mathbf{f} \tag{34}$$

where η is a positive scalar. Substitution of $\dot{\mathbf{x}}$ of the last equation into Eq. (33) yields

$$\dot{v} = -\frac{1}{2}\eta\mathbf{f}^T\mathbf{f} = -\eta v < 0 \tag{35}$$

Remarks

(1) $\dot{v} = -\eta v$ guarantees an exponential decrease of v.

(2) $\dot{v} = 0$ if and only if $\mathbf{f}^T\mathbf{f} = 0$ which implies $\mathbf{f} = 0$.

(3) Observe that $\dot{\mathbf{x}} \to 0$ as $\mathbf{f} \to 0$. Therefore $\mathbf{x}^*$ which satisfies $\mathbf{f}(\mathbf{x}^*) = 0$ is the asymptotic solution of the differential equation associated with $\mathbf{x}$.

(4) Let $\dot{\mathbf{x}}$ define a smooth vector field on $\Re^n$. Assume that $\mathbf{x}^*$ is the associated equilibrium point. Let $\Omega \subset \Re^n$ be a compact neighborhood of $\mathbf{x}^*$. Then v is a Lyapunov function on Ω and $\mathbf{x}^*$ is a stable equilibrium point.

(5) Consider the Euler integration scheme for which $\dot{\mathbf{x}}$ is approximated by

$$\dot{\mathbf{x}} \cong \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{\Delta t} \tag{36}$$

Using Eq. (34) one gets

$$\frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{\Delta t} = -\frac{1}{2}\eta J^{-1}(\mathbf{x}_k)\,\mathbf{f}(\mathbf{x}_k) \tag{37}$$

where from Eq. (29)

$$J(\mathbf{x}_k) = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{x}_k} \tag{38}$$

Therefore

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \frac{1}{2}\eta\Delta t J^{-1}(\mathbf{x}_k)\,\mathbf{f}(\mathbf{x}_k) \tag{39}$$

Observe that a selection of the integration step as $\Delta t = \frac{2}{\eta}$ leads to the well-known iterative Newton-Raphson algorithm of Eq. (28).

VII. EXAMPLES

In this section we use the algorithms discussed in the preceding sections to solve the same problem. The first solution uses the Vectorized Phase Measurements algorithm where the phase measurements are converted to vector measurements and QUEST is used to obtain $\mathbf{q}^*$. The second solution uses the Iterative Eigenproblem algorithm. Next the problem is solved using the Discrete Newton-Raphson algorithm. We show that even though the initial guess of the quaternion is the same as the one used by the Eigenproblem algorithm, the algorithm settles on the wrong quaternion. In fact that quaternion yields a local maximum rather than the desired minimum. Following the failure of this algorithm we solve the problem using the Continuous Newton-Raphson algorithm and show that this algorithm converges to the correct solution even when using the same initial quaternion. We also show that when we start the Discrete Newton-Raphson algorithm with an initial guess closer to the correct solution the algorithm converges.

VII.1 Problem Statement

Given are five GPS satellites, which are used to generate the measurements. The vectors $\mathbf{s}_i$, $i = 1, 2, \ldots, 5$ are the unit vectors to these satellites. In the reference coordinate system they have the following values:

$$\mathbf{s}_1 = \begin{bmatrix} -0.2650 \\ 0.4589 \\ 0.8480 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} -0.7002 \\ 0.0984 \\ 0.7071 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 0.5038 \\ 0.3027 \\ 0.8090 \end{bmatrix}, \mathbf{s}_4 = \begin{bmatrix} -0.3221 \\ -0.1571 \\ 0.9336 \end{bmatrix}, \mathbf{s}_5 = \begin{bmatrix} 0.3335 \\ -0.5337 \\ 0.7771 \end{bmatrix}$$

The vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are the x and y coordinate axes of the body system *a*, in which the computations are performed (see Section I). Thus

$$\mathbf{a}_1^T = [1 \quad 0 \quad 0] \qquad \mathbf{a}_2^T = [0 \quad 1 \quad 0]$$

The transformation matrix D_a^e, from the reference to the body coordinates and its corresponding quaternion are

$$D_a^e = \begin{bmatrix} 0.7127 & 0.6588 & 0.2409 \\ -0.6797 & 0.5635 & 0.4696 \\ 0.1737 & -0.4984 & 0.8494 \end{bmatrix}$$

$$\mathbf{q}^T = [0.2738, -0.0190, 0.3786, 0.8840]$$

The corresponding Euler angles of this attitude are

$$\psi = 42.7530° \quad \theta = -13.9390° \quad \phi = 28.9390°$$

where the order of rotations is "3-2-1".For this geometry the *nominal* phase measurements are

$$B_{11} = 0.3178 \quad B_{12} = -0.2639 \quad B_{13} = 0.7534$$
$$B_{14} = -0.1082 \quad B_{15} = 0.0732$$
$$B_{21} = 0.8369 \quad B_{22} = 0.8634 \quad B_{23} = 0.2081$$
$$B_{24} = 0.5688 \quad B_{25} = -0.1624$$

The corrupted phase measurements are generated using zero mean Gaussian error. The standard deviations of the error associated with each of the five satellites are

$$\sigma_1 = 0.01 \quad \sigma_2 = 0.05 \quad \sigma_3 = 0.03 \quad \sigma_4 = 0.02 \quad \sigma_5 = 0.02$$

The errors themselves are errors in B_{ji}, the projection of $\mathbf{s}_i$, the unit vector to satellite i, on the body coordinate system axis j. We wish to find the quaternion that minimizes the cost function defined in Eq. (21) (see also Eq. 19). The weights p_i used in Eqs. (1) and (17) are calculated via the relations

$$w_i = \frac{1}{\sigma_i^2} \;;\; W = \sum_{i=1}^{5} w_i \;;\; p_i = \frac{w_i}{W} \quad \text{for } i = 1,5$$

VII.2 Attitude Estimation Using Vectorized Phase Measurements

When vectorizing the noise free nominal phase measurements according to Eq. (2), and using QUEST to compute the optimal quaternion $\mathbf{q}^*$, the achieved accuracy is

$$|\mathbf{q} - \mathbf{q}^*| < 1 \cdot 10^{-15}$$

The same computation for the noisy phase measurements leads to the following quaternion error

$$|\mathbf{q}-\mathbf{q}^*| < 0.0160\ ,\ \varphi_e = 1.8305^\circ$$

where φ_e is the Euler angle associated with the quaternion error $\mathbf{q}_e = \mathbf{q}\tilde{\mathbf{q}}^*$, and $\tilde{\mathbf{q}}^*$ denotes the adjoint of $\mathbf{q}^*$. The corresponding errors in yaw, pitch, and roll are

$\delta\psi = -0.1526^\circ \quad \delta\theta = 0.3889^\circ \quad \delta\phi = -1.7453^\circ$

VII.3 Attitude Estimation Using Direct phase measurements

Instead of using the vectorized measurements and consequently the QUEST algorithm, here the attitude quaternion is estimated using the algorithms developed for finding $\mathbf{q}^*$ directly from the phase measurements themselves. These algorithms yield the quaternion, which minimizes J of Eq. (21).

VII.3.1 Attitude Estimation Using the Iterative Eigenproblem Algorithm

The iteration starts with $\hat{\lambda}_0 = 0$ and $\hat{\mathbf{q}}_0^T = [0.4266, 0.1606, 0.0454, 0.8889]$ with corresponding initial angular error $\varphi_{e_0} = 47.1081^\circ$. This choice of the initial quaternion is associated with the following initial angular errors:

$\delta\psi = 29.7684^\circ \quad \delta\theta = -28.2260^\circ \quad \delta\phi = -23.9697^\circ$

After 4 iterations the solution settles on

$\hat{\mathbf{q}}_4^T = [0.2815,\ -0.0167,\ 0.3809,\ 0.8806]$

which yields a cost function value of 0.0003. The corresponding final attitude estimation error in terms of yaw, pitch and roll angles is

$\delta\psi = -0.2418^\circ \quad \delta\theta = 0.1736^\circ \quad \delta\phi = -0.9402^\circ$

The absolute value of the difference between the final quaternion and the true quaternion associated with this case is

$$|\mathbf{q}-\hat{\mathbf{q}}_4| = 0.0091\ ,\ \varphi_{e_4} = 1.0405^\circ$$

VII.3.2 Attitude Estimation Using the Discrete Newton-Raphson algorithm

Starting the iterations with the same initial conditions as before, the solution settles after 20 iterations on

$\hat{\mathbf{q}}_{20}^T = [0.8391,\ 0.0674,\ -0.1815,\ -0.5083]$

with corresponding angular error $\varphi_{e_{20}} = 146.3321^\circ$. The associated attitude estimation error is

$\delta\psi = 24.9133^\circ \quad \delta\theta = -27.5905^\circ \quad \delta\phi = -35.6287^\circ$

In this case the solution converges to the wrong stationary point, which is a maximum rather than a minimum of the cost function. The situation is described in Figure 1.

In drawing Fig. 1, one is faced with the problem of drawing a $5^{\underline{th}}$ dimensional shape. (Note that the value of λ does not affect the plot as long as the unity quaternion length constraint is maintained.) Next, it was observed that the basic shape of the cost function i.e.- the valley and hill characteristic- is invariant to the value of q_3 as long as it satisfies the unity property of the quaternion. The figure is drawn with $q_3 = 0.3809$ which is the third component of the minimizing quaternion, $\mathbf{q}^*$. The value of q_4 is determined by the values of q_1, q_2 and q_3. The calculated values of q_1 and q_2 used in the figure satisfy the constraint $q_1^2 + q_2^2 = 1 - q_3^2 - q_4^2$. In Fig. 1 we see the minimum point of J. Owing to the fact that we set q_3 to be a component of the minimizing quaternion, the maximum point on which the algorithm settles in this example is not the one shown in this figure. However, the shape of the surface is similar to the one obtained when the surface contains the maximum point.

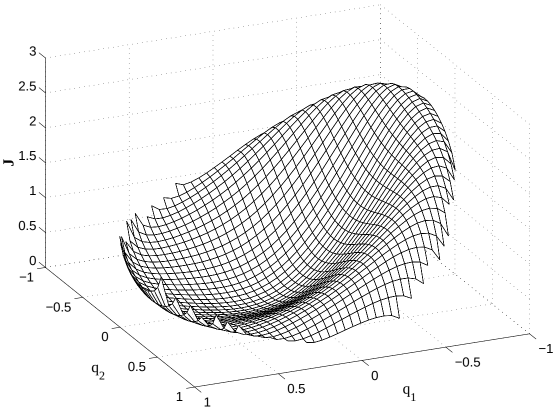

Figure 1: Typical behavior of the cost function

$$J(\mathbf{q}) = \mathbf{q}^T C(\mathbf{q})\mathbf{q}\ .$$

VII.3.3 Attitude Estimation Using the Continuous Newton-Raphson algorithm

The exact same data that has been used previously is used here. Because (a) the quartic cost function has one minimum and one maximum and, (b) $\lambda_0 = 0$ initializes the algorithm near the minimum point, then $J(\mathbf{x}_k)$ is positive definite and the continuous Newton-Raphson algorithm has a guaranteed exponential convergence to that minimum.

The ODE is solved using $\eta = 10^8$, which sets the time constant of Eq. (35) to 10^{-8}, and a 4th order Runge-Kutta integration scheme. Observe that a different selection of the integration scheme can be used. The selection depends on the stiffness of the ODE and a proper selection may simplify the implementation. In our case the stiffness is defined by η that can be set to any desired value. It takes about 10 time constants for the algorithm to reach the steady state value of the ODE, which is the minimum value of the cost function. The estimated attitude almost coincides with that of the Iterative Eigenproblem Algorithm. The differences in the final Euler angles between both algorithms are of the order of 10^{-7}. The convergence of the Continuous Newton-Raphson algorithm versus that of the Iterative Eigenproblem and the Discrete Newton–Raphson algorithms is presented in Figures 2. For the sake of comparison, the run time of the continuous Newton–Raphson algorithm is converted to iterations using $\Delta t = 2/\eta$, $\eta = 10^8$ where a time interval of Δt corresponds to one iteration. As is well known, the discrete Newton–Raphson algorithm is sensitive to the initial conditions. The initial conditions determine if the algorithm will converge and to which extreme point. To demonstrate the case consider the same data as before but with different initial conditions; $\hat{\mathbf{q}}_0 = [-0.0628, 0.2392, 0.1624, 0.9552]$ and $\hat{\lambda}_0 = 0$ with corresponding initial angular error

$\varphi_{e_0} = 55.7138°$. The convergence of the three algorithms in this case is described in Figure 3. All three algorithms converge to the minimum point with accuracy similar to that obtained before.

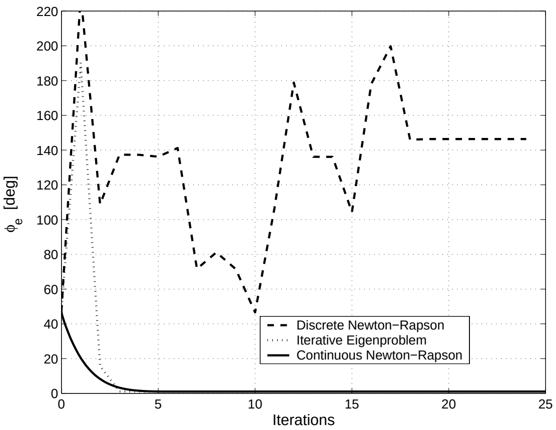

Figure 2: Performance Comparison of the Iterative Algorithms.

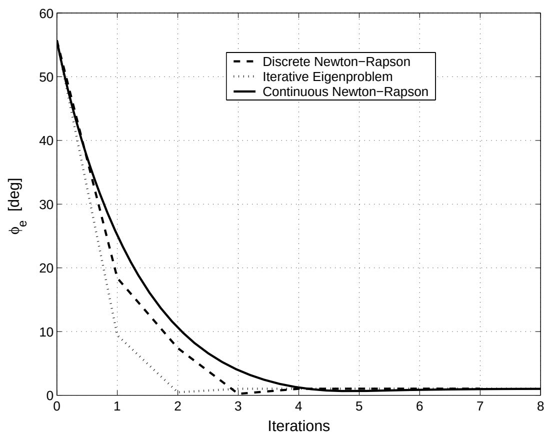

Figure 3: Performance Comparison of the Iterative Algorithms Using Initial Conditions Different from Those Shown in Fig. 2.

VIII. CONCLUSIONS

This paper presents algorithms for attitude determination using phase difference between GPS signals arriving at different antennae. Since the number of measurements is greater than the number of the unknown attitude parameters, and since the phase measurements are corrupted by noise, it is advantageous to find the attitude as a least squares fit.

To meet this end, the problem of attitude estimation is posed as a parameter optimization problem where a new quaternion-based quartic cost function is used. Then a Lagrange function is defined that included the quaternion unity-constraint. The conditions required for the Lagrange function to be stationary yield five nonlinear algebraic equations with five unknowns, four of which are the components of the optimal quaternion. Four of the equations have the form of eigenvalue/eigenvector problem. This structure leads to the derivation of a corresponding iterative algorithm.

Two additional algorithms are developed for solving the set of the five nonlinear algebraic equations. One is based on the well-known iterative Newton-Raphson algorithm. The other one is a continuous version of the latter algorithm with a guaranteed exponential convergence to the nearest local minimum located on the gradient direction in regions where the associated Hessian matrix is positive definite.

In all the examined examples, where the algorithms converged, we observed that not only they reached the solution very fast. Their accuracy was better than that of QUEST, particularly when the elevation of the GPS satellites was low. This was so not because of a deficiency in the q-Method or in QUEST, which implements this method, but because the measurement errors were amplified by low elevation when the phase measurements were converted to vector measurements, a step always needed when QUEST is used. It is proved in Ref. 14 that the transformation to vectors produces a correlated noise matrix, whereas Wahba's problem assumes an isotropic measurement error covariance matrix.

Finally, it should be noted that the algorithms presented in this work cover a deficiency in earlier work in that they are also applicable to attitude determination systems employing planar antenna arrays.

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